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A RANGE PROBLEM FOR HOMOGENEOUS, HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction A range problem will be considered for the partial differential equation $Pu = f$ where $u(x;t)$ and $f(x;t)$ are real-valued functions on $\mathbf{R}^n \times \mathbf{R}$ and $P = P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\right)$ is linear, homogeneous with constant coefficients, and hyperbolic with respect to t. The question to be considered is: if f is supported in the bounded set Ω and u is known for values of t such that $\mathbf{R}^n \times \{t\}$ is disjoint from Ω, can f be determined? For simplicity, it will be assumed herein that u vanishes when $t < \inf\{\tau : \Omega \cap (\mathbf{R}^n \times \{\tau\}) \neq \emptyset\}$. For a physical system modeled by the classical wave equation, this question is equivalent to asking if a force of finite extent and duration can be found from the subsequent disturbance that it generates.

One elementary observation regarding this question can be made immediately. Whereas u can be found from f by classical, explicit formulas, f is not uniquely determined by the values of u outside Ω. Indeed, for v also supported in $\Omega, f + Pv$ yields a solution which coincides with those values of u.

The main result of this paper is the following theorem. $L_0^2(\Omega)$ will denote the square-integrable, real-valued functions having support in Ω.

THEOREM 1.1. *Let Ω be an open, convex, and bounded subset of $\mathbf{R}^n \times \mathbf{R}, f \in L_0^2(\Omega)$, and $P = P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\right)$ be linear with constant coefficients, homogeneous, and hyperbolic with respect to t. Suppose $u(x;t)$ vanishes for large negative values of t and satisfies $Pu = f$. Then, for $t > T = \sup\{\tau : \Omega \cap (\mathbf{R}^n \times \{\tau\}) \neq \emptyset\}$, a representative of the class $[f] = \{f + Pv : v, Pv \in L_0^2(\Omega)\}$ can be computed from the Cauchy data for u on $\mathbf{R}^n \times \{t\}$. Furthermore,*

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this computation can be implemented from Cauchy data or, if t is large enough, partial data (e.g. the values of u, but not its normal derivatives) for u on a proper subset of its support in $\mathbf{R}^n \times \{t\}$.

The computations consist of two main steps. The first, described in §2, obtains the Radon transform of f in the characteristic directions for P from the Cauchy data for u on $\mathbf{R}^n \times \{t\}$. The second step, outlined in §3, uses this Radon transform data to compute a representative of [f] via the Kaczmarz method, a sequence of projections in the Hilbert space $L_0^2(\Omega)$. Also an alternate to the Kaczmarz method is described in §3 for the case where Ω is the open unit ball in $\mathbf{R}^n \times \mathbf{R}$.

2. Calculations with the Radon Transform. The Radon transform on Euclidean space has been used extensively in studying hyperbolic linear partial differential equations, for example in [**3**] and [**4**]. Its standard properties that are listed and subsequently used here are discussed in [**7**].

A point in the Euclidean space $\mathbf{R}^n \times \mathbf{R}$ will be denoted by $(x;t) = (x_1,\dots,x_n,t)$. The standard inner product on either $\mathbf{R}^n$ or $\mathbf{R}^n \times \mathbf{R}$ will be denoted by the usual dot product notation (i.e., $x \cdot y$ or $(x;t)\cdot(y;s)$). The norm of $x \in \mathbf{R}^n$ will be denoted by $|x|$.

The Radon transform of an integrable function $g : \mathbf{R}^n \to \mathbf{R}$ is given by

$$\begin{aligned} Rg(\theta,s) &= R_\theta g(s) \\ &= \int_{\{x\cdot\theta=s\}} g(x)dx \text{ where } s \in \mathbf{R} \text{ and } \theta \in S^{n-1} = \{|x|=1\}. \end{aligned}$$

That is, $R_\theta g(s)$ is the integral of g over the hyperplane $\{x \cdot \theta = s\}$ in $\mathbf{R}^n$ which contains $s\theta$ and is orthogonal to the direction θ. If the Fourier transform of g is defined to be

$$\hat{g}(\zeta) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-ix\cdot\zeta} g(x)\,dx \quad \text{for } \zeta \in \mathbf{R}^n,$$

then the Fubini theorem shows that

$$(R_\theta g)\hat{}\,(\tau) = (2\pi)^{(n-1)/2}\hat{g}(\tau\theta) \quad \text{for } \tau \in \mathbf{R}. \tag{2.1}$$

