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ON WEIGHTED ORLICZ SEQUENCE SPACES AND THEIR SUBSPACES

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ABSTRACT. Weighted Orlicz sequence spaces $\ell^\phi(a)$ containing an isomorphic copy of ℓ^∞ and c_0 are characterized by means of suitable conditions on the Orlicz function ϕ and the weight sequence (a_n) for $a_n \to 0$. This extends a result of B. Turett [**14**] for Orlicz spaces $L^\phi(\mu)$ over atomless measures of the case of purely atomic probability measures. As an application, the spaces $\ell^\phi(a)$ which are B-convex are determined. Also, a question of W. Luxemburg [**11**] on inclusions of spaces $\ell^\phi(a)$ for sequences (a_n) slowly decreasing to 0 is answered.

1. Introduction. Recent years have seen a quite profound analysis of the relationship between Orlicz spaces and the spaces $\ell^p, 1 \leq p \leq \infty$. In this direction a well-known result is that every Orlicz space always contains an isomorphic copy of some ℓ^p. A deeper analysis by J. Lindenstrauss and L. Tzafriri [**9, 10**] determined the set of all numbers p such that ℓ^p can be isomorphically embedded into an Orlicz sequence space ℓ^ϕ. For Orlicz spaces of functions $L^\phi(\Omega)$, B. Turett [**14, 15**] has characterized, in terms of the Orlicz function ϕ, the spaces $L^\phi(\Omega)$ for atomless finite measures containing an isomorphic copy of ℓ^∞ and c_0.

In this paper, we analyze these topics for *weighted Orlicz sequence spaces* $\ell^\phi(a)$ when $a_n \to 0$ or $a_n \to \infty$. The results answer the following general question: for which class of weight sequences (a_n) can the suitable characterizations of the non-atomic case be extended to the spaces $\ell^\phi(a)$?

In a more precise way, we study the class of weighted Orlicz sequence spaces $\ell^\phi(a)$ where (a_n) is of finite sum and

$$\underline{\lim}_{n\to\infty} \frac{\sum_{k=n+1}^\infty a_k}{a_n} > 0$$

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(it seems that this is the greatest class for which most positive results can be established) and give an inclusion theorem for this class, answering a question of W. Luxemburg [**11**, p. 40]. Also, we characterize, in terms of the function ϕ, the spaces $\ell^\phi(a)$ containing an isomorphic copy of ℓ^∞ and c_0 (Theorem 3). This extends the above mentioned result of B. Turrett [**14**, **15**] for $L^\phi(\Omega)$ to the case of purely atomic probability measures. An easy consequence of this theorem is a characterization of the B-convex [**3**] spaces $\ell^\phi(a)$ generalizing a result of Denker and Kombrink [**1**].

On the other hand, the problem of characterizing in terms of the function ϕ, when $\ell^\phi(a)$ contains a copy of ℓ^p has a negative answer for the class of weights verifying (*). Thus, in §4, we show functions ϕ nonequivalent to the function $\psi(x) = x^p$ and weight sequences (a_n) slowly decreasing to 0 such that $\ell^\phi(a)$ is isomorphic to $\ell^p (0 < p < \infty)$. Finally, the results are complemented with several counterexamples.

2. Notations and preliminary results. Let us start with some notation and definitions. An *Orlicz function* ϕ is a non-decreasing function $\phi : \mathbf{R}^+ \to \mathbf{R}^+$, left continuous for $t > 0$, continuous at 0 and such that $\phi(0) = 0$. Given a poisitive measure space (Ω, μ) and an Orlicz function ϕ, the *Orlicz space* $L^\phi(\Omega)$ is defined as the set of equivalence classes of μ-measurable scalar functions on (Ω, μ) such that $\int_\Omega \phi(|f|/s)d\mu < \infty$ for some $s > 0$. The space $L^\phi(\Omega)$ becomes a linear metric space when it is endowed with the F-norm $|f| = \inf\{s > 0 : \int_\Omega \phi(|f|/s)d\mu \leq s\}$, and has as a basis of neighbourhoods at 0 the sets $s.B^\phi_s(0)$ for $s > 0$, where $B^\phi_s(0) = \{f \in L^\phi(\Omega) : \int_\Omega \phi(|f|) \leq s\}$. If ϕ is convex, then $B^\phi_s(0)$ is a convex bounded neighbourhood of 0 and $L^\phi(\Omega)$ with the Minkowski functional of $B^\phi_1(0)$ - the so-called *Luxemburg norm* - $|f|_\phi = \inf\{s > 0 = \int_\Omega \phi(|f|/s) \leq 1\}$ is a Banach space.

