ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 14, Number 4, Fall 1984

ALGEBRAICITY VERSUS ANALYTICITY

JACEK BOCHNAK

Dedicated to the memory of Gus Efroymson

This paper presents a review of results concerning the problem of transforming an analytic set or function into an algebraic one. We are mainly interested in the real case, but we shall make also a few remarks about the complex (local) case.

LOCAL THEORY. Let $O_x(\mathbf{R}^n)$ (resp. $N_x(\mathbf{R}^n)$) be the ring of germs of real analytic (resp. Nash) functions at $x \in \mathbf{R}^n$. If $x = 0$ we simply note $O_0(\mathbf{R}^n) = O(n)$ and $N_0(\mathbf{R}^n) = N(n)$. Recall that a germ $f \in O(n)$ is said to be Nash if $P(x, f(x)) \equiv 0$ for some polynomial $P(X, Y) \in \mathbf{R}[x_1, \ldots, X_n, Y]$, $P \not\equiv 0$ [11]. It is well known that $N_0(n)$ is the henselization of $\mathbf{R}[X_1, \ldots, X_n]$ at the origin (we shall not use this fact). The graph of $f \in N(n)$ and its set of zeros are semi-algebraic germs in $\mathbf{R}^{n+1}$ and $\mathbf{R}^n$ respectively.

We say that two function-germs f, g: $(\mathbf{R}^n, x) \to \mathbf{R}$ at $x \in \mathbf{R}^n$ (resp. two set-germs $F, G \subset \mathbf{R}^n$ at $x \in \mathbf{R}^n$) are C^ν-equivalent ($\nu = 0, 1, \ldots, \infty, \omega$) if there is a local C^ν-diffeomorphism σ: $(\mathbf{R}^n, x) \to (\mathbf{R}^n, x)$such that $f \circ \sigma = g$ (resp. $\sigma(F) = G$).

PROBLEM. Describe analytic germs (of sets or of functions) C^ν-equivalent to algebraic or Nash ones.

Of course by an algebraic (resp. Nash) set we understand a set of the form $\bigcap_{i=1}^k f_i^{-1}(0)$, $f_i \in \mathbf{R}[X_1, \ldots, X_n]$ (resp. $f_i \in N(n)$). An algebraic set is obviously a Nash one. It will be shown later that any Nash setgerm is C^ω-equivalent to an algebraic one (Theorem 2). The first example of an analytic germ not C^1-equivalent to an algebraic one has been given by Whitney.

EXAMPLE 1. (Whitney [40]). The set of zeros of the germ $f \in O_0(\mathbf{R}^3)$ defined by the formula $f(t, x, y) = xy(y - x)(y - (3 + t)x)(y - 4e^t x)$ is not C^1-equivalent to a Nash germ (and hence, a fortiori, to an algebraic one). Observe that the structure of the set $V = f^{-1}(0)$ is very simple, i.e., V is a union of five non-singular surfaces intersecting along the t-axis. The proof of non-equivalence is based on the following remark. The

Copyright © 1984 Rocky Mountain Mathematics Consortium

largest cross-ratio of the first four surfaces at $(t, 0, 0)$ is $3 + t$; the largest cross-ratio of the first three and the last surface is $4e^t$. Each cross-ratio is intrinsically related to the variety at the given point (i.e., is invariant by C^1-local-diffeomorphisms $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$). It follows that if V were C^1-equivalent to a Nash germ, the function e^t would be algebraic, whereas e^t is transcendental.

Whitney considers his example rather in the holomorphic case, but the proof is the same in both cases.

EXAMPLE 2. (Tougeron [35] p. 220) is a modification of the previous one. The germ of zeros of the function $f \in O_0(\mathbf{R}^3)$ given by the formula

$$f(x, y, z) = z(x^2+y^2+2z)(x^2+y^2+z)(x^2+y^2-(1+x)z)(x^2+y^2-2e^xz)$$

is not C^ω-equivalent to a Nash germ. However for any $\nu \in \mathbf{N}$ the germ $f^{-1}(0)$ is C^ν-equivalent to an algebraic set. Observe that here $f^{-1}(0)$ is a surface with an isolated singular point at $0 \in \mathbf{R}^3$.

REMARK. Obviously one can consider the analogous definitions and problems for the complex data, i.e., over $\mathbf{C}$.

1. C^ν-equivalence of analytic sets, $\nu < \infty$. Recall that a set $\{E_i\}_{i=1,\dots,k}$ of linear subspaces of $\mathbf{R}^n$ is said to be in general position if codim $\bigcap_{i=1}^k E_i = \sum_{i=1}^k \operatorname{codim} E_i$. A finite collection of smooth submanifolds $M_\alpha \subset \mathbf{R}^n$, $\alpha \in \Lambda$, is in general position at a point $x \in \bigcup_{\alpha\in\Lambda} M_\alpha$ if the family of tangent spaces $\{T_xM_\alpha : \alpha \in \Lambda_x\}$ is in general position, where $\Lambda_x = \{\alpha \in \Lambda : x \in M_\alpha\}$.

THEOREM 1. *Let $V \subset (\mathbf{R}^n, 0)$ be a coherent, real analytic germ at $0 \in R^n$, and let $\nu \in \mathbf{N}$. Assume that V is a union of irreducible components $V = V_1 \cup \cdots \cup V_d$ satisfying the following conditions*:

(i) *each set $V_i \backslash \{0\}$ is an analytic submanifold, $i = 1, \dots, d$;*

(ii) *the family of submanifolds $\{V_i\backslash\{0\}\}_{i=1,\dots,d}$ is in general position at each point of $V\backslash\{0\}$.*

Then there is a local C^ν-diffeomorphism σ: $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, analytic outside of 0, such that $\sigma(V)$ is a germ of an algebraic subset of $\mathbf{R}^n$.

(NOTE. For $d = 1$ the theorem is due to Tougeron [38] (with the conclusion that $\sigma(V)$ is Nash).)

It is useful to formulate a more general result (in more algebraic form) which, together with Theorem 2 of the next section, implies immediately Theorem 1. First more notation.

Let $I_1, \dots, I_d$ be a finite family of ideals of $O(n)$ and let $k_1, \dots, k_d$ be a sequence of positive integers. Let $G = \prod_{i=1}^d G_i \subset GL(k, \mathbf{R})$, where $G_i = GL(k_i, \mathbf{R})$ and $k = k_1 + \cdots + k_d$, and

$$f = (f_{11}, \dots, f_{1k_1}, \dots, f_{d1}, \dots, f_{dk_d}) \in \underbrace{O(n) \times \cdots \times O(n)}_{k} = \bigoplus_k O(n).$$

Let $\{A_i\}$, $i = 1, \ldots, q$, $q = k_1^2 + \cdots + k_d^2$ be a vector basis of the tangent space $T_E G$ to G at $E = E_k$ (= the identity $k \times k$ matrix), and let $J_{(k_1,\ldots,k_d)}(f)$ be the ideal of $O(n)$ generated by $k \times k$ minors of the $k \times (n+q)$ matrix $L_G(f)$ with $A_1 \cdot f, \ldots, A_q \cdot f, \partial f/\partial x_1, \ldots, \partial f/\partial x_n$ as columns

$$L_G(f) = \left(A_1 \cdot f, \ldots, A_q \cdot f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Let $J_{(k_1,\ldots,k_d)}(I_1, \ldots, I_d)$ be the ideal of $O(n)$ generated by $J_{(k_1,\ldots,k_d)}(f)$, where $f = (f_{11}, \ldots, f_{1k_1}, \ldots, f_{d1}, \ldots, f_{dk_d})$, $f_{ij} \in I_i$. Given an ideal I of $O(n)$, let $Z(I)$ be its set of zeros. We say that I is elliptic if $Z(I) \subset \{0\}$.

