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QUASI–COHERENT MODULES ON QUASI–AFFINE SCHEMES

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ABSTRACT. It is shown that a quasi–coherent sheaf of modules on a quasi–compact open subset of an affine scheme can be realized as an object in a subcategory of a module category. In particular, the modules of sections is canonically isomorphic to a (torsion theoretic) localized module. This generalizes the noetherian case of P.–J. Cahen. A few simple examples exploit this relationship.

1. Introduction. If A is a noetherian ring and U is an open subset of $X = \operatorname{Spec} A$, then P.–J. Cahen [**1**, Theorem 6.1] has shown, by torsion theoretic methods, that for any A-–module M, the module of sections $r(u, \tilde{M})$ of the quasi–coherent Q_x–module $\tilde{M}$ is the module of quotients $Q_U(M) = \lim \rightarrow \operatorname{Hom}(I, M)$ where the direct limit is taken over the set $\phi_u = \{I \subseteq A | \forall p \in U,\ I \not\subseteq p\}$. Our aim is to generalize this result to an arbitrary (commutative) ring in the case U is a quasi–compact open subset of $\operatorname{Spec} A$.

We show that for any such U : 1) every quasi–coherent Q_U–module F is the restriction to U of some quasi–coherent $\mathcal{Q}_X$–module $\tilde{M}$; 2) if $\tilde{M}$ is any extension of F, the module of sections $\Gamma(U, F) = \Gamma(U, \tilde{M})$ is just the module of quotients $Q_U(M) = \lim \rightarrow \operatorname{Hom}(I, \overline{M})$ where $\overline{M} = M/T_u(M)$ and $T_U(M) = \{x \in M | (0 : x)\varepsilon_U^{\phi}\}$ is the torsion submodule of M with respect to the torsion class T_U; and 3) the category of quasi–coherent $\mathcal{Q}_U$–modules is equivalent to the category $(A, T_U) - \text{mod}$. Here $(A, T_U) - \text{mod}$ is the full subcategory $\{M \in A - \text{mod} \,|\, \phi_M : M = \operatorname{Hom}(A, M) \rightarrow Q_U(M)$ is an isomorphism $\}$. As a corollary, torsion theoretic methods in $(A, T_U) - \text{mod}$ yield interesting proofs of generalizations of standard theorems in algebraic geometry as well as new theorems in this class of $\mathcal{Q}_U$–modules. We give an example of the latter by characterizing the injective objects in the category of quasi–coherent $\mathcal{Q}_U$–modules.

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2. Torsion theory. Much of the torsion theory background may be found in the text [**4**], but we provide a brief summary for the reader's convenience. If U is a subset of Spec A, we denote the torsion class $T_U = \{M \in A - \text{mod} \,|\, \forall p \in U, M_p = 0\}$, the filter $\phi_U = \{I \subseteq A | A/I \in T_U\}$, and the localization functor Q_U defined by $Q_U(M) = \lim \to \text{Hom}\,(I, \overline{M})$ (taken over all $I \in \phi_U$). The A-module $Q_U(M)$ has a natural $Q_U(A)$-module structure. Those A-modules M such that the natural homomorphism $\phi_M : M \to Q_U(M)$ is an isomorphism (with their A-linear maps) determine a full subcategory of $A-\text{mod}$ denoted by $(A, T_U)-\text{mod}$. This category may be regarded as a (full) subcategory of $Q_U(A)-\text{mod}$. The module of quotients $Q_U(M)$ is uniquely determined (up to canonical isomorphism) by the following properties: if $\phi : M \to N$ is an A-module homomorphism with $\ker\phi \in T_U, \text{coker}\,\phi \in T_U$, and $N \in (A, T_U) - \text{mod}$, then $N = Q_U(M)$. Since $\ker\phi_M = T_U(M)$ and $\text{coker}\,\phi_M = T_U(E(\overline{M})/\overline{M})$ where $E(\overline{M})$ is the injective envelope of $\overline{M}, M \in (A, T_U) - \text{mod}$ if and only if M and $E(M)/M$ are T_U-torsionfree. We note that the class of torsionfree modules is closed under submodules, direct products, extensions, and injective envelopes. It follows that $(A, T_U)-\text{mod}$ is closed under direct products (as a subcategory) and, since Q_U is left exact, kernels.