For arbitrary purely atomic σ-finite measure spaces we obtain the *Orlicz sequence spaces* $\ell^\phi(a)$ of the sequences $x = (x_n)_{n\in\mathbf{N}} \in \mathbf{K}^\mathbf{N}$ such that $\sum_{n=1}^\infty \phi(|x_n|/s)a_n < \infty$ for some $s > 0$, where $(a_n)_{n\in\mathbf{N}}$ is a sequence of positive scalars (a *weight sequence*). The symbol $h^\phi(a)$ denotes the closed subspace of the sequences $x = (x_n)_{n\in\mathbf{N}}$ such that $\sum_{n=1}^\infty \phi(|x_n|/s)a_n < \infty$ for every $s > 0$, having an unconditional Schauder basis in the sequence of unit vectors (e_n). In the case that $0 < \underline{\lim} a_n \leq \overline{\lim} a_n < \infty$ we simply write ℓ^ϕ, as usual. Recall that a function ϕ satisfies the Δ_2-*condition at* ∞ if there exist constants $M > 0$ and $s_0 > 0$ with $\phi(s_0) > 0$ such that $\phi(2s) \leq M\phi(s)$ if $s \geq s_0$. If ϕ and ψ are Orlicz functions, we write $\phi \prec \psi$ at ∞ when there exist constants $K > 0$, $r > 0$ and $s_0 > 0$ such that $\phi(s) \leq K\psi(rs)$ for every $s \geq s_0 > 0$. We say that ϕ and ψ are *equivalent at* $\infty, \phi \sim \psi$, if $\phi \prec \psi$ and $\psi \prec \phi$ at ∞. The reader is referred to [**8**, **12**, **19**] for a detailed exposition of the basic properties of Orlicz spaces.

It is clear that the spaces $\ell^\phi(a)$ are included in the general category of *modular sequence spaces* or *Musielak-Orlicz spaces* $\ell^{(\phi_n)}$ [**9**, **12**]. For these spaces, $\ell^{(\phi_n)}$, generated from a sequence (ϕ_n) of Orlicz functions, J. Woo [**17**] has introduced the notion of *uniform* Δ_2-*condition* which plays a remarkable role in the study of the properties of the modular space $\ell^{(\phi_n)}$. However, this notion has the disadvantage of not being preserved by equivalences, i.e., two different Orlicz function sequences can define the same modular space $\ell^{(\phi_n)}$ while the first satisfies the uniform Δ_2-condition and the second does not satisfy it (see, e.g., [**9**, p. 167]). This is the reason why the criteria based on the uniform Δ_2-condition are not easy to handle when applied to specific cases. Here we have avoided use of this condition, and, on the contrary, the ordinary Δ_2-condition has been used as often as possible.

The results of this paper are centered on the spaces $\ell^\phi(a)$ when $a_n \to 0$ with $\sum_{n=1}^\infty a_n < \infty$. By symmetry we have similar results for the case of weight sequences $a_n \to \infty$, by replacing the behavior of ϕ near to ∞ by its behavior near to 0. The statements of these results are omitted, as well as the results for the case $a_n \to 0$ with $\Sigma a_n = \infty$, since, for this class, they are easily obtainable by using a certian universal property of $\ell^\phi(a)$ (see [**4**, **5**], namely $\ell^\phi(a)$ always contains a copy of $\ell^\phi(b)$ for any arbitrary sequence of weights (b_n).

PROPOSITION 1. *Let* ϕ *and* ψ *be Orlicz functions and let* (a_n) *be a weight sequence of finite sums. If* $\underline{\lim}_{n\to\infty}(\sum_{k=n+1}^\infty a_k)/a_n > 0$, *then* $\ell^\psi(a) \hookrightarrow \ell^\phi(a)$ *if and only if* $\phi \prec \psi$ *at* ∞.

PROOF. Let $\ell^\psi(a) \hookrightarrow \ell^\phi(a)$. W.l.o.g. assume that (a_n) is decreasing. By the hypotheses, there exists a $\lambda \in (0,1)$ such that, for every $n, \sum_{k=n+1}^\infty a_k > \lambda a_n$, so if μ is the measure on $\mathbf{N}$ defined by $\mu(n) = a_n$, then, by Theorem I of [**13**] (see also [**7**, Theorem 3], μ has the Darboux

θ-property for the function $\theta(s) = \lambda s$; i.e., for $0 < \alpha' \leq \Sigma a)_n = \alpha$, there exists an $A \subset \mathbf{N}$ such that $\lambda\alpha' \leq \mu(A) \leq \alpha'$. Let $r = \alpha\phi(s_0)$ for $s_0 > 0$ with $\psi(s_0) > 0$, and consider the $\lambda r.B^{\phi}_{\lambda r}(0)$ neighbourhood of 0 in $\ell^{\phi}(a)$. As the inclusion $\ell^{\psi}(a) \hookrightarrow \ell^{\phi}(a)$ is continuous, by the closed graph theorem, there is a neighbourhood $r'.B^{\psi}_{r'}(0)$ of 0 in $\ell^{\psi}(a)$ such that $r'.B^{\psi}_{r'}(0) \subset \lambda r.B^{\phi}_{\lambda r}(0)$. Now, if $s > 0$ and $\alpha' = r/\phi(s) \leq \alpha$, for some $A \subset \mathbf{N}$, we have $\lambda r/\phi(s) \leq \mu(A) \leq r/\phi(s)$, and if $f = srX_A$ we deduce that

$$\sum_{n=1}^{\infty} \phi\Big(\frac{|f(n)|}{\lambda r}\Big) a_n \geq \sum_{n=1}^{\infty} \phi\Big(\frac{|f(n)|}{r}\Big) a_n = \phi(s)\mu(A) \geq \lambda r.$$

Thus, f belongs neither to $\lambda r B^{\phi}_{\lambda r}(0)$ nor to $r' B^{\psi}_{r'}(0)$, and $\sum_{n=1}^{\infty} \psi(|f(n)| /r)a_n = \psi(rs/r')\mu(A) > r', \phi(s)\mu(A) \leq r < (r/r')\psi(rs/r')\mu(A)$. Therefore, if $k = r/r'$, then $\phi(s) \leq k\psi(ks)$ for all $s > s_0 > 0$.