It is easy to see that a coherent irreducible germ $(V, 0) \subset (\mathbf{R}^n, 0)$ has an isolated singularity at 0 if and only if the ideal $J_{(k)}(I)$ is elliptic; $k =$ codim V and $I = \mathrm{id}_0(V) = \{\varphi \in O(n): \varphi(V) = 0\}$. More generally we have

Lemma 1. *Let $V_1, \ldots, V_d$ be a finite family of irreducible, coherent analytic germs at $0 \in \mathbf{R}^n$. Let $I_i = id_0(V_i)$ be the ideal of V_i and let $k_i =$* codim *V_i, $i = 1, \ldots, k$. Then the following conditions are equivalent*:

(i) *each $V_i \backslash \{0\}$ is an analytic manifold and the family $V_1, \ldots, V_d$ is in general position at each point of $V \backslash \{0\}$, $V = V_1 \cup \cdots \cup V_d$.*

(ii) $Z(J_{(k_1,\ldots,k_d)}(I_1, \ldots, I_d)) \subset \{0\}$.

Proof. This follows from the above observation concerning the ideal $J_{(k)}(I)$ together with some elementary argument of linear algebra.

Since any Nash set-germ is C^ω-equivalent to an algebraic one (cf. Theorem 2), Theorem 1 is an immediate consequence of the following statement. Let $E_\nu = E_\nu(n)$ be the ring of germs of C^ν-functions $(\mathbf{R}^n, 0) \to \mathbf{R}$, $0 \leqq \nu \leqq \infty$.

Theorem 1′. *Let $V_1, \ldots, V_d$ be a family of coherent irreducible analytic germs at $0 \in \mathbf{R}^n$, $I_i = id_0(V_i)$, $k_i =$* codim *V_i. Assume that the ideal $J_{(k_1,\ldots,k_d)}(I_1, \ldots, I_d)$ is elliptic. Let $\nu \in \mathbf{N}$. Then there is a local C^ν-diffeomorphism τ: $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, analytic outside of 0, such that the ideal $\tau^*(I_i)E_\nu$ of E_ν is generated by Nash functions, $i = 1, \ldots, d$.*

Notation. $\tau^*: E_\nu \to E_\nu$ is an isomorphism defined by $\tau^*(\varphi) = \varphi \circ \tau$.

The proof of this theorem for $d = 1$ is given by Tougeron ([**38**], Th. 2.4′) and uses in an essential way his generalization of M. Artin's theorem on the solutions of analytic equations. The proof of Theorem 1′ for $d > 1$ goes along similar lines, but is not quite an automatic extension of Tougeron's proof. For $d = 1$ the assumption concerning the mutual behaviour of V_i is, of course, empty. On the contrary, for $d > 1$ such a behaviour must be taken under consideration, as Example 1 above shows. In his original statement (for $d = 1$) Tougeron uses the ideal $J_k(I)$ which, in general, is different from our $J_{(k)}(I)$, although both have the same radical. The proof of Theorem 1 will appear in [**43**].

Let us consider the special case of Theorem 1 (or 1′), when the V_i's are complete intersections.

THEOREM 1″. *Let* $V \subset (\mathbf{R}^n, 0)$ *be a coherent analytic germ at* $0 \in \mathbf{R}^n$, *and let* $\nu \in \mathbf{N}$. *Assume that* $V = V_1 \cup \cdots \cup V_d$ *is a union of irreducible components satisfying the following conditions*:

(i) *each* V_i *is a complete intersection* (*i.e., the ideal* $\mathrm{id}_0(V_i)$ *is generated by* codim V_i *elements*) *and* $V_i \backslash \{0\}$ *is an analytic submanifold*, $i = 1, \ldots, d$;

(ii) *the family* $\{V_i \backslash \{0\}\}$ *is in general position at each point of* $V \backslash \{0\}$. *Then there is a local* C^ν*-diffeomorphism* τ: $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$, *analytic outside of* $0 \in \mathbf{R}^n$, *such that each ideal* $\tau^*(\mathrm{id}_0(V_i))E_\nu$ *is generated by polynomials. In particular* V *is* C^ν*-equivalent to an algebraic set* $Z \subset \mathbf{R}^n$. *Moreover* $\tau(V_i) = Z_i$, *where the* Z_i's *are irreducible* (*algebraic*) *components of* Z.

REMARK. In Theorem 1″, V itself need not be a complete intersection. Let, for example, $V_1 = \{x \in \mathbf{R}^4 \colon x_1 = x_2 = 0\}$ and $V_2 = \{x \in \mathbf{R}^4 \colon x_3 = x_4 = 0\}$. Then $V = V_1 \cup V_2$ is not a complete intersection [**19**].

Theorem 1″ (stated explicitly for $d = 1$ in [**35**]) is a consequence of another result of Tougeron. We are going to explain this result which certainly deserves to be better known.

Let G be a q-dimensional Lie group in $GL(k, \mathbf{R})$. Let $G^\nu(n)$ be the group of germs at $0 \in \mathbf{R}^n$ of C^ν-maps $(\mathbf{R}^n, 0) \to (G, E)$ (where $E =$ the identity matrix of $GL(k, \mathbf{R})$), and let $\mathrm{Diff}^\nu(n)$ be the group of germs at $0 \in \mathbf{R}^n$ of local C^ν-diffeomorphisms $(\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$; here ν may be $0, 1, \ldots, \infty$ or ω. The set $\Omega^\nu(n) = G^\nu(n) \times \mathrm{Diff}^\nu(n)$ is a group with the multiplication defined by $(g_1, \tau_1) \cdot (g_2, \tau_2) = (g_1 \cdot (g_2 \circ \tau_1^{-1}), \tau_1 \circ \tau_2))$, acting on $\oplus_k E_\nu(n)$ as follows: for $(g, \tau) \in \Omega^\nu(n)$ and $f \in \oplus_k E_\nu(n)$, $(g, \tau) \cdot f$ is the germ at 0 of the mapping $x \to g(x) \cdot (f \circ \tau^{-1}(x))$. For $f \in \oplus_k O(n)$, $jl(f)$ will denote the l-jet of f at $0 \in \mathbf{R}^n$. A germ $f \in \oplus_k O(n)$ is said to be C^ν-finitely G-determined if there exists an $l \in \mathbf{N}$ such that for any $h \in \oplus_k O(n)$, with $j^l(h) = j^l(f)$, one has $h = (g, \tau) \cdot f$ for some $(g, \tau) \in \Omega^\nu(n)$. Obviously an $\Omega^\nu(n)$-orbit of a C^ν finitely G-determined map-germ contains a polynomial.

The following characterization of finitely G-determined maps is due to Tougeron. Let $\{A_1, \ldots, A_q\}$ be a basis over $\mathbf{R}$ of the Lie algebra $T_E G$ of G. For $f \in \oplus_k O(n)$, $f(0) = 0$, let M_f be $k \times (q + n)$ matrix with $A_1 \cdot f, \ldots, A_q \cdot f, \partial f/\partial x_1, \ldots, \partial f/\partial x_1$ as columns. Let $I_G(f)$ be an ideal of $O(n)$ generated by the $k \times k$ minors of this matrix.

THEOREM I. (Tougeron). *Let* f: $(\mathbf{R}^n, 0) \to (\mathbf{R}^k, 0)$ *be an analytic map-germ and let* $G \subset GL(k, \mathbf{R})$ *be a Lie group. Then*

(i) ([36] p. 169) *f is C^ω-finitely G-determined if and only if $I_G(f)$ contains a power of the maximal ideal of $O(n)$.*

(ii) [37] *For any $\nu \in \mathbf{N}$, f is C^ν-finitely G-determined if $I_G(f)$ is elliptic.*

REMARK. Theorem I (i) also holds true for holomorphic germs and complex Lie groups.