The action of the dual of R on $h : S^{n-1} \times \mathbf{R} \to \mathbf{R}$ is given by

$$R^*h(x) = \int_{S^{n-1}} h(\theta, x\cdot\theta)d\theta \quad \text{for } x \in \mathbf{R}^n.$$

If g is integrable and square-integrable, then

$$2^{-1}(2\pi)^{-1} \wedge^{(n-1)/2} R^* \wedge^{(n-1)/2} Rg = g \text{ where} \tag{2.2}$$

$$(\wedge g)(\zeta) = \mid \zeta \mid \hat{g}(\zeta). \tag{2.3}$$

Note that $\wedge$ acts on Rg by fixing θ as in (2.1). Partial differentiation and the Radon transform are related by

$$R_\theta \frac{\partial g}{\partial x_i} = \theta_i \frac{d}{ds}(R_\theta g). \tag{2.4}$$

The following calculations will use the Radon transform both of a function $g(x;t)$ on $\mathbf{R}^n \times \mathbf{R} \approx \mathbf{R}^{n+1}$ and of its restriction $g(\cdot\ ;t)$ to $\mathbf{R}^n \times \{t\} \approx \mathbf{R}^n$. The latter transform will be denoted by $R_\theta g(s,t)$ with $\theta \in S^{n-1}$ and $s \in \mathbf{R}$ and the former by $R_\varphi g(s)$ with $\varphi \in S^n$ and $s \in \mathbf{R}$.

Denote the symbol of $P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t}\right)$ by $p(\xi;t) = p(\xi_1, \ldots, \xi_n, t)$. By "hyperbolic of order m with respect to t, it is meant that for each $\theta \in S^{n-1}, p(\theta;\beta) = 0$ has nonzero real solutions $\beta_1(\theta) < \cdots < \beta_m(\theta)$. These conditions are often described by the term strictly hyperbolic. Thus, to each $\theta \in S^{n-1}$, there correspond unit characteristic directions $\varphi_i(\theta) = (1+\beta_i(\theta)^2)^{-1/2}(\theta;\beta_i(\theta)), i = 1, \ldots, m$. Where context permits, the dependence of β_i and φ_i on θ will not be explicitly denoted. The definition of hyperbolicity implies that m is even, $\beta_i(\theta) = -\beta_{m-i}(-\theta)$, and $\varphi_i(\theta) = -\varphi_{m-i}(-\theta)$. Hence P has distinct characteristic cones $\Gamma_i = \{r\varphi_i : r \in \mathbf{R}, \theta \in S^{n-1}\}$ for $i = 1, \ldots, m/2$. The union of these cones will be denoted by Γ.

The calculation begins by applying R_θ to $Pu = f$ to obtain

$$P_\theta\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) R_\theta u(s,t) = R_\theta f(s,t) \tag{2.5}$$

where the symbol of P_θ is $p_\theta(\alpha, \beta) = p(\alpha\theta; \beta)$. P_θ has order m and is hyperbolic with respect to t because $p_\theta(1, \beta_j) = p(\theta; \beta_j) = 0$ for $j = 1, \dots, m$. Thus $R_\theta u$ can be expressed in terms of $R_\theta f$ by a variant of the method of Herglotz [3] for $n = 1$ and an application of Duhamel's principle, as outlined below.

Herglotz's method solves the Cauchy problem

$$P_\theta\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) v = 0$$

$$(2.6) \qquad v(s,0) = \frac{\partial v}{\partial t}(s,0) = \cdots = \frac{\partial^{m-2} v}{\partial t^{m-2}}(s,0) = 0$$

$$\frac{\partial^{m-1} v}{\partial t^{m-1}}(s,0) = h(s)$$

where h is a rapidly decreasing function. For any sufficiently differentiable function H of one variable

$$P_\theta\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) H(s + \beta_i t) = p_\theta(1, \beta_i) H^{(m)}(s + \beta_i t) = 0.$$