The second implication is obviously valid for any arbitrary sequence (a_n) of finite sum. □

The next proposition has, as a consequence, that the class of the sequences of weights such that $\underline{\lim}_{n\to\infty} (\sum_{k=n+1}^{\infty} a_k)/a_n > 0$ is the greatest class of which the above proposition holds.

PROPOSITION 2. *If (a_n) is a sequence of finite sum with $\underline{\lim}_{n\to\infty} (\sum_{k=n+1}^{\infty} a_k)/a_n = 0$, then there exist Orlicz functions ϕ and ψ such that $\ell^{\psi}(a) \hookrightarrow \ell^{\phi}(a)$ and $\phi < \psi$ at ∞ does not hold.*

PROOF. From the hypotheses it follows that there is a subsequence $\Big(b_{n_k}/a_{n_k}\Big)_{k=1}^{\infty}$, where b_n denotes $\sum_{m=n+1}^{\infty} a_m$ such that $b_{n_k}/a_{n_k} < 1/2^k$. Consider $(t_k)_{k=1}^{\infty}$ with $t_k \to \infty$ such that $\sum_{n=1}^{\infty} t_k b_{n_k}/a_{n_k} < \infty$ and with the Orlicz functions:

$$\psi(s) = \begin{cases} \frac{1}{a_{n_1}} s, & \text{if } s \in [0,1] \\ \frac{1}{a_{n_k}}, & \text{if } s \in (k!, (k+1)!], \quad k \geq 1, \end{cases}$$

and

$$\phi(s) = \begin{cases} \frac{t_1}{a_{n_1}} s, & \text{if } s \in [0,1] \\ \frac{t_k}{a_{n_k}}, & \text{if } s \in (k!, (k+1)!], \quad k \geq 1. \end{cases}$$

Let us show that $\ell^{\psi}(a) \hookrightarrow \ell^{\phi}(a)$. If $x = (x_n) \in \ell^{\psi}(a)$, there exist $s > 0$ and $N \in \mathbf{N}$ such that $\psi(|x_n|/s)a_n \leq 1$ for every $n \geq N$. If $I_k = \{i \geq N : |x_i|/s \in (k!, (k+1)!]\}$, then $a_i\psi(|x_i|/s) = a_i/a_{n_k} < 1$ for every $i \in I_k$. Thus $i > n_k$ for every $i \in I_k$ and $\sum_{i \in I_k} a_i \leq b_{n_k}$. Hence $x = (x_n) \in \ell^{\phi}(a)$, since

$$\begin{aligned}\sum_{n=1}^{\infty} \phi\Big(\frac{|x_n|}{s}\Big)a_n &\leq \sum_{n \notin UI_k} a_n\phi\Big(\frac{|x_n|}{s}\Big) + \sum_{k=1}^{\infty} \phi\Big(\frac{|x_k|}{s}\Big) \sum_{j \in I_k} a_j \\ &\leq \sum_{n=1}^{\infty} \frac{a_n}{a_{n_1}} t_1 + \sum_{k=1}^{\infty} \frac{t_k}{an_k} b_{n_k} < \infty.\end{aligned}$$

However, $\psi \prec \phi$ at ∞ does not hold because $\overline{\lim}_{s \to \infty}(\phi(s)/\psi(\lambda s)) = \infty$ for every $\lambda > 0$. □

REMARKS (1). It is clear that this class of sequences of weights (a_n) with $\underline{\lim}_{n \to \infty}(\sum_{k=n+1}^{\infty} a_k)/a_n > 0$ is bigger than the class of sequences *slowly decreasing* to 0, i.e., such that $\underline{\lim}_{n \to \infty} a_{n+1}/a_n > 0$, considered for example in [**11, 1**]. In fact, it may happen that $\underline{\lim}_{n \to \infty}(\sum_{k=n+1}^{\infty} a_k)/a_n = \infty$ and $\underline{\lim}_{n \to \infty} a_{n+1}/a_n = 0$, as the following example shows:

Let $c_k = \sum_{j=0}^{k} 2^{\frac{j(j+1)}{2}}, k \in \mathbf{N}$. Define $(a_k)_{k \in \mathbf{N}}$ by

$$a_0 = 1, \qquad a_n = \frac{1}{2^{k(k+3)/2}} \text{ if } c_{k-1} \leq n < c_k.$$

It can be easily checked that the sequence (a_n) satisfies the required conditions.

(2). If $\ell_0^{\phi}(a)$ is the Orlicz class defined by $\{x = (x_n)_{n \in \mathbf{N}} : \Sigma\phi(|x_n|)a_n < \infty\}$, then, as a direct consequence of Proposition 1, we obtain that $\ell_0^{\psi}(a) \hookrightarrow \ell_0^{\phi}(a)$ if and only if there exist $K > 0$ and $s_0 > 0$ such that $\phi(s) \leq K\psi(s)$ for every $s > s_0 > 0$. These results answer a question of *W*. Luxemburg in [**11**; p. 40, Remark 1].