PROOF (of Theorem 1″). Let $k_i = \operatorname{codim} V_i$ and let $f_{i1}, \dots, f_{ik_i}$ be a system of generators of the ideal $\mathrm{id}_0(V_i)$, $i = 1, \dots, d$. Let $f = (f_{11}, \dots, f_{1k_1}, \dots, f_{d1}, \dots, f_{dk_d})$. The ideal $J_{(k_1,\dots,k_d)}(f)$ defined earlier is precisely the ideal $I_G(f)$, described above, for $G = \prod_{i=1}^d G_i \subset GL(k, \mathbf{R})$, $k = k_1 + \cdots + k_d$, $G_i = GL(k_i, \mathbf{R})$. The assumptions (i) and (ii) of Theorem 1″ imply (and in fact are equivalent to) the ellipticity of the ideal $J_{(k_1,\dots,k_d)}(f)$. Hence Theorem 1″ follows directly from Theorem I (ii).

The main unsolved problem related to this section is, of course, whether or not any real analytic set-germ is C^0-equivalent to an algebraic (or Nash) one.

A claim concerning this problem is given, without proof, in [24]. Any germ of a complex analytic surface in $\mathbf{C}^3$ is C^0-equivalent to an algebraic subset of $\mathbf{C}^3$ [25].

2. C^ω-equivalence of analytic sets. In this section we shall also consider complex analytic germs. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ and let $O_0(\mathbf{K}^n) = O(n)$ ($N_0(\mathbf{K}^n) = N(n)$, etc.) be the ring of germs of $\mathbf{K}$-analytic functions (Nash functions over $\mathbf{K}$, etc.). Of course by "$\mathbf{K}$-analytic" we mean "holomorphic" if $\mathbf{K} = \mathbf{C}$, and "real analytic" if $\mathbf{K} = \mathbf{R}$.

THEOREM 2. *Let $(V, 0) \subset (\mathbf{K}^n, 0)$ be a germ of a Nash set. Then there is a local $\mathbf{K}$-analytic isomorphism $\sigma\colon (\mathbf{K}^n, 0) \to (\mathbf{K}^n, 0)$ such that the ideal $\mathrm{id}_0(\sigma(V))$ of $O(n)$ of germs vanishing on $\sigma(V)$ is generated by polynomials. In particular V is analytically equivalent to a germ of an algebraic subset of $\mathbf{K}^n$.*

NOTE. The theorem seems to be known to a few specialists (certainly it's known to Michael Artin), but no proof has ever been published. One can obtain a better result ([43]), i.e., if $V = V_1 \cup \cdots \cup V_q \subset (\mathbf{K}^n, 0)$ is a Nash germ with irreducible analytic components $V_1, \dots, V_q$, then there is a local $\mathbf{K}$-analytic isomorphsim $\sigma\colon (\mathbf{K}^n, 0) \to (\mathbf{K}^n, 0)$ such that, for any $i = 1, \dots, q$, the ideal $\mathrm{id}_0(\sigma(V_i))$ is generated by polynomials. In particular each component $\sigma(V_i)$ of $\sigma(V)$ is algebraic.

Now let us consider the following conjecture.

CONJECTURE 1. Let $(V, 0) \subset (\mathbf{K}^n, 0)$ be a germ of a coherent analytic set at $0 \in \mathbf{K}^n$. Assume that $V = V_1 \cup \cdots \cup V_d$ is a union of irreducible components V_i, satisfying the following conditions:

(i) each set $V_{i\mathbf{C}}\backslash\{0\}$ is a holomorphic submanifold in $\mathbf{C}^n$, $i = 1, \ldots, d$;

(ii) the family of submanifolds $\{V_{i\mathbf{C}}\backslash\{0\}\}_{i=1,\ldots,d}$ is in general position at each point of $V_{\mathbf{C}}\backslash\{0\}$.

Then V is C^ω-equivalent to an algebraic subset of $\mathbf{K}^n$, i.e., there is a local analytic isomorphism σ: $(K^n, 0) \to (\mathbf{K}^n, 0)$ such that $\sigma(V)$ is a germ of an algebraic subset of $\mathbf{K}^n$.

NOTATION. $V_{\mathbf{C}} = V$ when $\mathbf{K} = \mathbf{C}$, and $V_{\mathbf{C}}$ = the complexification of V if $\mathbf{K} = \mathbf{R}$.

The conjecture is known to be true in several important cases.

THEOREM 3 (Artin [5], Tougeron [38]). *Let $V \subset (\mathbf{K}^n, 0)$ be a coherent analytic germ (of pure dimension). Assume that $V_{\mathbf{C}}$ has an isolated singular point at $0 \in \mathbf{C}^n$. Then V is C^ω-equivalent to a germ of an algebraic subset of $\mathbf{K}^n$.*

The proof of this theorem is rather complicated and combines very refined algebraic and analytic technique. One proves first that V is C^ω-equivalent to a Nash germ, and after that one applies Theorem 2.

THEOREM 4. *Let $V \subset (\mathbf{K}^n, 0)$ be a coherent analytic germ. Assume that $V = V_1 \cup \cdots \cup V_d$ is a union of irreducible components satisfying:*

(i) *each set $V_{i\mathbf{C}}\backslash\{0\}$ is a holomorphic submanifold, and each V_i is a complete intersection, $i = 1, \ldots, d$;*

(ii) *the family $\{V_{i\mathbf{C}}\backslash\{0\}\}_{i=1,\ldots,d}$ is in general position at each point of $V_{\mathbf{C}}\backslash\{0\}$.*

Then there is a local analytic isomorphism σ: $(\mathbf{K}^n, 0) \to (\mathbf{K}^n, 0)$ such that each ideal $\sigma^*(\mathrm{id}_0(V_i))$ is generated by polynomials. In particular V is C^ω-equivalent to an algebraic set.

PROOF. Using the notation as in the proof of Theorem 1″ (with $\mathbf{R}$ replaced by $\mathbf{K}$), the assumptions (i) and (ii) imply that the ideal $J_{(k_1,\ldots,k_d)}(f) = I_G(f)$ contains a power of the maximal ideal of $O_0(\mathbf{K}^n)$. Hence the theorem follows directly from Tougeron's Theorem I (i).

3. C^ν-Equivalence of analytic function-germs, $O \leqq \nu \leqq \omega$. Let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. For $f \in O_0(\mathbf{K}^n)$ let $f_{\mathbf{C}} = f$ if $\mathbf{K} = \mathbf{C}$, and let $f_{\mathbf{C}}$ be the complexification of f if $\mathbf{K} = \mathbf{R}$.

THEOREM 5 ([13], [17]). *Let $f \in O_0(\mathbf{K}^n)$ be a germ of a $\mathbf{K}$-analytic function, $f(0) = 0$, $f = \varepsilon \prod_{i=1}^{p} f_i^{n_i}$, where $f_i \in O_0(\mathbf{K}^n)$ are irreducible, relatively prime factors of f, $\varepsilon = \pm 1$. Assume that each $f_{i\mathbf{C}}^{-1}(0)\backslash\{0\}$, $i = 1, \ldots, p$, is a complex analytic submanifold, and that $f_{\mathbf{C}}^{-1}(0)$ is a normal crossing at each point $\neq 0$. Then f is $\mathbf{K}$-analytically equivalent to a polynomial in $\mathbf{K}[X_1, \ldots, X_n]$.*

COROLLARY 1 [28], [17]. *Any analytic function-germ f: $(\mathbf{K}^2, 0) \to \mathbf{K}$ of two variables is $\mathbf{K}$-analytically equivalent to a polynomial.*