Define $k(s,t) = \sum_{i=1}^{m} H(s + \beta_i t) / \frac{\partial p_\theta}{\partial \beta}(1, \beta_i)$. Then, by the identity

$$\sum_{i=1}^{m} \beta_i^\ell \Big/ \frac{\partial p_\theta}{\partial \beta}(1, \beta_i) = \begin{cases} 0 & \text{for } \ell < m-1 \\ p_\theta(0,1)^{-1} & \text{for } \ell = m-1 \end{cases}$$

it follows that

$$P_\theta\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) k = 0$$

$$\frac{\partial^i}{\partial t^i} k(s,0) = 0 \qquad \text{for } j = 1, \dots, m-2, \text{ and}$$

$$\frac{\partial^{m-1}}{\partial t^{m-1}} k(s,0) = \frac{H^{(m-1)}(s)}{p_\theta(0,1)}$$

The solution to (2.6) is obtained by setting $H(s) = (p_\theta(0,1)/m!) s^m$ sgn s, where sgn $0 = 0$ and sgn $s = s / \mid s \mid$ if $s \neq 0$, and $v(s,t) =$

$\int_{-\infty}^{\infty} h''(y)k(s-y,t)dy$. The $(m-1)$ order initial condition is satisfied because

$$\frac{\partial^{m-1}}{\partial t^{m-1}} v(s,0) = \int_{-\infty}^{\infty} h''(y) \mid s-y \mid dy = h(s).$$

Using the solution in conjunction with Duhamel's principle gives the following solution to (2.5):

$$R_\theta u(s,t) = \frac{p_\theta(0,1)}{m!} \int_{-\infty}^{t} \int_{-\infty}^{\infty} \frac{\partial^2}{\partial y^2} R_\theta f(y,\tau) \sum_{i=1}^{m} \frac{\Big(s-y+\beta_i(t-\tau)\Big)^m}{\frac{\partial p_\theta}{\partial \beta}(1,\beta_i)} \times$$

$$\times \operatorname{sgn}\, (s-y+\beta_i(t-\tau))\, dy\, d\tau.$$

Integrating by parts twice yields

$$\begin{aligned} R_\theta u(s,t) &= \frac{p_\theta(0,1)}{(m-2)!} \int_{-\infty}^{t} \int_{-\infty}^{\infty} R_\theta f(y,\tau) \\ &\sum_{i=1}^{m} \frac{(s-y+\beta_i(t-\tau))^{m-2} \operatorname{sgn}\, ((s-y+\beta_i(t-\tau))}{\frac{\partial p_\theta}{\partial \beta_i}(1,\beta_i)}\, dy\, d\tau. \end{aligned} \tag{2.7}$$

In order to complete the first step of calculation, assume that $t > T =$ sup $\{\tau : \Omega \cap (\mathbf{R}^n \times \{\tau\}) \neq \emptyset\}$. For such t, repeated differentiation of (2.7) with respect to s and t shows that

$$\frac{\partial^{m-2-k}}{\partial s^{m-2-k}} \frac{\partial^k}{\partial t^k} R_\theta u(s,t) = p_\theta(0,1) \sum_{i=1}^{m} \frac{\beta_i^k}{\frac{\partial p_\theta}{\partial \beta}(1,\beta_i)} \int_{-\infty}^{t}$$

$$\left[\int_{-\infty}^{s+\beta_i(t-\tau)} R_\theta f(y,\tau)\, dy - \int_{s+\beta_i(t-\tau)}^{\infty} R_\theta f(y,\tau)\, dy \right] d\tau$$

for $k = 0, 1, \ldots, m-2$, and therefore that

$$\begin{aligned} &\frac{\partial^{m-1-k}}{\partial s^{m-1-k}} \frac{\partial^k}{\partial t^k} R_\theta u(s,t) \\ &= p_\theta(0,1) \sum_{i=1}^{m} \frac{2\beta_i^k}{\frac{\partial p_\theta}{\partial \beta}(1,\beta_i)} \int_{-\infty}^{t} R_\theta f(s+\beta_i(t-\tau),\tau) d\tau \end{aligned} \tag{2.8}$$

for $k = 0, 1, \ldots, m-1$.