3. Subspaces isomorphic to $\ell^{\infty}, \mathbf{c}_0$ and ℓ^1.

THEOREM 3. *Let ϕ be a convex Orlicz function and let (a_n) be a weight sequence of finite sum with $\underline{\lim}_{n\to\infty}\left(\sum_{k=n+1}^{\infty} a_k\right)/a_n > 0$. Then the following statements are equivalent:*

(1) *ϕ verifies the Δ_2-condition at ∞,*

(2) *$\ell^\phi(a) = h^\phi(a)$,*

(3) *$\ell^\phi(a)$ is separable, and*

(4) *$\ell^\phi(a)$ contains no isomorphic copy of ℓ^∞.*

This theorem constitutes an extension to the case of purely atomic finite measures of a result of Turett ([**14** Theorem 4], [**15**, p. 33]). In the proof we shall make use of the following technical lemma.

LEMMA 4. *Let (a_n) be a finite sum with $\sum_{k=n+1}^{\infty} a_k > 2ca_n$ for $0 < 2c < 1$. If ϕ does not satisfy the Δ_2-condition at ∞, then there exist sequences (m_k), (p_k) of natural numbers and (t_k) of positive scalars such that*

(1) *$c^{k+1} < (\sum_{j=m_k}^{m_k+p_k} a_j)\phi(t_k) \le c^k$, and*

(2) *$\phi(st_k) > (1/c^{k+1}\phi(t_k)$ for every $k \in \mathbf{N}$.*

PROOF. We proceed by induction. As ϕ does not fulfill the Δ_2-condition at ∞, we can take a $t_1 > 0$ in such a way that $\phi(2t_1) \ge (1/c^2)\phi(t_1)$ and $\phi(t_1) > 2c/b_1$.

Let $b_n = \sum_{k=n+1}^{\infty} a_k$ and $N_1 = \{n > 1$: there exists $q \ge 1$ such that $\sum_{j=n}^{n+q} a_j\phi(t_1) > c^2\} \subseteq \mathbf{N}$. N_1 is non empty since $2 \in N_1$: if $\varepsilon = c^2/\phi(t_1)$ we can find a $j \in \mathbf{N}$ with $b_{j+1} < \varepsilon$, so $(b_1 - b_{j+1})\phi(t_1) > 2c^2 - \varepsilon\phi(t_1) = c^2$. Moreover, N_1 is bounded above by $j+2$, because $(b_{n-1} - b_{n+q})\phi(t_1) \le b_{j+1}\phi(t_1) < \varepsilon\phi(t_1) = c^2$ for every $n \ge j+2$ and $q \ge 1$. Let $m_1 = \sup N_1$. By the definition of N_1, there is a $q_1 \ge 1$ with $(\sum_{j=m_1}^{m_1+q_1} a_j)\phi(t_1) > c^2$. Now consider the set $Y_1 = \{q \in N : c^2 < (\sum_{j=m_1}^{m_1+q} a_j)\phi(t_1) \le c\}$ and write $p_1 = \inf Y_1$ if $Y_1 \ne \emptyset$ and $p_1 = 0$ otherwise. Clearly, if $Y_1 \ne \emptyset$, then (1) holds. Let us show that if $p_1 = 0$ then $c^2 < a_{m_1}\phi(t_1) \le c$. Otherwise, for some q

sufficiently big, we would have

$$\begin{aligned}(b_{m_1} - b_{m_1+q_1})\phi(t_1) =& (b_{m_1-1} - b_{m_1+q_1}\phi(t_1) \\ &+ (b_{m_1} - b_{m_1-1})\phi(t_1) > c - c^2 > c^2\end{aligned}$$

since $Y_1 = \emptyset$. So $m_1 + 1 \in N_1$ in contradiction to the choice of m_1.

Now suppose we have $m_1 < m_2 < \cdots < m_k, p_1, \dots, p_k$ and $t_1, \dots, t_k$ such that (1) and (2). Choose t_{k+1} such that

$$\phi(2t_{k+1}) > \frac{1}{c^{k+2}}\phi(t_{k+1})$$

and

$$\phi(t_{k+1}) > 2\frac{c^{k+2}}{b_{n_k}}$$

where $n_k = m_k + p_k$. Define

$$N_{k+1} = \{n > n_k : \text{ there exists } p \geq 1 \text{ with } (\sum_{j=n}^{n+p} a_j)\phi(t_{k+1}) > c^{k+2}\}.$$

It is clear that $n_k + 1 \in N_{k+1}, c^{k+2}/\phi(t_{k+1})$ is an upper bound of N_{k+1} and if $m_{k+1} = \sup N_{k+1}$, then $m_{k+1} > n_k \geq m_k$.

