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APPROXIMATION OF ZOLOTAREV TYPE

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1. Introduction. We consider generalized Zolotarev polynomials which minimize the expression

$$\sup_{x\in[-1,1]} |ax^{m+k+1} + x^{m+1} + p(x)|$$

where $a \in \mathbf{R}, m, k \in \mathbf{Z}, k > 0, m \geq 0$, and where p is a polynomial of degree $\leq m$. Thus, the two highest terms (a and 1) are prescribed (with a gap of length k between these terms). Their structure is quite complicated, hence we exhibit approximations which can replace the generalized Zolotarev polynomials for many purposes. Our investigations are based on complex approximation and related to the Carathéodory-Fejér method. Therefore, we first treat complex variable approximation problems (approximation by a modified finite Laurent series on the unit circle), thereby extending investigations by Al'per [**2**] and Rivlin [**13**]. Further we determine the Caratheéodory-Fejér approximant to the function $az^k + 1$, and then we truncate the corresponding Caratheódory-Fejér series in a modified way in order to get bounds for the generalized Zolotarev polynomials.

2. Complex approximation. For $-1 < b < 1$, and $a := -b/(1-b^2)$, we consider the functions

$$G(z) := \frac{1}{1-b^2}\cdot\frac{1-bz}{1-b/z} = az + \sum_{n=0}^{\infty} b^n z^{-n}$$

and

$$H(z) := z^{m+1}\cdot G(z^k) = az^{m+1+k} + \sum_{n=0}^{\infty} b^n z^{m+1-kn}$$

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(where $k, m \in \mathbf{Z}$, and $k > 0, m \geq 0$), and approximate them in the sense of the Chebyshev norm on the unit circle, i.e.,

$$||F|| := \sup_{|z|=1} |F(z)|,$$

from the subspace

$$\mathbf{P}(m+2,\infty) := \operatorname{span}(z^{m+2}, z^{m+3}, z^{m+4}, \dots).$$

The following result extends investigations by Al'per [**2**] and Rivlin [**13**]; see also Klotz [**9**] and Trefethen [**16**].

LEMMA 1. *The unique best Chebyshev approximant to H from $\mathbf{P}(m+2,\infty)$ on $|z| = 1$ is given by $P^* = 0$.*

PROOF . Suppose there exists a (nontrivial) $P \in \mathbf{P}(m+2,\infty)$ with

$$||H - P|| < ||H|| \quad (\neq 0).$$

Then H and P have the same winding number (of the image curve with respect to zero; cf. Henrici [**8**, p. 277]). This is a consequence of an extended version of Rouché's theorem, see, e.g., Saks-Zygmund [**14**, p. 193] (also a direct proof can be given by the usual homotopy method for the corresponding integral). We see easily that H has winding number $m+1$ while P as a non-zero member of $\mathbf{P}(m+2,\infty)$ has a winding number $\geq m+2$. This contradiction shows that $P^* = 0$ is proximinal.

Now suppose that a certain $P \in \mathbf{P}(m+2,\infty)$ gives a best approximation to H:

$$||H - P|| = ||H||.$$

Since $|H(z)| = ||H||$ for all $|z| = 1$ (note that G is a Blaschke function), by the strict convexity of the disk,

$$\left|H(z) - \frac{1}{2}P(z)\right| < ||H|| \quad \text{if } P(z) \neq 0.$$

Hence one would obtain an approximation better than allowed if P has no zeros (on $|z| = 1$); a better approximation also could be achieved in the case where P has only a finite number of zeros (this is shown by employing a suitable correction polynomial which improves the approximation at these critical points). Thus P has an infinite number of zeros, assuring $P = 0$ and unicity.

Next we consider the polynomials $P_r \in \mathbf{P}(m+1-k(r-1), \infty)$, for $(r = 0, 1, 2, \dots)$ given by

$$\begin{aligned} P_r(z) :&= H(z) - b^r z^{-kr} \cdot H(z) \\ &= az^{m+1+k} + \sum_{n=0}^{r-1} b^n z^{m+1-kn} - ab^r z^{m+1-k(r-1)}. \end{aligned}$$

Thus P_r is a modified partial sum of the Laurent series for H: the last coefficient is replaced by

$$b^{r-1} - ab^r = b^{r-1}(1-ab) = \frac{b^{r-1}}{1-b^2}.$$

PROPOSITION 2. *The best Chebyshev approximant to H on $|z| = 1$ from $\mathbf{P}(j, \infty)$, where*

$$m+1-kr < j \le m+1-k(r-1,) \text{ for } r > 0,$$

$$m+1 < j, \text{ for } r = 0$$

is given by P_r (and uniquely determined). Further, we have

$$\|H - P_r\| = \frac{|b|^r}{1-b^2}. \tag{1}$$

Note that, for $r = 0$, we have $P_0 = 0$.