Because $R_\theta f(\sigma, \tau) = 0$ for $\tau > t > T$, the i^{th} integral in (2.8) is, up to the factor $(1+\beta_i^2)^{-1/2}$, the integral of $R_\theta f(\sigma, \tau)$ along the line $\tau = t - (\sigma - s)/\beta_i$. Thus, writing $R_\theta f$ as an integral over $\theta^\perp$ and setting $\varphi_i = (\theta; \beta_i) \big/ \sqrt{1+\beta_i^2}$, it follows by the Fubini theorems that

$$\text{(2.9)} \qquad \int_{-\infty}^{t} R_\theta f(s + \beta_i(t-\tau), \tau)\, d\tau = \sqrt{1+\beta_i^2} R_{\varphi_i} f\left(\frac{s+\beta_i t}{\sqrt{1+\beta_i^2}}\right).$$

Substituting (2.9) into (2.8) gives a system of linear equations

(2.10)
$$\frac{\partial^{m-1-k}}{\partial s^{m-1-k}} \frac{\partial^k}{\partial t^k} R_\theta u(s,t) = \sum_{i=1}^{m} \beta_i^k \left[\frac{2p_\theta(0,1)\sqrt{1+\beta_i^2}}{\frac{\partial p_\theta}{\partial \beta}(1, \beta_i)} R_{\varphi_i} f\left(\frac{s+\beta_i t}{\sqrt{1+\beta_i^2}}\right)\right],$$

for $k = 0, 1, \ldots, m-1$, which has matrix $\left(\beta_i^{j-1}\right)_{1 \le i,j \le m}$ and is invertible because the β_i are distinct and non-zero. Inversion of this system yields the following proposition. To see the validity of the proposition for $f \in L_0^2(\Omega)$, observe that the integrals in (2.7) make sense for such a function and that the derivatives in (2.8) and (2.10) can be taken in the sense of distributions.

PROPOSITION 2.11. *For $f \in L_0^2(\Omega)$ and $t > T$, the Cauchy data for u on $\mathbf{R}^n \times \{t\}$ allows the computation of*

$$\{R_\varphi f : \varphi \in \Gamma\}.$$

A closer examination of (2.10) allows a somewhat more precise statement. For $\theta \in S^{n-1}$ and $t \in \mathbf{R}$, let $\Omega_i(\theta, t) = \{(x;t) : \min_\Omega\{(y;\tau)\cdot\varphi_i\} \le (x;t)\cdot\varphi_i \le \max_\Omega\{(y;\tau)\cdot\varphi_i\}\}$; i.e., the strip $\Omega_i(\theta,t)$ is the intersection of $\mathbf{R}^n \times \{t\}$ with the union of all hyperplanes normal to φ_i and meeting Ω. Because $\varphi_i(\theta) = -\varphi_{m-i}(-\theta)$, it follows that $\Omega_i(\theta,t) = \Omega_{m-i}(-\theta,t)$. Furthermore, when t is sufficiently large, the $\Omega_i(\theta,t)$ are pairwise disjoint.

LEMMA 2.12. *Let $\theta \in S^{n-1}$ and $t > T$.*

a) *Fix* $1 \le i \le m$, *and let* S_i *be an open strip in* $\mathbf{R} \times \{t\}$ *which contains* $\Omega_i(\theta, t)$. *Then there exists a neighborhood* Σ_i *of* φ_i *in* Γ_i *such that* $\{R_{\tilde{\varphi}_i} f : \tilde{\varphi}_i \in \Sigma_i\}$ *can be computed from Cauchy data for* u *on* S_i.

b) *If* t *is large enough so that the* $\Omega_i(\theta, t)$ *are pairwise disjoint, then* $R_{\varphi_i} f, i = 1, \ldots, m$, *can be found from the values of* $\frac{\partial^k}{\partial t^k} u$ *on the corresponding* $\Omega_i(\theta, t)$ *for any fixed* $0 \le k \le m - 1$.

PROOF. Since u has compact support K in $\mathbf{R}^n \times \{t\}$, there in a neighborhood Λ_i of θ in S^{n-1} such that for each $\tilde{\theta} \in \Lambda_i$, $\Omega_i(\tilde{\theta}, t) \cap K$ is contained in S_i. Let $\Sigma_i = \{\tilde{\varphi}_i : \tilde{\theta} \in \Lambda_i\}$. Cauchy data for u on S_i allows the computation of $R_{\tilde{\theta}} u$ on each $(n-1)$ dimensional hyperplane in $\Omega_i(\tilde{\theta}, t)$ for each $\tilde{\theta} \in \Sigma_i$ and hence, via (2.10), of $R_{\tilde{\varphi}_i} f$ throughout its support for each $\tilde{\varphi}_i \in \Sigma_i$. This establishes a).

If the $\Omega_i(\theta, t)$ are pairwise disjoint, then for each $s \in \mathbf{R}$ and $0 \le k \le m-1$, at most one sum and on the right-hand side of (2.10) is non-zero. Varying s in the k^{th} equation of (2.10) establishes b).