3. Quasi-affine schemes. If $\phi : A \to B$ is a ring homomorphism, then we shall use ${}^a\phi : \text{Spec}\, B \to \text{Spec}\, A$ for the natural induced map.

LEMMA 1. *Let $\phi A \to B$ be a ring homomorphism, M a B-module, N an A-submodule of M. If, for all contracted primes $p \in Im^a\phi$, we have $N_p = 0$, then $N = 0$.*

PROOF. Let $X \in N \subseteq M$. Write $Ax = A/I, Bx = B/J$. Since $Ax \subseteq Bx, I = \phi^{-1}(J)$. If $x \neq 0$, then $J \neq B$, so choose a prime $q \supseteq J$. Then $\phi^{-1}(q) = p \supseteq I$ and $0 \neq (A/I)_p \subseteq N_p = 0$, a contradiction.

THEOREM 2. *Let U be a quasi-compact open subset of Spec A.*

1) *Every quasi-coherent $\mathcal{Q}_U$-module F can be represented as $F = \tilde{M}|_U$ for some A-module M.*

2) *The category of quasi-coherent $\mathcal{Q}_U$-modules is equivalent to the category $(A, T_U) - mod$. Whenever $\tilde{M}$ extends F, then $\Gamma(U, F) =$*

$Q_U(M)$; the functors $\Gamma(U,)$ and $M \to \tilde{M}|_U$ define the equivalence.

PROOF. 1) If $U_i = D(f_i), f_i \in A$, is a finite cover of U, let $U_{ij} = U_i \bigcap U_j = D(f_i f_j)$. Then, as in [**2**; Proposition II 5.8], there is an exact sequence of sheaves on X

$$0 \to \tau_* F \to \overset{\oplus}{i} \tau_*(F|_{U_i}) \to \overset{\oplus}{i,j} \tau_*(F|U_{ij}).$$

Now $\tau_*(F|_{U_i})$ and $\tau_*(F|_{U_{ij}})$ are quasi-coherent, hence so is $\tau_* F$. Write $\tau_* F = \tilde{M}$ for some A-module M. It follows that $F = \tau_* F|_U = \tilde{M}|_U$.

2) We first show that for any A-module $M, \Gamma(U, \tilde{M}) = Q_U(M)$. To this end we consider the restriction map $\phi : M = \Gamma(X, \tilde{M}) \to \Gamma(U, \tilde{M})$ and show that, for any $D(f) \subseteq U$, the map $\phi \otimes A_f : M_f \to \Gamma(U, \tilde{M})_f$ is an isomorphism. Note that if $x \in \ker \phi$, then x vanishes on U, hence on $D(f)$, so that $A_f x = 0$. Thus, by flatness, $M_f \to \Gamma(U, \tilde{M})_f$ is an injection. Now let $t \in \Gamma(U, \tilde{M})$. We claim that, for some integer a $f^a t$ is in the image of ϕ, i.e., can be extended to all of X. Since $t|_{D(f)} \in M_f$, there is an integer m such that $f^{m_t}|_{D(f)}$ can be extended to a section $S \in M$ (0ver X). Write $U = \bigcup U_i$, the U_i as above. Now $s - f^m t = 0$ on $D(f) \cap U_i = D(f f_i)$, so we can choose r_i so that $f^{r_i} s = f^{r_i + m} t$ on U_i. Let r be the maximum of the r_i (we are assuming a finite covering). Since $f^r s$ and $f^{r+m} t$ agree everywhere on U, and $\tilde{M}$ is a sheaf, $\phi(f^r s) = f^{r+m} t$ as claimed. Hence $M_f \to \Gamma(U, \tilde{M})_f$ is an isomorphism for each $D(f) \subseteq U$. It follows that $M_p \to \Gamma(U, \tilde{M})_p$ is an isomorphism for each $p \in U$, so that the kernel and cokernel of ϕ are T_U-torsion. Next we show that $N = \Gamma(U, \tilde{M}) \in (A, T_U) - \text{mod}$, i.e., N and $E(N)/N$ are T_U-torsionfree. If $D(f) \subseteq U$, then every A_f-module is torsionfree, for its torsion submodule vanishes on $U \supseteq D(f)$ and by Lemma 1 is thus zero. Hence all terms of an A_f-injective resolution of M_f are torsionfree. As localization at f is a flat epimorphism, any A_f-injective module is A-injective. This yields that M_f and $E(M_f)/M_f$, where $E(M_f)$ is the A-injective envelope of M_f, are also torsionfree. By previous remarks in §2, $M_f \in (A, T_U) - \text{mod}$. Now, if $U = \cup D(f_\alpha)$ is covering by special open sets $D(f_\alpha)$, then $\Gamma(U, \tilde{M})$ is the kernel of a map $\Pi\, M_{f_\alpha} \to \Pi M_{f_\alpha f_\beta}$. But $(A, T_U) - \text{mod}$ is closed under products and kernels (as a subcategory of A-modules) so $\Gamma(U, \tilde{M}) \in (A, T_U) - \text{mod}$. This proves that $\Gamma(U, \tilde{M})$ is canonically isomorphic to $Q_U(M)$. The remainder of 2) follows readily,