Moreover, if

$$Y_{k+1} = \{q \geq 1 : c^{k+2} < \Big(\sum_{j=m_{k+1}}^{m_{k+1}+q} a_j\Big)\phi(t_{k+1}) \leq c^{k+1}\}$$

and $p_{k+1} =$ inf Y_{k+1} if $Y_{k+1} \neq \emptyset$ and $p_{k+1} = 0$ otherwise, then

$$c^{k+2} < \Big(\sum_{j=m_{k+1}}^{m_{k+1}+p_{k+1}} a_j\Big)\ (t_{k+1}) \leq c^{k+1}.$$

This ends the proof of the lemma. □

PROOF OF THEOREM 3. (1) ⇒ (2). It follows from Proposition 1 with a standard argument (see [**11, 12**]).

(2) $\Rightarrow$ (3) $\Rightarrow$ (4). Immediate.

(4) $\Rightarrow$ (1). We assume that ϕ does not satisfy the Δ_2-condition at ∞. Then there exist $0 < c < 1/2$ with $\sum_{k=n+1}^{\infty} a_k > 2ca_n$ for every $n \in \mathbf{N}$, and sequences $(m_k), (p_k)$ and (t_k) like in Lemma 4. Define the operator $T : \ell^\infty \to \ell^\phi(a)$ by $T(x_n) = y_n$, where

$$y_n = \begin{cases} t_k x_k & \text{if } n \in [m_k, m_k + p_k] \text{ for some } k \\ 0 & \text{otherwise .} \end{cases}$$

Clearly, T is linear and one-to-one. If $S_1 = |x|_\infty$, then

$$\sum_{n=1}^{\infty} \phi\Big(\frac{|y_n|}{s_1}\Big) a_n \leq \sum_{k=1}^{\infty} \phi(t_k)\Big(\sum_{j=m_k}^{m_k+p_k} a_j\Big) \leq \sum_{k=1}^{\infty} c^k = \frac{c}{1-c} < 1$$

while, for $s_2 = |x|_\infty/4$, there is a $q \in \mathbf{N}$ with $|x_q| > 2s_2$ and

$$\sum_{n=1}^{\infty} \phi\Big(\frac{|y_n|}{s_n}\Big) a_n = \sum_{k+1}^{\infty} \Big(\sum_{j=m_k}^{m_k+p_k} a_j\Big) \phi\Big(\frac{t_k|x_k|}{s_2}\Big)$$
$$\geq \left(\sum_{j=m_q}^{m_q+p_q} a_j\right) \phi(2t_2) > 1.$$

Hence, $|x|_\infty/4 \leq |T(x)|_\phi \leq |x|_\infty$ and T is a topological isomorphism. □0

REMARK. It is clear that the isomorphism T of the previous theorem between ℓ^∞ and $T(\ell^\infty)$ is a Riesz-isomorphism that, in general, is not onto, i.e., $\ell^\phi(a) \neq \ell^\infty$. For example, if $\phi(x) = e^x - 1$ and $a_n = 1/n \log^2 n$ is a sequence of weights, using a recent result of W. Wnuk [**18**, Theorem 2], we get that $\ell^\phi(a)$ is not Riesz isomorphic to ℓ^∞.

If, in the above theorem, we assume that ϕ has a Young conjugate we get that "$\ell^\phi(a)$ *contains no copy of* c_0" is a fifth equivalent statement. This is due to the fact that $\ell^\phi(a)$ is a pre-dual space, by Proposition 5, together with a general property of pre-dual spaces (see [**9**, p. 103]).

PROPOSITION 5. *If ϕ is a convex Orlicz function with Young conjugate ψ, then the topological dual of $h^\phi(a)$ is isometrically isomorphic* *to $\ell^{\psi}(a)$ for any arbitrary weight sequence (a_n).*

PROOF. First we define the sequence of functions $\phi_n(s) = a_n\phi(r_n s)$ for $s \geq 0$ and (r_n) such that $a_n\phi(r_n) = 1$ and then consider the modular sequence space

$$\ell^{(\phi_n)} = \left\{x = (x_n) : \sum_{n=1}^{\infty} \phi_n\Big(\frac{|x_n|}{s}\Big) < \infty \text{ for some } s > 0\right\}$$

endowed with the usual norm

$$|x|_{\phi_n} = \inf\left\{s > 0 : \sum_{n=1}^{\infty} \phi_n\Big(\frac{|x_n|}{s}\Big) \leq 1\right\},$$

and also the corresponding closed subspace $h^{(\phi_n)}$. It is easy to check that $\ell^{(}\phi_n)$ and $\ell^{\phi}(a)$ are isometrically isomorphic: take the operator $T : \ell^{(}\phi_n) \to \ell^{\phi}(a)$ defined by $T(x_n) = (r_n x_n)$. Now, by a result of Woo (see [**9**, p. 168]) we have that $(h^{(\phi_n)})' \cong \ell^{(\psi_n)}$ (Here (ψ_n) denotes the sequence of Young conjugate functions of (ϕ_n)). A simple computation in our case gives the following expression for the functions ψ_n:

$$\psi_n(s) = a_n\psi\Big(\frac{1}{r_n a_n}s\Big) \text{ for } s \geq 0.$$

Finally we have that the spaces $\ell^{(\psi_n)}$ and $\ell^{\psi}(a)$ are isometrically isomorphic by means of the operator

$$T : \ell^{\psi}(a) \to \ell^{(\psi_n)} \text{ defined by } T(x_n) = (r_n a_n x_n).$$