The next lemma will be needed in the following section. While it follows from well-known general results in partial differential equations, the preceding computations and a characterization of the range of the Radon transform establish it directly.

LEMMA 2.13. *Let* $P\left(\frac{\partial}{\partial x}; \frac{\partial}{\partial t}\right)$ *and* Ω *be as in Theorem (1.1), and let* $f \in L_0^2(\Omega)$ *satisfy* $R_\varphi f = 0$ *for all* $\varphi \in \Gamma$. *Then there exists* $u \in L_0^2(\Omega)$ *with* $Pu = f$.

PROOF. Denote the right-hand side of (2.7) by $g(\theta, s, t)$. If a function $u(x; t)$ can be defined by $R_\theta u(s, t) = g(\theta, s, t)$, then $Pu = f$. To do so, $g(\theta, s, t)$ must be in the range of R for each t.

By the characterization of the range of R given in [**7**], it is sufficient to verify for $f \in L_0^2(\Omega)$ that

(2.14a) $g(\theta, s, t)$ vanishes when $\{(x; t) : x \cdot \theta = s\}$ is disjoint from the compact set $\Omega_t = \{(x; t) : ((x; t) - \Gamma) \cap \Omega \neq \emptyset\}$,

(2.14b) g is jointly even in θ and s, and for each θ and $t, h(s) = g(\theta, s, t)$ and $\Lambda^{(n-1)/2} h$ are square integrable, and

(2.14c) for each $t \in \mathbf{R}$ and each non-negative integer j, $\int_{-\infty}^{\infty} g(\theta, s, t) s^j ds$ is the restriction to S^{n-1} of a homogeneous polynomial of degree j. When these properties hold, $g(\theta, s, t)$ is the Radon transform with respect to θ some function in $L_0^2(\Omega_t)$. Because $R_\theta f$ satisfies these properties, it is straightforward to verify that g also does. Therefore, u is well defined by (2.7) and is square integrable with compact support in $\mathbf{R}^n \times \{t\}$ for each t.

Clearly $R_\theta u$ and hence, by the Radon inversion formula (2.2), u vanish when $t < inf\{\tau : \Omega \cap (\mathbf{R}^n \times \{\tau\}) \neq \emptyset\}$. If $R_\varphi f = 0$ for all $\varphi \in \Gamma$, then (2.10) shows that $R_\theta u$ and hence u have trivial Cauchy data when $t > T$. Hence, $u(x; t)$ has compact support and is therefore square-integrable on $\mathbf{R}^n \times \mathbf{R}$. An elementary lemma on page 80 of [**2**] then shows that u is supported in Ω.

REMARK 1) If P is the classical linear wave operator, this lemma characterizes the forcing functions in $L_0^2(\Omega)$ which propagate no disturbance beyond their support.

2) Implicit in the proof of Lemma (2.13) is the fact that, given f square integrable with compact support, a solution to $Pu = f$ is defined by (2.7) and the Radon inversion formula (2.2). Indeed, applying a variant of the Radon inversion formula to the solution v defined for (2.6) yields Herglotz's solution to the Cauchy problem for P given in [**3**].

3) Let K be compact in Ω. The restriction to K of any $g \in C_0^\infty(\Omega)$ can be extended by the following construction to a function f such that $Rf = 0$ on Γ as in (2.3). Let w be the solution to $Pw = g$ (given by Herglotz's solution), and let $h \in C_0^\infty(\Omega)$ with $h = 1$ on K. Then let $f = hg$.

3. Constructing a Function in [f]. For $f \in L_0^2(\Omega)$, the lemma (2.13) shows that $[f] = \{f + g : g \in L_0^2(\Omega)$ and $R_\varphi g = 0$ for all $\varphi \in \Gamma\}$. Thus, the basic problem of finding f from Cauchy data for u on $\mathbf{R}^n \times \{t\}$ for $t > T$ admits at best the partial solution of finding a function in $[f]$. Such a function $f_1 \in [f]$ can be constructed by the Kaczmarz method which is discussed extensively in [**6**] and [**7**]. This algorithm constructs f_1 as the limit of a sequence of projections in $L_0^2(\Omega)$ of an arbitrary $g \in L_0^2(\Omega)$. If $g = 0$, then the result will be f_0, the function in $[f]$ of minimum norm. The projections can be computed from the data

$\{R_\varphi f \mid \varphi \in \Gamma\}$, as is noted in iii) of the following lemma. The proof of the lemma is elementary and essentially identical to that given in [**1**] for the case where Ω is the unit disc in $\mathbf{R}^2$.