THEOREM 6. *Let* $f \in O_0(\mathbf{R}^n)$, $f(0) = 0$, $f = \varepsilon\prod_{i=1}^p f_i^{n_i}$, *where* $f_i \in O_0(\mathbf{R}^n)$ *are irreducible, relatively prime factors of* f, $\varepsilon = \pm 1$. *Assume that each factor* f_i *has an isolated critical point at* $0 \in \mathbf{R}^n$, *and that* $f^{-1}(0)$ *is a normal crossing at each point* $\neq 0$. *Let* $\nu \in \mathbf{N}$. *Then* f *is* C^ν*-equivalent to a polynomial function.*

Both theorems, which are probably the most general known results concerning the algebraicity of analytic function-germs, are an immediate consequence of a characterization (published in **[17]** (complex case) and **[13]** (Th. 1.1 and Th. 1.3)) of a so-called weakly finitely determined germs, introduced by Cerveau and Mattei. Let us recall this notion which will be also used in section 6. Let f: $(\mathbf{K}^n, 0) \to (\mathbf{K}, 0)$ be a germ of a $\mathbf{K}$-analytic function and let $f = \varepsilon\prod_{i=1}^p f^{n_i}_i$, $\varepsilon = \pm 1$, f_i irreducible, relatively prime factors of f. The germ f is said to be weakly finitely determined if there exists an $r \in \mathbf{N}$ such that any $g \in O_0(\mathbf{K}^n)$ with $g = \varepsilon\prod_{i=1}^p g_i^{n_i}$, $j^r(f_i) = j^r(g_i)$, $i = 1, \ldots, p$, is C^ω-equivalent to f. Weakly finitely determined germs are precisely the germs satisfying the assumptions of Theorem 5. The related theory of finite determinacy of functions and sufficiency of jets is investigated in an excellent review by C.T.C. Wall **[39]**, also containing very complete references.

Now let us consider the following problem. Assume that for an $f \in O_0(\mathbf{K}^n)$, the germ $f^{-1}(0)$ is algebraic (or Nash). Is f equivalent to a germ of a polynomial?

THEOREM 7 **[43]**. *Let* f: $(\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ *be an analytic germ. If* $f^{-1}(0)$ *is semi-algebraic, then* f *is* C^0*-equivalent to a polynomial function.*

Let V be a germ at $0 \in \mathbf{K}^n$ of a coherent $\mathbf{K}$-analytic hypersurface in $\mathbf{K}^n$. Let $f \in O_0(\mathbf{K}^n)$ and let $\Delta(f_x)$ be the jacobian ideal of the germ $f_x \in O_x(\mathbf{K}^n)$, $x \in \mathbf{K}^n$, i.e., $\Delta(f_x) = \sum_{i=1}^n(\partial f/\partial x_i)\, O_x(\mathbf{K}^n)$. Define

$$\textstyle\sum_\Delta(f) = \{x \in f^{-1}(0) : f_x \notin \Delta(f_x)\}.$$

$\sum_\Delta(f)$ is an analytic subset of codim $\geqq 2$ in $\mathbf{K}^n$.

THEOREM 8. *Let* f: $(\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ *be a holomorphic germ. If* $V = f^{-1}(0)$ *is Nash and* $\sum_\Delta(f) \subset \{0\}$, *then* f *is holomorphically equivalent to a polynomial.*

PROOF. Using the remark after Theorem 2, we may assume that V is algebraic and that $f = \varphi g$, for some $g \in \mathbf{C}[X_1, \ldots, X_n]$ and some $\varphi \in O_0(\mathbf{C}^n)$, $\varphi(0) \neq 0$. The assumption $\sum_\Delta(f)/fM^{q-1} \subset \{0\}$ implies that $f^{q-1} \subset \Delta(f)$ for some $q \in \mathbf{N}$, where M is the maximal ideal of $O_0(\mathbf{C}^n)$. Let w be a complex polynomial of n variables, such that $j^q(w) = j^q(\varphi)$. Put $\rho = wg$ and $u = \varphi/w \in O_0(\mathbf{C}^n)$. Then $f = u\rho$, where $\rho \in \mathbf{C}[X_1, \ldots, X_n]$ and $j^q(u) = 1$. It follows from Lemma 1.3 in **[13]** that $f = u\rho$ and ρ are holomorphically equivalent.

COROLLARY 2. *Let* f: $(\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ *be a holomorphic germ. If* $f^{-1}(0)$ *is a Nash set and* $f \in \Delta(f)$, *then* f *is holomorphically equivalent to a polynomial.*

REMARKS. (i) Theorem 8 and Corollary 2 hold true for real analytic germs, assuming that any irreducible factor φ of f has codim $\varphi^{-1}(0) = 1$.
(ii) Is Theorem 8 true without the assumption $\Sigma_\Delta(f) \subset \{0\}$?

Open problem. Let f: $(\mathbf{K}^n, 0) \to (\mathbf{K}, 0)$ be a germ of an analytic function. Assume that $f = \varphi g$, where $\varphi \in O_0(\mathbf{K}^n)$, $\varphi(0) \neq 0$, and $g \in \mathbf{K}[X_1, \ldots, X_n]$. Is f equivalent to a polynomial? (See addendum located after references.)

GLOBAL THEORY. In this section M will be always an algebraic manifold, i.e., a non-singular real algebraic subset of $\mathbf{R}^n$. As usual, we say that two sets F, $G \subset M$ (resp. two real-valued functions f, g: $M \to \mathbf{R}$) are C^ν-equivalent if there is a global C^ν-diffeomorphism τ: $M \to M$ such that $\tau(F) = G$ (resp. $f \circ \tau = g$); $\nu = 0, 1, \ldots, \infty, \omega$.

Our main concern now is the problem of equivalence between global real analytic and real algebraic sets, as well as the analogous question for functions. We shall deal with the elements of the following rings of real-valued functions on M.

$P[M]$ = ring of polynomial functions, i.e., $P[M] = \mathbf{R}[X_1, \ldots, X_n]$/ideal of M,

$R(M)$ = ring of entire rational functions, i.e., $R(M) = \{f/g : f, g \in P[M], g^{-1}(0) = \varnothing\}$,

$N(M)$ = ring of Nash functions ([**11**], [**45**]),

$O(M)$ = ring of analytic functions.

Of course $P[M] \subset R(M) \subset N(M) \subset O(M)$. A subset $A \subset M$ is called algebraic, Nash or analytic if A is of the form $A = f^{-1}(0)$ for some f in $P[M]$, $N(M)$ or $O(M)$ respectively.

4. Topology of real algebraic varieties. [**1**]–[**4**], [**6**]–[**10**], [**21**], [**22**], [**32**]–[**34**], [**46**]. Several very good accounts concerning the topology of real algebraic sets are now available ([**1**], [**21**], [**34**], [**46**] and the original papers listed in the references), but let us metnion briefly some results closely related to the problem of equivalence of analytic, Nash and algebraic sets. One of the main questions is, of course,

(A) Which topological spaces are homeomorphic to real algebraic sets of $\mathbf{R}^n$?

The following fundamental theorem of Tognoli has greatly stimulated the researches during the past decade.

THEOREM 9. (Tognoli [**32**], [**22**]) *Any compact smooth manifold is* C^∞ *diffeomorphic to a non-singular algebraic subset of* $\mathbf{R}^n$.

Tognoli's result was conjectured by J. Nash in 1952 (who showed a weaker result) and is based on two fundamental theories unknown in 1952—transversality and cobordism.

A beautiful characterization of compact algebraic sets with isolated singularities has been discovered recently by Akbulut and King.