for if $F = \tilde{M}|_U$, then $\Gamma(U, \tilde{M}|_U) = \Gamma(U, \tilde{M}) = Q_U(M)$. The map $M \to Q_U(M)$ yields the isomorphism $F = \tilde{M}|_U \to \tilde{Q_U}(M)|_U$ since each map on stalks $M_p \to Q_U(M)_p$ is an isomorphism.

COROLLARY 3. *If U is a quasi–compact open subset of $\operatorname{Spec} A$ and F is an injective object in the category of quasi–coherent $\mathcal{Q}_U$–modules, then $F = \tilde{E}|_U$ where E is a T_U–torsionfree, injective A–module.*

PROOF. This follows from [4, Proposition X 1.7] where it is shown that injective objects of $(A, T) - \text{mod}$ are just the T–torsionfree, injective A–modules.

A torsion class T is called stable if it is also closed under injective envelpoes, or, equivalently, if $T(E)$ is always a direct summand of E whenever E is injective. In this case, $\overline{E}$, as a direct summand of E, is also injective (and torsionfree) and thus $Q(E) = Q(\overline{E}) = \overline{E}$. If A is a noetherian ring, then T_U is stable for any U since a module and its injective envelope have the same associated primes, while $M \in T_U$ if and only if $U \cap \operatorname{Ass} M = \emptyset$.

COROLLARY 4. If T_U is stable for a quasi–compact open U, and E is any injective A–module then $\tilde{E}|_U$ is an injective in the category of quasi–coherent $\mathcal{Q}_U$–modules.

PROOF. $\tilde{E}|_U = \tilde{Q}_U(E)|_U$ by Theorem 2. Stability says that $Q_U(E) = \overline{E}$ is injective (and torsionfree) so that [4, Proposition X 1.7] again gives the result.

COROLLARY 5. *If T_U is stable for a quasi–compact open U, then the homomorphism $E = \Gamma(X, \tilde{E}) \to \Gamma(U, \tilde{E})$ is a surjection for any injective module E.*

PROOF. The restriction map is just $\phi_E : E \to Q_U(E)$, which is surjective since $Q_U(E) = \overline{E}$.

REMARKS 6. i) Theorem 2 is indeed a generalization of Cahen's result, for if A is noetherian, then $Q(M) = \lim \operatorname{Hom}(I, M)$ by [**4**, Proposition IX 1.7].

ii) It is the author's contention that the above results can be further generalized to arbitrary quasi–compact subsets of $\operatorname{Spec} A$, stable under generalization. In [**3**, Proposition IV 2.5], D. Lazard has examined the case for (induced) affines. It is not known if any of the results above can be extended to arbitrary open U, but it should be noted, in this regard, for a domain $A, \Gamma(U) = \bigcap_{p \in U} A_p = Q_U(A)$.

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