LEMMA 3.1. *Let Ω be a bounded, open subset of $R^n \times \mathbf{R}, \chi_\Omega$ denote the characteristic function on $\Omega, \varphi \in S^n, \mu_\varphi(s) = [R_\varphi \chi_\Omega(s)]^{-1}$, and $I_\varphi =$ closure $\{s \in \mathbf{R} : \mathbf{R}_\varphi \chi_\Omega(s) \neq 0\}$. Then*

i) *$R_\varphi : L_0^2(\Omega) \to L_0^2(I_\varphi, \mu_\varphi(s)ds)$ is a continuous operator with adjoint R_φ^* given by*

$$R_\varphi^* h(x;t) = \begin{cases} h((x;t)\cdot\varphi)\mu_\varphi((x;t)\cdot\varphi) & \text{for } (x;t) \in \Omega \\ 0 & \text{for } (x;t) \notin \Omega, \end{cases}$$

ii) *The orthogonal complement of ker (R_φ), the null-space of R_φ, is*

$$N_\varphi = \{g \in L_0^2(\Omega) : g(x;t) = \tilde{g}((x;t)\cdot\varphi) \text{ for some } \tilde{g} \in L_0^2(I_\varphi, \mu_\varphi(s)ds)\},$$

iii) *P_φ, the orthogonal projection of $L_0^2(\Omega)$ onto the translated subspace $f +$ ker (R_φ), is given by*

$$P_\varphi g(x;t) = \Big(g + R_\varphi^* R_\varphi(f-g)\Big)(x;t)$$

$$= \begin{cases} g(x;t) + [(R_\varphi f - R_\varphi g)((x;t)\cdot\varphi)\mu_\varphi((x;t)\cdot\varphi)] & \text{if } (x;t) \in \Omega \\ 0 & \text{if } (x;t) \notin \Omega. \end{cases}$$

Given a countable set of directions $\{\varphi_1, \varphi_2, \varphi_3, \dots\}$ in S^n, let $P_k = P_{\varphi_1} P_{\varphi_2} \dots P_{\varphi_k}$. The Kaczmarz method consists of constructing the sequence $\{P_k^j g : j = 1, 2, 3, \dots\}$ which converges in $L_0^2(\Omega)$ to g_k, the orthogonal projection of g onto $f + \cap_{i=1}^k \ker(R_{\varphi_i})$. Conditions on the directions $\varphi_1, \dots, \varphi_k$ and the domain Ω which guarantee a geometric rate of convergence are given in [**5**].

It is an elementary exercise in Hilbert space theory to show that the sequence $\{g_k\}_{k=1,2,3,\dots}$ converges to g_0, the orthogonal projection of g onto $f + \cap_{i=1}^\infty \ker(R_{\varphi_i})$, and that projections of the initial choice $g = 0$ yield g_k and g_0 which have the minimum norms possible.

For integrable h supported in Ω, the fact that $\hat{h}$ is real analytic and the identity (2.1) show that $\{\varphi \in S^n : R_\varphi h = 0\}$ is an analytic variety in S^n. Therefore, if the directions $\{\varphi_1, \varphi_2, \varphi_3, \dots\}$ are dense in an open subset of $\Gamma, \cap_{i=1}^{\infty}$ ker $(R_{\varphi_i}) = \cap_{\varphi \in \Gamma}$ ker (R_φ), and consequently $[f] = f + \cap_{i=1}^{\infty}$ ker (R_{φ_i}). Thus, the function g_0 produced as the limit of the projections of the Kaczmarz method, which can be computed from the data $\{R_\varphi f : \varphi \in \Gamma\}$ and hence from the Cauchy data for u on $\mathbf{R}^n \times \{t\}$ for $t > T$, is in $[f]$.

By Lemma (2.12), the data $\{R_\varphi f : \varphi \in \Gamma\}$ needed for finding a representative of $[f]$ by the Kaczmarz method can be found from $\frac{\partial^k u}{\partial t^k}$ on $\mathbf{R}^n \times \{t\}$ when t is sufficiently large and $0 \le k \le m-1$ is fixed. It remains to establish the assertion of Theorem (1.1) that this computation can be performed from Cauchy data for u, or partial data when t is large, on a proper subset of its support in $\mathbf{R}^n \times \{t\}$. The argument below applies to either case.