THEOREM 10 [3]. *For a compact topological space, the following conditions are equivalent*:

(i) *X is homeomorphic to a real algebraic set with isolated singularities.*

(ii) *X is homeomorphic to the quotient space obtained by taking a compact smooth manifold Y and collapsing each K_i to a point, where K_i, $i = 1, \ldots, l$, is a collection of disjoint smooth subpolyhedra of Y.*

COROLLARY 3. *Any (global) compact, purely dimensional analytic set X, with a finite set $\sum$ of singular points, is homeomorphic to an algebraic set.*

PROOF. Let $\sum = \{a_1, \ldots, a_k\}$. Using Hironaka's resolution of singularities one obtains a continuous map φ: $Y \to X$, with Y a compact analytic manifold with some analytic subsets (and hence smooth subpolyhedra) $K_i = \varphi^{-1}(a_i)$, $i = 1, \ldots, k$, so that $\varphi | Y \backslash \bigcup_{i=1}^k K_i$: $Y \backslash \bigcup_{i=1}^k K_i \to X \backslash \sum$ is a homeomorphism. So X is obtained from Y by collapsing each K_i to a point, and hence by Theorem 10, is homeomorphic to an algebraic set.

The major open question is, of course,

whether any compact analytic set is homeomorphic to an algebraic one?

It is enough to solve the problem of topological characterization of algebraic sets only for compact spaces, since it was shown [3] that a topological space X is homeomorphic to an algebraic set if and only if X is locally compact and the one point compactification of X is homeomorphic to a real algebraic set.

By a well known theorem of Lojasiewicz [23] any real analytic set is triangulable (an easier theorem on triangulability of algebraic sets is proven in an elegant paper of Hironaka [20]). The essential problem therefore, would be to decide which polyhedrons are homeomorphic to algebraic sets. A necessary conditions is

THEOREM 11 (Sullivan [31]). *A real analytic set is locally homeomorphic to the cone over a polyhedron with even Euler characteristic.*

Sullivan's conditions fully characterise algebraic sets of dimension $\leqq 2$.

THEOREM 12 (Benedetti-Dedo [9]). *A compact polyhedron X of dimension $\leqq 2$ is homeomorphic to a real algebraic set if and only if X satisfies Sullivan's even Euler characteristic condition.*

COROLLARY 4. *Any compact* 2 *dimensional analytic set is homeomorphic to an algebraic one.*

Unfortunately Theorem 12 is false if $\dim X \geqq 3$ [**1**]. Recently Akbulut and King have described a very large class of topological spaces (A-spaces [**4**]) homeomorphic to algebraic sets. The class of A-spaces contains, for example, any compact PL manifold.

(B) Homology classes represented by algebraic sets. Let $M \subset \mathbf{R}^n$ be a compact algebraic m-manifold and let $Z \subset M$ be an analytic subset of dimension k. Taking any triangulation of M compatible with Z, one can show ([**16**], [**18**]) that every $(k-1)$-simplex contained in Z is a face of an even number of k-simplexes of Z. This implies that Z determines a homology class $[Z] \in H_k(M, \mathbf{Z}_2)$. Let $H_k^{alg}(M, \mathbf{Z}_2)$ be the subgroup of $H_k(M, \mathbf{Z}_2)$ of homology classes represented by k-dimensional algebraic subsets of M, and let $H_*^{alg}(M, \mathbf{Z}_2) = \oplus_{k=0}^m H_k^{alg}(M, \mathbf{Z}_2)$. Let $H_{alg}^*(M, \mathbf{Z}_2)$ be the subgroup of $H^*(M, \mathbf{Z}_2)$ corresponding to $H_*^{alg}(M, \mathbf{Z}_2)$ by the Poincaré duality. In fact H_{alg}^* is a subring of H^*[**8**]. The question whether a homology class is in H_*^{alg} appears to be essential. This is however not always the case.

Positive results. A continuous vector bundle $\xi = \{\pi\colon E \to M\}$ over M, of rank k, is said to be strongly algebraic if E is an affine algebraic variety over $\mathbf{R}$, and the projection π, as well as the transition functions of ξ, are regular morphisms [**6**]. It should be mentioned that there exists an algebraic vector bundle (i.e., a bundle with the total space being a real algebraic variety in the sense of Serre) which is not strongly algebraic [**33**]. This class of vector bundles is important because of the following result: $H_{alg}^*(M, \mathbf{Z}_2)$ contains the subring generated by Stiefel-Whitney classes of all strongly algebraic vector bundles over M [**8**].

THEOREM 13 [**6**], [**8**]. *Let X be a compact, connected smooth manifold. Then there is an algebraic manifold M diffeomorphic to X, such that*

(i) *Any continuous vector bundle (of finite rank) is C^0-isomorphic to a strongly algebraic one.*

(ii) *$H_{alg}^*(M, \mathbf{Z}_2)$ contains the subring generated by the Stiefel-Whitney classes of all continuous vector bundles over M and by the set of classes representable by smooth submanifolds of M.*

THEOREM 14 [**8**], [**14**], [**30**]. *If M is a connected, compact algebraic m-manifold, then $H_{m-1}^{alg}(M, \mathbf{Z}_2) = H_{m-1}(M, \mathbf{Z}_2)$ if and only if any continuous line bundle over M is isomorphic to a strongly algebraic one.*

COROLLARY 5 [**2**], [**7**]. *For any compact, connected smooth m-manifold X there exists an algebraic manifold M diffeomorphic to X, such that $H_{m-1}^{alg}(M, \mathbf{Z}_2) = H_{m-1}(M, \mathbf{Z}_2)$.*

Negative results.

THEOREM 15 (Benedetti-Tognoli [**8**], Risler [**26**], Silhol [**44**]). *For any* $m \geqq 2$ *there is a compact, connected algebraic m-manifold* M, *such that* $H_{m-1}^{alg}(M, \mathbf{Z}_2) \neq H_{m-1}(M, \mathbf{Z}_2)$.

In fact the result is more precise: For any $p \geqq 3$ and any smooth compact, connected p-manifold X there is an algebraic m-manifold, $m = p + 1$, differomorphic to $X \times S^1$ such that $H_{m-1}^{alg}(M, \mathbf{Z}_2) \neq H_{m-1}(M, \mathbf{Z}_2)$. The proof for $X = S^3$ is given in [**8**] and the general case goes along the same line, based on two observations

(α) If V is a compact non-singular irreducible real algebraic curve $V \subset \mathbf{R}^2$, with two connected components V_1 and V_2, then the line bundle ξ over $V \times S^1$ which is trivial over $V_1 \times S^1$ but non trivial over $V_2 \times S^1$, is not isomorphic to a strongly algebraic one.

(β) One can find an algebraic m-manifold M containing $V \times S^1$, diffeomorphic to $X \times S^1$, and a line bundle $\bar{\xi}$ over M, such that $\bar{\xi}|V \times S^1 \cong \xi$.

Hence $\bar{\xi}$ is a line bundle which cannot be isomorphic to a strongly algebraic one, and therefore by Theorem 14, $H_{m-1}^{alg}(M, \mathbf{Z}_2) \neq H_{m-1}(M, \mathbf{Z}_2)$. A similar idea also works for $m = 3$, at least for some X. Unexpectedly the case of dimension 2 in Theorem 15 is particularly hard and one uses a different method to show the following.

THEOREM 16 (Risler [**26**]). *For any* $1 \leqq k \leqq 9$ *there is an orientable, compact connected non singular algebraic surface* $T_k \subset \mathbf{R}^n$, *of genus* k, *such that* $H_1^{alg}(T_k, \mathbf{Z}_2) \neq H_1(T_k, \mathbf{Z}_2)$ (*in fact* $\dim_{\mathbf{Z}_2} H_1^{alg}(T_k, \mathbf{Z}_2) \leqq 1$).

Finally let us mention a result which will not be used in this review, but which is very important and clarifies the situation concerning the topology of real algebraic sets.

THEOREM 17 (Benedetti-Dedó [**10**]). *For any* $m \geqq 11$ *there is a compact, connected smooth m-manifold X such that for any algebraic manifold M homeomorphic to X*, $H_{m-2}^{alg}(M, \mathbf{Z}_2) \neq H_{m-2}(M, \mathbf{Z}_2)$.