□

As a consequence we obtain an improvement of the results given by Luxemburg [**11**, p. 60] and Denker-Kombrink [**1**, Theorem 2]:

PROPOSITION 6. *Let ϕ be a convex Orlicz function with Young conjugate ψ, and let (a_n) be a weight sequence with $\Sigma a_n < \infty$ and $\underline{\lim}_{n\to\infty}(\sum_{k=n+1}^{\infty} a_k)/a_n > 0$. The following conditions are equivalent :*

(1) $\ell^\phi(a)$ *is reflexive,*

(2) ϕ *and* ψ *satisfy the* Δ_2*-condition at* ∞,

(3) $\ell^\phi(a)$ *is uniformly convexifiable,*

(4) $\ell^\phi(a)$ *is B-convex, and*

(5) $\ell^\phi(a)$ *contains no isomorphic copy of* ℓ^1.

PROOF. (1) $\Leftrightarrow$ (2). If the Δ_2-condition at ∞ does not hold for ϕ, then, by Theorem 4, $\ell^\infty \underset{\sim}{\subset} \ell^\phi(a)$ and so $\ell^\phi(a)$ is not reflexive. If the Δ_2-condition holds for ϕ but not for ψ, then $(\ell^\phi(a))' \cong \ell^\psi(a) \underset{\sim}{\supset} \ell^\infty$, and thus $(\ell^\phi(a))'$ is not reflexive and neither is $\ell^\phi(a)$. The remaining implication follows from the last proposition.

(2) $\Rightarrow$ (3). Following Akimovich (see [**1**]) we construct a function $\overline{\phi}$ which is equivalent to ϕ at ∞ such that, for every $0 < r < 1$, there is $0 < t < 1$ such that $\overline{\phi}(s + rs/2) \leq (1-t)(\overline{\phi}(s) + \overline{\phi}(rs))/2$ for all $s \geq 0$. Now, by a result of Luxemburg [**11**, p. 64], $\ell^{\overline{\phi}}(a)$ is uniformly convex, and so $\ell^\phi(a)$ is uniformly convexifiable.

(3) $\Rightarrow$ (4). Well-known (see [**3**]).

(4) $\Rightarrow$ (2). If ϕ does not satisfy the Δ_2-condition at ∞, then as ℓ^∞ is not B-convex and $\ell^\infty \underset{\sim}{\subset} \ell^\phi(a)$, we have that $\ell^\phi(a)$ cannot be B-convex. If ϕ fulfils the Δ_2-condition and ψ does not, then $\ell^\psi(a)$ is not B-convex since $(\ell^\phi(a))' \simeq \ell^\psi(a) \underset{\sim}{\supset} \ell^\infty$. So the pre-dual $\ell^\phi(a)$ is not B-convex either.

(1) $\Rightarrow$ (5). Trivial.

(5) $\Rightarrow$ (2). If the Δ_2-condition does not hold for ϕ, $\ell^\phi(a) \underset{\sim}{\supset} \ell^\infty \underset{\sim}{\supset} \ell^1$. Moreover, if it holds for ϕ but not for ψ, then $(\ell^\phi(a))' \simeq \ell^\psi(a) \underset{\sim}{\supset} \ell^\infty \underset{\sim}{\supset} c_0$ and, by [**9**, p. 103], $\ell^\phi(a)$ contains a complemented copy of ℓ^1.

4. Counterexamples. The following examples show that, in general, Theorem 3 and Proposition 6 cannot be extended to wider classes of weight sequences.

EXAMPLE 1. A separable Orlicz sequence space $\ell^\phi(a)$ such that ϕ

does not satisfy the Δ_2-condition at ∞ and (a_n) is a weight sequence with $\Sigma a_n < \infty$ and

$$\varliminf_{n\to\infty} \frac{\sum_{k=n+1}^{\infty} a_k}{a_n} = 0.$$

Let (d_n) be an increasing sequence of positive scalars such that $d_1 = 1$ and $\sum_{n+1}^{\infty} d_n/d_{n+1} = D < 1/2$ (for example, $d_n = 2^{2^n}, n > 1$).

Define the Orlicz function

$$\phi(s) = \begin{cases} s^2 & \text{if } 0 \le s \le 1 \\ A_n s + B_n & \text{if } d_n \le s \le d_{n+1}, \end{cases}$$

where $A_n = d_{n+1} + d_n$ and $B_n = -d_{n+1}d_n$ for $n \in \mathbf{N}$. Clearly, ϕ is a convex Orlicz function which does not verify the Δ_2-condition at ∞ since $\phi(2dn)/\phi(dn) = 2 + d_{n+1}/d_n$. Consider the space $\ell^\phi(a)$ where (a_n) is the sequence of weights given by $a_n = 1/\phi(d_{n+1}$. We shall show that $\ell^\phi(a)$ is isomorphic to ℓ^1. By a similar argument to the one in Proposition 5, we have that $\ell^\phi(a)$ is isomorphic to $\ell^{(\phi_n)}$ for the (ϕ_n) sequence of Orlicz functions defined by $\phi_n(s) = \phi(d_{n+1}s)/\phi(d_{n+1})$ for $n \in \mathbf{N}$. Now, observe that

$$\sup_{0\le s\le 1} |\phi_n(s) - s| < 2\frac{d_n}{d_{n+1}},$$

since if $0 \le s \le d_n/d_{n+1}, |\phi_n(s) - s| \le s + s < 2d_n/d_{n+1}$ and if $d_n/d_{n+1} \le s \le 1$, then

$$|\phi_n(s) - s| = \left|\frac{A_n d_{n+1}s + B_n}{d_{n+1}^2} - s\right| = \frac{d_{n+1}d_n(1-s)}{d_{n+1}^2} < 2\frac{d_n}{d_{n+1}}.$$