Let $S_i, i = 1, \dots, m/2$, be strips as described in Lemma (2.12) (which may or may not correspond to different $\theta \in S^{n-1}$), and let $\Sigma_i, i = 1, \dots, m/2$, be the neighborhoods in the Γ_i given in that lemma. Because each cone Γ_i is an irreducible variety, their union Γ is the analytic variety determined by $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_{m/2}$. Thus $\cap\{$ ker $(R_\varphi) : \varphi \in \Gamma\} = \cap\{$ ker $(R_\varphi) : \varphi \in \Sigma\}$ and $[f] = f + \cap\{$ ker $(R_\varphi) : \varphi \in \Sigma\}$. By lemma 2.12, the data $\{R_\varphi f; \varphi \in \Sigma\}$ needed to compute $[f]$ by the Kaczmarz method can be found from Cauchy data for u (or partial data when t is large) on $S = S_1 \cup \dots \cup S_{m/2}$.

REMARK. In the case that P is the wave operator, S is a single strip in $\mathbf{R} \times \{t\}$ which need only be wider than $\sqrt{2} \cdot$(diameter Ω). For general P, S can be confined to a half-space of $\mathbf{R}^n \times \{t\}$.

The special case $\Omega = B^{n+1} = \{|\ (x;t)\ | < 1\}$ can be analyzed by calculations similar to those for $n = 1$ in [1]. In this case, $I_\varphi = [-1, 1]$ and $\mu_\varphi(s) = \Omega_n^{-1}(1-s^2)^{-n/2}$ where Ω_n is the volume of B^n. Let $C_m(s)$ denote the Jacobi polynomical of degree m with $\alpha = \beta = n/2$, i.e., the Gegenbauer polynomial of appropriate index (see [**8**]). By construction, $\{C_0, C_1, C_2, \dots\}$ is an orthonormal basis for $L^2([-1,1], (1-s^2)^{n/2} ds)$ with C_m orthogonal to s^k for each integer $0 \le k < m$. For $\varphi \in S^n$, let

$$C_{m,\varphi}(x;t) = \Omega_n^{-1/2} C_m((x;t) \cdot \varphi). \tag{3.3}$$

Then $\{C_{0,\varphi}, C_{1,\varphi}, C_{3,\varphi}, \dots\}$ is an orthonormal basis for $N_\varphi =$ ker $(R_\varphi)^\perp =$ range R_φ^*. Using the orthogonality of $C_m(s)$ and s^k for any

$k < m$, direct calculations show that for $\varphi_1, \varphi_2, \in S^n$,

$$R^*_{\varphi_2} R_{\varphi_2} C_{m,\varphi_1} = \frac{C_m(\varphi_1 \cdot \varphi_2)}{C_m(1)} C_{m,\varphi_2}. \tag{3.4}$$

Therefore the inner product of C_{m,φ_1} and C_{ℓ,φ_2} in $L^2_0(B^{n+1})$ is

$$\begin{aligned} \left\langle C_{m,\varphi_1}, C_{\ell,\varphi_2} \right\rangle &= \left\langle C_{m,\varphi_1}, R^*_{\varphi_2} R_{\varphi_2} C_{\ell,\varphi_2} \right\rangle \\ &= \left\langle R^*_{\varphi_2} R_{\varphi_2} C_{m,\varphi_1}, C_{\ell,\varphi_2} \right\rangle \\ &= \begin{cases} 0 & \text{if } m \neq \ell \\ \frac{C_m(\varphi_1 \cdot \varphi_2)}{C_m(1)} & \text{if } m = \ell \end{cases} \end{aligned} \tag{3.5}$$

These observations allow direct (rather than iterative) computation of approximations to the function in $[f]$ of minimum norm.