After these preparations we may start the discussion of the problem of global equivalence of analytic and algebraic sets and functions.

5. Equivalence of analytic and algebraic sets.

(A) *Non singular case.* Let $X \subset \mathbf{R}^{m+k}$ be an analytic compact m-submanifold of $\mathbf{R}^{m+k}$. The proof of Tognoli's Theorem 9 shows more: if $k > m$ then there is an analytic diffeotopy of $\mathbf{R}^{m+k}$, arbitrarily close to the identity, transforming X onto a non-singular algebraic subset of $\mathbf{R}^{m+k}$. However it is an open challenging question whether X can be realized as an algebraic subset in the same $\mathbf{R}^{m+k}$, even if k is not greater than m. In particular it is unknown whether any compact non-orientable 3-manifold in $\mathbf{R}^5$ can be realized as an algebraic subset of $\mathbf{R}^5$. Recently two results concerning this problem have been found.

THEOREM 18. *Let $X \subset \mathbf{R}^{m+k}$ be a compact analytic m-submanifold. Then there is a C^ω-diffeomorphism of $\mathbf{R}^{m+k}$, arbitrarily close to the identity, transforming X onto a non-singular algebraic subset of $\mathbf{R}^{m+k}$ in each of the following cases*

(i) (Ivanov [**21**]) *if* $2k > m + 1$, *or*

(ii) (Bochnak and Kucharz [**15**]) *if X has trivial normal bundle and* $k = 1, 2, 4$ or 8.

In fact, Theorem 18 (ii) is more general and holds true also with $\mathbf{R}^{m+k}$ replaced by any compact algebraic $m + k$-manifold with all vector bundles strongly algebraic [**15**].

EXAMPLE 3. Let us consider again the example mentioned in section 4 of an algebraic manifold M diffeomorphic to $S^3 \times S^1$, with $H_3^{alg}(M, \mathbf{Z}_2) = 0$. Let $Z \subset M$ be an analytic submanifold of codim 1, diffeomorphic to S^3, with non-trivial homology class $[Z] \in H_3(M, \mathbf{Z}_2) = \mathbf{Z}_2$. It is clear that there is no homeomorphism of M transforming Z onto an algebraic set.

However for non-singular hypersurfaces, the non-algebraicity of its homology class is the unique obstruction to equivalence with an algebraic set. More precisely,

THEOREM 19 ([**1**], [**8**], [**13**]). *Let M be a compact algebraic m-manifold and let $\varepsilon > 0$. Then the following conditions are equivalent*

(i) *any compact codim* 1 *analytic submanifold of M is analytically ε-isotopic to an algebraic non-singular subset of M*;

(ii) $H_{m-1}^{alg}(M, \mathbf{Z}_2) = H_{m-1}(M, \mathbf{Z}_2)$.

Finally let us mention that any codim 1 submanifold of a compact algebraic manifold M is ε-isotopic to a Nash non-singular subset of M.

REMARK. Although the statements of Theorems 18 and 19 are formulated for analytic submanifolds and C^ω-equivalence, they are also valid for C^∞-manifolds and C^∞-equivalence.

(B) *Singular case.* In this section we shall consider the question of equivalence of analytic and algebraic hypersurfaces (possibly with singularities) of a compact, connected algebraic m-manifold M. By a hypersurface we always understand a closed analytic set of codim 1 at each point. All results of this section are proved in [**13**].

Let $Z \subset M$ be an analytic hypersurface of M which is C^k-equivalent to an algebraic one. Then $[\sigma(Z)] \in H_{m-1}^{alg}(M, \mathbf{Z}_2)$ for some $\sigma \in \mathrm{Diff}^k(M)$, and at each point $x \in Z$, the germ Z_x is locally C^k-equivalent to an algebraic subset of M. This suggests the following

Local-Global Problem. Assume $H_{m-1}^{alg}(M, \mathbf{Z}_2). = H_{m-1}(M, \mathbf{Z}_2)$. Let Z be a real analytic hypersurface of M, which is at each point $x \in Z$ locally C^k-equivalent to an algebraic set $A^x \subset M$ (A^x depends on x). Is Z globally C^k-equivalent to an algebraic hypersurface of M?

It appears that the answer is affirmative for a reasonably large class of hypersurfaces. Example 4 shows that the assumption $H_{m-1}^{alg} = H_{m-1}$ is essential.

Given an analytic set Z of M let us define

$$\sum^{an}(Z) = \{x \in Z : \forall f \in O(M) \text{ if } Z \subset f^{-1}(0) \text{ and } Z_x = f_x^{-1}(0), \text{ then } d_x f = 0\}$$

(of course Z_x, f_x denotes the germ of Z, f at x). If Z is a hypersurface, then $\sum^{an}(Z)$ is the set of "analytically" singular points of Z. One can also define

$$\sum(Z) = \{x \in Z \colon Z_x \text{ is not an analytic submanifold of codim } 1\}$$

which is, for hypersurfaces, the set of "topologically" singular points of Z. Although $\sum \subset \sum^{an}$, in general $\sum \neq \sum^{an}$. If $\sum$ is discrete, then $\sum = \sum^{an}$ if and only if Z is coherent.

THEOREM 20 ([**13**] Part II). *Let Z be an analytic hypersurface of a compact, connected algebraic m-manifold M, with $\sum^{an}(Z)$ finite and $[Z] \in H_{m-1}^{alg}(M, \mathbf{Z}_2)$. Assume that for any $x \in \sum^{an}(Z)$ there is an algebraic set $A^x \subset M$, such that the germs of Z and A^x at x are C^ω-equivalent. Then Z is globally C^ω-equivalent to an algebraic hypersurface of M.*

The proof of Theorem 20 uses a criterion of algebraicity formulated in [**13**], Part I, Th. 1, and Theorem 2 of this paper. The assumption of Theorem 20 can be even weakened; it sufficies to assume that for each $x \in \sum^{an}(Z)$ the germ Z_x is C^ω-equivalent to a Nash germ (instead of to an algebraic one). In particular we have

THEOREM 21 ([**13**] Part II). *Let Z be an analytic hypersurface of M. Assume that Z is semi-algebraic, $\sum^{an}(Z)$ is finite and $[Z] \in H_{m-1}^{alg}(M, \mathbf{Z}_2)$. Then Z is globally C^ω-equivalent to an algebraic hypersurface of M.*

In both Theorems 20 and 21, M can be replaced by $\mathbf{R}^n$ (assuming Z compact). Hypersurfaces considered in Theorems 20 and 21 have finite sets of singular points. There is however a large class of hypersurfaces with $\sum(Z)$ of dimension > 0, which are C^ω-equivalent to algebraic ones. We say that a germ of a complex analytic hypersurface V_x in a complex analytic manifold has a singular point of type L at x, if V_x is a normal crossing at each point different from x and if each irreducible analytic component of V_x has an isolated singular point at x (or is regular at x). We say that a real analytic hypersurface-germ V_x has a singular point of type L at x if it is so for its complexification germ $V_{x,\mathbf{C}}$(observe that singularities of type L appear precisely in Theorem 5).

EXAMPLE 4. The hypersurface $V = \{(x^2 + y^2 - z^2)(x^4 - y^4 + z^4) = 0\} \subset \mathbf{R}^3$ has a singular point of type L at 0.

A real analytic hypersurface $Z \subset M$ is called of type L if the set of singular points of any global analytic irreducible component of Z is finite, and if Z_x has a singular point of type L at each $x \in \Sigma(Z)$. (Notice that $\Sigma(Z)$ need not be discrete).