Hence, ℓ^1 and $\ell^{(\phi_n)}$ are the same sets. Let us show that the identity map between ℓ^1 and $\ell^{(\phi_n)}$ is an isomorphism. Let $A = 1/1 - 2D$ and B be such that $B(1-2D) > 1$. Then, for $s_\varepsilon = |x|_1 + \varepsilon A^{-1}$,

$$\sum_{n=1}^{\infty} \phi_n\Big(\frac{|x_n|}{As_\varepsilon}\Big) \le \frac{1}{A}\sum_{n=1}^{\infty} \phi_n\Big(\frac{|x_n|}{s_\varepsilon}\Big) \le \frac{1}{A}\Big(\sum_{n=1}^{\infty} \frac{|x_n|}{s_\varepsilon} + 2\frac{d_n}{d_{n+1}}\Big)$$
$$\le \frac{1+2D}{A} = 1 - 4D^2 < 1,$$

so $As_\varepsilon \geq |x|_{\phi_n}$ for all $\varepsilon > 0$ and hence $|x|_{\phi_n} \leq A|x|_1$. On the other hand, if $\delta_\varepsilon = |x|_1 - \varepsilon B$ and we assume that $|x_n| \leq \delta_\varepsilon$ for every $n \in \mathbf{N}$, then

$$\sum_{n=1}^{\infty} \phi_n\Big(\frac{|x_n|}{\delta_\epsilon B^{-1}}\Big) \geq B \sum_{n=1}^{\infty} \phi_n\Big(\frac{|x_n|}{\delta_\varepsilon}\Big) \geq B \sum_{n=1}^{\infty} \Big(\frac{|x_n|}{\delta_\epsilon} - 2\frac{d_n}{d_{n+1}}\Big)$$
$$\geq B\Big(\frac{|x_1|}{\delta_\epsilon} - 2D\Big) > B(1-2D) > 1.$$

If there exists a natural number n_0 with $|x_{n_0}| \geq \delta_\varepsilon$, then we also have

$$\sum_{n=1}^{\infty} \phi_n\Big(\frac{|x_n|}{\delta_\epsilon B^{-1}}\Big) \geq B\phi_{n_1}\Big(\frac{|x_{n_0}}{\delta_\epsilon}\Big) \geq B\phi_{n_0}(1) = B > 1,$$

so $\delta_\epsilon B^{-1} < |x|_{\phi_n}$ and hence $|x|_1 B^{-1} \leq |x|_{\phi_n}$.

EXAMPLE 2. A reflexive Orlicz sequence space $\ell^\phi(a)$ such that ϕ does not satisfy the Δ_2-condition at ∞ and (a_n) is a weight sequence with $\Sigma a_n < \infty$ and $\underline{\lim}_{n\to\infty} \sum_{k=n+1}^{\infty} a_k/a_n = 0$.

Let (d_n) be a sequence of scalars as in Example 1 and define

$$\phi(s) = \begin{cases} s^2, & \text{if } 0 \leq s \leq 1 \\ s^2 + A_n s + B_n, & \text{if } d_n \leq s \leq d_{n+1}, \end{cases}$$

where $A_n = d_n(d_{n+1} - d_{n-1}/d_{n+1} - d_n)$ and $B_n = -d_n d_{n+1}(d_n - d_{n-1}/d_{n+1} - d_n)$ for $n \geq 1 (A_0 = B_0 = d_0 = 0)$. First, ϕ is a convex Orlicz function not satisfying the Δ_2-condition at ∞ since

$$\frac{\phi(d_{n+1})}{\phi\Big(\frac{d_{n+1}}{2}\Big)} > 2(1 + \frac{d_{n+1}}{d_n}).$$

Consider $\ell^\phi(a)$ for $a_n = 1/\phi(d_{n+1})$. Then $\ell^\phi(a)$ is isomorphic to ℓ^2. In fact, reasoning as in the previous example, $\ell^\phi(a)$ is isomorphic to $\ell^{(\phi_n)}$ for $\phi_n(s) = \phi(d_{n+1}s)/\phi(d_{n+1})$ and $\ell^{\phi_n)}$ is isomorphic to ℓ^2. For this last case

$$\sup_{0\leq s\leq 1} |\phi_n(s) - s^2| \leq \frac{8}{3}\frac{d_n}{d_{n+1}} \quad \text{for } n \in \mathbf{N}.$$

REMARK. Recall that if the Orlicz function ϕ is such that $\underline{\lim}_{s\to\infty} \phi(2s)/\phi(s) = \infty$, then $\ell^{\phi}(a)$ always contains a copy of ℓ^{∞} for every weight sequence (a_n) of finite sum [**2**, Prop. 4].