PROPOSITION 3.6 *Let f_0 be the function in $[f] = f + \cap\{\ ker\ (R_\varphi) : \varphi \in \Gamma\}$ of least norm. Then $f_0 = \sum_{m=0}^{\infty} f_{0,m}$ where the $f_{0,m}$ are pairwise orthogonal and for each m there is a finite set of directions $\{\varphi_1, \ldots, \varphi_{k(m)}\} \subset \Gamma$ with $f_{0,m} = \sum_{j=1}^{k(m)} \alpha_{j,m} C_{m,\varphi_j}$. Furthermore, the coefficients $\alpha_{j,m}$ can be computed directly from $\{R_{\varphi_i} f, \ldots, R_{\varphi_{k(m)}} f\}$.*

PROOF: Because N_m, the span of $\{C_{m,\varphi} : \varphi \in \Gamma\}$, is contained in the polynomials of degree m, N_m is finite dimensional, and there exist $\varphi_1, \ldots, \varphi_{k(m)} \in \Gamma$ such that $\{C_{m,\varphi_1}, \ldots, C_{m,\varphi_{k(m)}}\}$ is a basis for N_m. Let N denote the orthogonal complement of $\cap\{\ \ker R_\varphi : \varphi \in \Gamma\}$. Then N is the closure of $\sum_{\varphi\in\Gamma} N_\varphi$, and so it follows from (3.5) that $N = \sum_{m=0}^{\infty} N_m$ with the N_m pairwise orthogonal. Because f_0 is the orthogonal projection of the zero function onto $f + \cap\{\ \ker\ (R_\varphi) : \varphi \in \Gamma\}, f - f_0$ is in $\cap\{\ \ker\ (R_\varphi) : \varphi \in \Gamma\}$, and f_0 is in N.

Fix m and let $f_{0,m}$ denote the component of f_0 in N_m. Then for all $\varphi \in \Gamma$

$$\begin{aligned} 0 &= \left\langle f - f_0, C_{m,\varphi} \right\rangle \\ &= \int_{-}^{1} 1 R_\varphi f(s) C_m(s) ds - \left\langle f_{0,m}, C_{m,\varphi} \right\rangle. \end{aligned} \tag{3.7}$$

Write $f_{0,m} = \sum_{j=1}^{k(m)} \alpha_{j,m} C_{m,\varphi_j}$ and recall (3.5). Setting $\varphi = \varphi_i$, $i = 1, \ldots, k(m)$, in (3.7) yields the system of equations

$$C_m(1)^{-1} \sum_{j=1}^{m} C_m(\varphi_i \cdot \varphi_j)\alpha_{j,m} = \int_{-1}^{1} R_{\varphi_i} f(s) C_m(s) ds \tag{3.8}$$

$$\text{for } i = 1, \ldots, k(m).$$

The system (3.8) is clearly invertible because the set $\{C_{m,\varphi_1}, \ldots, C_{m,\varphi_{k(m)}}\}$ is a basis for N_m.

REMARK 3.9 An optimistic viewpoint would hold that a function in $[f]$ ought to be constructible via some method which is more direct that the Kaczmarz method. Since any function can be obtained from its Radon transform in all directions via the Radon inversion formula, a direct method results if the data $\{R_\varphi f : \varphi \in \Gamma\}$ can be extended to Rg for some $g \in L_0^2(\Omega)$ because then $g \in [f]$. Constructing such an extension appears to pose a difficult problem. Equation (2.1) shows that in terms of Fourier transforms this problem is: given $\hat{f}|_\Gamma$, find a real-analytic extension $\hat{g}$ on $\mathbf{R}^n \times \mathbf{R}$ which is square integrable and has complex analytic extension to $\mathbf{C}^{n+1}$ of exponential type corresponding to Ω.

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REFERENCES

1. C. Hamaker and D.C. Solmon, *The angles between the null spaces of x-rays*, J. Math. Anal. Appl. **62**, (1978), 1-23.

2. L. Hörmander. *Linear Partial Differential Operators.* Springer-Verlag, Berlin, 1976.

3. F. John. *Plane Waves and Spherical Means.* Springer-Verlag, New York, 1955.

4. P. Lax and R. Phillips. *Scattering Theory.* Academic Press, New York, 1967.

5. B. Petersen, K.T. Smith, and D.C. Solmon, *Sums of plane waves and the range of the Radon transform.* Math. Ann. **243** (1979), 153-161.

6. K.T. Smith, C. Hamaker, D.C. Solmon, and S.L. Wagner, *The divergent beam x-ray transform.* Rocky Mtn. J. Math. **10**. (1980), pp 253-283.

7. K.T. Smith, D.C. Solmon, and S.L. Wagner, *Practical and mathematical aspects of reconstructing objects from radiographs.* Bull. AMS. (1977), 1227-1270.

8. G.Szego. *Orthogonal Polynomials* (4th edition). AMS, Providence, 1975.

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