THEOREM 22 [13]. *Let M be a compact connected algebraic m-manifold. Then the following conditions are equivalent*:

(i) *any analytic hypersurface of type L of M is C^ω-equivalent to an algebraic subset of M*;

(ii) $H_{m-1}^{alg}(M, \mathbf{Z}_2) = H_{m-1}(M, \mathbf{Z}_2)$.

The case of C^ω-equivalence between analytic and Nash hypersurfaces is less complicated and the Nash version of the Local-Global Problem always has an affirmative solution, provided $\Sigma^{an}(Z)$ is finite.

THEOREM 23 [13]. *Let M be a compact algebraic manifold and let $Z \subset M$ be an analytic hypersurface with $\Sigma^{an}(Z)$ finite. Assume that at each singular point x, the germ Z_x is C^ω-equivalent to a Nash germ. Then Z is globally C^ω-equivalent to a Nash hypersurface of M.*

It can also be shown that any hypersurface of type L is C^ω-equivalent to a Nash set.

Also the case of C^ν-equivalence, $\nu < \infty$, has a satisfying solution for hypersurfaces with $\Sigma^{an}(Z)$ finite.

THEOREM 24 [13]. *Let $Z \subset M$ be an analytic hypersurface of a compact connected algebraic m-manifold M. Assume $\Sigma^{an}(Z)$ finite and $[Z] \in H_{m-1}^{alg}(M, \mathbf{Z}_2)$. Let $\nu \in \mathbf{N}$. Then there is a C^ν-diffeomorphism $\sigma: M \to M$ such that $\sigma(Z)$ is an algebraic subset of M. Moreover σ can be chosen arbitrarily close to the identity and C^∞ outside of $\Sigma^{an}(Z)$.*

We do not formulate any theorem concerning the case of global equivalence of analytic and algebraic sets of codim > 1, but several results in this direction are known ([12] and [13] Part II).

6. Global equivalence of analytic and algebraic functions [13], [28], [29]. In this section we shall quote several theorems describing global analytic functions equivalent to rational, polynomial or Nash ones. The proofs and more results in this direction are given in [13], [28], [29].

Let M be a compact connected algebraic m-manifold. Most results concern the class of so-called functions of type L, introduced in [13].

Let f: $M \to \mathbf{R}$ be a real analytic function, C_f the set of its critical values, and let $S_f = f^{-1}(C_f)$. We say that f is a function of type L if the set of singular points of any global analytic irreducible component of S_f is finite and if at each point $x \in M$ the germ $f_x - f(x) \in O_x(M)$ is weakly finitely determined (cf. §3 for the definition of this notion).

EXAMPLE 5. If dim $M \leqq 2$, then any analytic function on M is of type L.

THEOREM 25. [13]. (i) *Any analytic function $f \in O(M)$ of type L is C^ω-equivalent to an entire rational one if and only if $H_{m-1}^{alg}(M, \mathbf{Z}_2) = H_{m-1}(M, \mathbf{Z}_2)$.*

(ii) *Any analytic function of type L is C^ω-equivalent to a Nash function.*

COROLLARY 6. [13], [29]. *If* dim $M = 2$ *then the following conditions are equivalent*:

(i) *any analytic function on M is C^ω-equivalent to an entire rational function*;

(ii) $H_1^{alg}(M, \mathbf{Z}_2) = H_1(M, \mathbf{Z}_2)$.

COROLLARY 7 [13]. *Given a smooth compact connected manifold X, we may find an algebraic manifold $M \subset \mathbf{R}^n$ diffeomorphic to X such that any analytic function $f \in O(M)$ of type L is C^ω-equivalent to an entire rational function on M.*

PROOF. Follows from Theorem 25 and Corollary 5.

It would be very interesting to decide whether any rational function $f \in R(M)$ is C^ω (or even C^0) equivalent to a polynomial. This is not known even for dim $M = 2$ (except for $M = S^2$ or $P^2(\mathbf{R})$, cf. Theorem 27 below).

THEOREM 26. *Let $X \subset \mathbf{R}^n$ be a connected (not necessarily compact) algebraic manifold and let f: $X \to \mathbf{R}$ be an analytic function. Assume that the set Ω of critical points of f is is finite and that at each point $x \in \Omega$ the Milnor number of the germ f_x is finite. Then f is globally C^ω-equivalent to a polynomial in each of the following cases*:

(i) *if X is compact* [12], *or*

(ii) *if f is proper and $X = \mathbf{R}^n$, $n \neq 4, 5$ (or, more generally, if X has a good behaviour "at infinity")* [29].

THEOREM 27 [13], [29]. *If M is an algebraic manifold homeomorphic to S^2 or $P^2(\mathbf{R})$, then any analytic function on M is C^ω-equivalent to a polynomial.*

Now let us consider the Local-Global Problem for functions.

Let $f \in O(M)$ be a function, which is at each point $x \in M$ locally C^k-equivalent to a rational (resp. polynomial) function h^x on M (h^x depends on x). Is f globally C^k-equivalent to a rational (resp. polynomial) function on M?

THEOREM 28 ([**13**] Part II). *Let f: $M \to \mathbf{R}$ be an analytic function on a compact, connected algebraic manifold M. Assume that the set of critical points $\sum_f$ of f is finite and that for each $x \in \sum_f$ there exist a Zariski open neighborhood U_x of x, a rational function $\phi^x \in R(U_x)$* with $\sum_{\phi^x} = \{x\}$, *and an orientation preserving local C^ω diffeomorphism σ*: $(M, x) \to (M, x)$, *such that $f_x \circ \sigma = \phi^x_x$. Then f is C^ω equivalent to an entire rational function.*

Theorem 28 also holds true for $M = \mathbf{R}^m$, assuming f proper and $m \neq 4$ or 5 [**29**]. For $k < \infty$ one can show

THEOREM 29 ([**13**] I). *Let f: $M \to \mathbf{R}$ be an analytic function and let $k \in \mathbf{N}$. If the set of critical points of f is finite, then f is C^k-equivalent to a polynomial function.*

Finally we have

THEOREM 30 [**28**]. *Let $f \in O(M)$ be a function with a finite set of critical points. If at each critical point x the germ f_x is C^ω-equivalent to a Nash germ, then f is globally C^ω-equivalent to a Nash function.*

The proofs of the results of this section are quite complicated. Without going into the details let us mention only that they are based(among other things) on a global version of Theorem I of §1, (formulated in [**13**] as Theorem 10), and on the following criterion of equivalence.

THEOREM 31 [**13**]. *Let f: $M \to \mathbf{R}$ be C^∞-function, $a_1, \ldots, a_k \in \mathbf{R}$, $S_f = f^{-1}(C_f) = S_1 \cup \cdots \cup S_k$, where each S_i is a union of some connected components of S_f, $S_i \cap S_j = \varnothing$* for $i \neq j$, *$f(S_i) = a_i$. Assume that there exist rational functions $\lambda_i \in R(M)$ (resp. Nash functions $\lambda_i \in N(M)$), $i = 1, \ldots, k$, such that $f = \lambda_i$ in a neighborhood of S_i and $\lambda_i^{-1}(a_i) = S_i$. Then f is C^∞-equivalent to a rational (resp. Nash) function. If $\lambda_i \in P[M]$ and $\lambda_{i\mathbf{C}}^{-1}(a_i) \cap \lambda_{j\mathbf{C}}^{-1}(a_j) = \varnothing$, $i \neq j$, then f is C^∞-equivalent to a polynomial.*

In the last statement $\lambda_{i\mathbf{C}}$ is the complexification of λ_i, i.e., the canonical polynomial extension of λ_i onto the algebraic complexification $M_\mathbf{C} \subset \mathbf{C}^n$ of $M \subset \mathbf{R}^n$.

AKNOWLEDGEMENT. The author would like to thank M. Artin and A. Grothendieck for helpful comments concerning Theorem 2, and W. Kucharz for his remarks about the subject of this paper.