It is well-known that every Orlicz space contains an isomorphic copy of some ℓ^p or c_0. A result of Lindenstrauss and Tzafriri [**9, 10**] gives a characterization of the set of numbers p such that ℓ^p can be isomorphically embedded into a normed Orlicz space: for the sequence spaces ℓ^{ϕ}, it is the interval $[\alpha_{\phi}, \beta_{\phi}]$ whose endpoints are the Matuszewska-Orlicz *indices* of the function ϕ at 0. In particular, ℓ^{ϕ} is isomorphic to ℓ^p if and only if ϕ is equivalent to the function $\psi(x) = x^p$ at 0. Now, for the spaces $\ell^{\phi}(a)$ we get the following result:

PROPOSITION 7. *If ϕ is an Orlicz function with the Δ_2-condition at ∞ and β_{ϕ}^{∞}-concave, then there exists a weight sequence (a_n) with $\Sigma a_n < \infty$ such that $\ell^{\phi}(a)$ is isomorphic to $\ell^{\beta_{\phi}^{\infty}}$.*

PROOF. Recall that $\beta_{\phi}^{\infty} = p$, the Matuszewska-Orlicz upper index of ϕ at ∞, is defined by

$$\beta_{\phi}^{\infty} = \underline{\lim}_{\lambda\to 0^+} \left(\log\left\{\underline{\lim}_{x\to\infty} \frac{\phi(\lambda x)}{\phi(x)}\right\} \Big/ \log\lambda\right).$$

As ϕ is p-concave at ∞, if we define $\gamma(\lambda) = \underline{\lim}_{1\leq\lambda x} \phi(\lambda x)/\phi(x)$ for $\lambda > 0$, then $\lambda^p \prec \gamma(\lambda)$ and so $\ell^{\gamma} \subseteq \ell^p$. On the other hand, as $\beta_{\phi}^{\infty} = p, \ell^p \subset \ell^{\gamma}$ as so $\ell^{\gamma} = \ell^p$. Now, given a sequence (x_n) of positive scalars such that $\Sigma|x_n|^p < \infty$, take (s_n) with $s_n \to \infty, s_n x_n \geq 1$ and such that

$$\gamma(x_n) \leq \frac{\phi(x_n s_n)}{\phi(s_n)} \leq M|x_n|^p$$

for all n, where M is a positive constant. Now consider the space $\ell^{\phi}(a)$ for the sequence of weights $a_n = 1/\phi(s_n)$. Then $\Sigma a_n < \infty$ since $\sum_{n=1}^{\infty} 1/\phi(s_n) \leq \sum_{n=1}^{\infty} M x_n^p/\phi(x_n s_n) < \infty$.

By the p-concavity of $\phi, \phi(xs_n)/x^p \leq \phi(x_n s_n)/x_n^p \leq M\phi(s_n)$ for every $x \geq x_n$. Now, as the Δ_2-condition at ∞ for ϕ holds, $\ell^\phi(a) = h^\phi(a)$ has in (f_n) for $f_n = s_n e_n$ a Schauder basis. Finally, it is not difficult to check that the canonical basis (e_n) of ℓ^p and the basis (f_n) of $\ell^\phi(a)$ are equivalent, and therefore $\ell^\phi(a)$ is isomorphic to ℓ^p. □0

A similar result can be proved for α_ϕ^∞-convex functions in a neighbourhood of ∞.

REMARK. Observe that from the last Proposition we can deduce that $\ell^\phi(a)$ with $\Sigma a_n < \infty$ may be locally convex even if ϕ is not equivalent to any convex function at ∞. On the other hand, this fact does not happen if $\Sigma a_n = \infty$ and $a_n \to 0$ (see [**6**] where a study of the spaces $\ell^\phi(a)$ within the theory of galbs of Turpin [**16**] is presented).

Finally, let us show that even if we restrict ourselves to the class of sequences (a_n) with $\underline{\lim}_{\to\infty} \sum_{k=n+1}^{\infty} a_k/a_n > 0$, the spaces $\ell^\phi(a)$ can be isomorphic to ℓ^p for ϕ nonequivalent to the function $\psi(x) = x^p$ at ∞. (In contrast with the results for the spaces ℓ^ϕ, cf. [**9**]).

EXAMPLE 3. An Orlicz sequence space $\ell^\phi(a)$ which is isomorphic to ℓ^p, with ϕ non-equivalent to the function x^p at ∞ and (a_n) is of finite sum and slowly decreasing to 0.

Let $0 < p < \infty$ and define

$$\phi(x) = \begin{cases} \frac{x^p}{\log 2}, & \text{if } x \in [0,1] \\ \frac{x^p}{\log (1+x)}, & \text{if } x \in [1,\infty]. \end{cases}$$

Then $\beta_\phi^\infty = p$ and ϕ is p-concave at ∞. If we consider the space $\ell^\phi(a)$ where $a_n = 1/\phi(2^{2n})$, then $\underline{\lim}_{n\to\infty} a_{n+1}/a_n = 1/4^p > 0$. Now, by a similar argument to that in the previous proposition, we get that $\ell^\phi(a)$ is isomorphic to ℓ^p.

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