REFERENCES

1. S. Akbulut and H. King, *Lectures on topology of real algebraic varieties*, Lecture notes, Institute for Advanced Study, Princeton, 1980/81 (mimeo).

2. ———, *A relative Nash theorem*, Transactions AMS **267** (2) (1981), 465–481.

3. ———, *Topology of real algebraic sets with isolated singularities*, Ann. of Math. **113** (1981), 425–446.

4. ———, *Real algebraic structures on topological spaces*, Publ. Math. IHES nr. **53** (1981), 79–162.

5. M. Artin, *Algebraic approximation of structures over complete local rings*, Publ. I.H.E.S. **36** (1969), 23–58.

6. R. Benedetti and R. Tognoli, *On real algebraic vector bundles*, Bull. Sc. Math. **104**, (1980), 89–112.

7. ———, *Théorèmes d'approximation en géométrie algébrique réelle*, Seminaire Risler, Publ. Math. Univ. Paris VII (1980).

8. ———, *Remarks and counterexamples in the theory of real algebraic vector bundles and cycles*, Géométrie Algébrique Réelle et Formes Quadratiques, Rennes 1981, Springer Lecture Notes **959**, 198–211.

9. R. Benedetti and M. Dedó, *The topology of two-dimensional real analytic varieties*, Ann. Mat. Pura Appl. (VI) **127** (1981), 141–171.

10. ——— and M. Dedó, *Counterexamples to representing homology classes by real algebraic subvarieties up to homeomorphism*, Compositio Math. (to appear).

11. J. Bochnak and G. Efroymson, *Real algebraic geometry and the* 17^{th} *Hilbert problem*, Math. Ann. **251** (1980), 213–241.

12. ———, W. Kucharz and M. Shiota, *On equivalence of ideals of global real analytic functions and the* 17^{th} *Hilbert problem*, Invent. Math. **63** (3), (1981), 403–421.

13. ———, *On algebraicity of global real analytic sets and functions* (Part I), Invent. Math. **70** (1), (1982), 115–156, (Part II in preparation).

14. ———, *The divisor class groups of some rings of global real analytic, Nash or rational regular functions*; Géométrie Algébrique Réelle et Formes Quadratiques, Rennes 1981, Springer Lecture Notes **959**, 218–248.

15. J. Bochnak and W. Kucharz, *On complete intersections in differential topology and analytic geometry*, preprint, Amsterdam 1982.

16. A. Borel and A. Haefliger, *La class d'homologie fondamentale d'un espace analytigue*, Bull. Soc. Math. France **87** (1961).

17. D. Cerveau and J. F. Mattei, *Integrales premieres multiformes*, preprint, University of Dijon (1981).

18. A. Dubson, $\mathbf{Z}_2$*-fundamental class of real analytic sets*, University of Paris (Orsay), preprint (1978).

19. R. Hartshorne, *Complete intersections and connectedness*, J. Amer. of Math. **84** (1962), 497–508.

20. H. Hironaka, *Triangulations of algebraic sets*, Algebraic Geometry, Arcata 1974, Amer. Math. Soc. Proc. Symp. Pure Math. **29** (1975), 165–184.

21. N. Ivanov, *Approximation of smooth manifolds by real algebraic sets*, Uspechi Matemat. Nauk **37** (1), (1982), 1–52 (in Russian).

22. H. King, *Approximating submanifolds of real projective space by varieties*, Topology **15** (1976), 81–85.

23. S. Lojasiewicz, *Triangulation of semi-analytic sets*, Annali Scu. Norm. Sup. Pisa **18** (1964), 449–474.

24. T. Mostowski, *A note on a decision procedure...*, Bull. Acad. Pol. Sci. **23** (12), (1975), 1229-1232.

25. A. Nobile, *Saturation of analytic rings*, Illinois Journal of Math. **24** (3), (1980), 485–525.

26. J. Risler, *Sur l'homologie des surfaces algébriques réelles*; Géométrie Algébrique Réelle et Formes Quadratiques, Springer Lecture Notes **959**, (1982), 381–385.

27. K. Saito, *Quasihomogene isolierte Singularitaten von Hyperflächen*, Invent. Math. **14** (1971), 123–142.

28. M. Shiota, *Equivalence of differentiable mappings and analytic mappings*, Publ. Math. I.H.E.S. **54** (1981), 38–122.

29. ———, *Equivalance of differentiable functions, rational functions and polynomials*, Ann. Inst. Fourier **32** (1982).

30. ———, *Real algebraic realization of characteristic classes*, Publ. R.I.M.S. **18** Kyoto University (1982), 995–1008.

31. D. Sullivan, *Combinatorial invariants of analytic spaces*, Proc. Liverpool Symp. Singularities I, Lecture Notes in Math. **192** (1971), Springer Verlag, 165–169.

32. A. Tognoli, *Su una congettura di Nash*, Annali Scu. Nor. Sup. Pisa **27** (1973), 167–185.

33. ———, *Algebraic geometry and Nash functions*, Institutiones math., Vol. III, Acad. Press London and New-York 1978.

34. ———, *Algebraic approximation of manifolds and spaces*, Seminaire Bourbaki 32 année (1979/80), **548**, Springer Lecture Notes.

35. J. Cl. Tougeron, *Ideaux de fonctions différentiables* I. Ann. Inst. Fourier **18** (1968), 177–240.

36. ———, *Ideaux de fonctions différentiables*, Springer 1972.

37. ———, *G-stabilité des germes d'applidations différentiables*, preprint, Université de Rennes (1971).

38. ———, *Solution d'un système d'équations analytiques réelles et applications*, Ann. Inst. Fourier **26** (1976), 109–135.

39. C. T. C. Wall, *Finite determinacy of smooth map-germs*, Bull. London Math. Soc. **13** (6) (1981), 481–539.

40. H. Whitney, *Local properties of analytic varieties, Differential and Combinatorial Topology*, Princeton Univ. Press 1965, 205–244.

41. R. Palais, *Real algebraic differential topology*, Publish or Perish Inc. 1980.

42. M. Artin and B. Mazur, *On periodic points*, Ann. of Math. **81** (1965), 82–99.

43. J. Bochnak and W. Kucharz, *Local algebraicity of analytic sets*, J. reine angev. Math., to appear.

44. R. Silhol, *A bound on the order of $H^{(a)}_{n-1}(X, \mathbf{Z}/2)$ on a real algebraic variety*; Géométrie Algébrique Réelle et Formes Quadratiques, Lecture Notes in Math. **959**, Springer Verlag 1982, 443–450.

45. Géométrie Algébrique Réelle et Formes Quadratiques, Proc. Rennes 1981, Lecture Notes in Math. **959**, Springer Verlag 1982.

VRIJE UNIVERSITEIT DEPARTMENT OF MATHEMATICS, P.O. BOX 7161, 1007 MC AMSTERDAM, THE NETHERLANDS

Added in proof.

1. The problem stated at the end of section 3 has been negatively solved by R. Pellikaan. He has shown that the germ f: $(\mathbf{K}^3, 0) \to (\mathbf{K}, 0)$ defined by $f(x,y,z) = xyz(x^4 + y^4 + x^3yz + x^2y^3z)e^z$ is not analytically equivalent to a polynomial (preprint University of Utrecht 1984).

The question remains open whether any Nash germ $(\mathbf{K}^n, 0) \to (\mathbf{K}, 0)$ is C^ω equivalent to a polynomial.

2. Theorem 12 has been proved independently by Akbulut and King in [46]. "The topology of real algebraic sets", L'Enseignements Math. 29 (1983), 221–261.

3. H. King has shown that Corollary 4 is valid also in domlnsion 3 (cf. "Topology of real algebraic sets"-preprint 1982).