The proof is an easy adaptation of Lemma 1. In order to get (1), we observe that

$$\begin{aligned} \|H - P_r\| &= \sup_{|z|=1} |b^r z^{-kr} H(z)| \\ &= \sup_{|z|=1} \left| b^r \cdot z^{-kr} \cdot z^{m+1} \cdot \frac{1}{1-b^2} \cdot \frac{1-bz^k}{1-b/z^k} \right| \\ &= \frac{|b|^r}{1-b^2}. \end{aligned}$$

3. Carathéodory-Fejér approximants. The functions G and H introduced in §2 can be considered from the point of view of Carathéodory-Fejér approximation (see, e.g., Gutknecht-Trefethen [**6**, Theorem 1.1], and also Carathéodory-Fejér [**4**] and Schur [**15**]). Indeed, the unique Carathéodory-Fejér approximant to $az + 1$ (for $a \neq 0$) is given by the series

$$\sum_{n=1}^{\infty} b^n z^{-n} \quad \left(\text{where } b = \frac{1 - \sqrt{1+4a^2}}{2a}\right).$$

Then the Hankel matrix belonging to this problem is

$$\begin{pmatrix} 1 & a \\ a & 0 \end{pmatrix}$$

with largest eigenvalue (in absolute value)

$$\lambda = \frac{1}{2} + \frac{1}{2}\sqrt{1+4a^2} = \frac{1}{1-b^2}.$$

Hence among all expansions with leading coefficients a and $1, G(z)$ yields the minimal norm on $|z| = 1$, namely $(1 - b^2)^{-1}$ (see also Gutknecht - Trefethen [**6**]). This minimum property can be extended as follows:

Let us determine the Carathéodory-Fejér approximant to the function $az^k + 1 (k \in \mathbf{N})$. Here we have

PROPOSITION 3. *Let $-1 < b < 1, a := -b/(1-b^2)$, and $k \in \mathbf{N}$. Then the unique Carathéodory-Fejér approximant to $az^k + 1$ is given by*

$$\sum_{n=1}^{\infty} b^n z^{-kn}.$$

PROOF . The corresponding Hankel matrix to this problem is

$$\begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 & a \\ 0 & 0 & \cdot & \cdot & \cdot & a & 0 \\ \cdot & \cdot & & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \cdot & \cdot \\ 0 & a & \cdot & \cdot & \cdot & 0 & 0 \\ a & 0 & \cdot & \cdot & \cdot & 0 & 0 \end{pmatrix}$$

with spectral radius $1/2 + 1/2(1+4a^2)^{1/2} = 1/(1-b^2)$. Since $G(z^k)$ has modulus $1/(1-b^2)$, the proposition is settled, by the unicity of the Carathéodory-Fejér approximant. □

4. Polynomials of Zolotarev type. We consider the Zolotarev polynomials in the following way: Given $a \in \mathbf{R}$, and $m = 0, 1, 2, \ldots$, then

$$Z_a := Z_{a,m+2} := aT_{m+2} + T_{m+1} + q^*,$$

where q^* is the uniquely determined algebraic polynomial of degree $\leq m$ such that

$$||Z_{a,m+2}||_\infty \leq ||aT_{m+2} + T_{m+1} + q||_\infty$$

for all polynomials q of degree $\leq m$, where $||\cdot||_\infty$ is the sup-norm on [-1,1].

In general, the Zolotarev polynomials are rather complicated to describe explicitly (for their connection with elliptic functions, cf. Achieser [**1**], Carlson-Todd [**5**] and the literature quoted there). Thus good approximants to $Z_{a,m+2}$ are of interest.

In [**7**] we proved the inclusion

$$\frac{1-b^{2m+2}}{1-b^2} \leq ||Z_{a,m+2}||_\infty \leq \frac{1+b^{2m+2}}{1-b^2} \tag{2}$$

for $b = (1-(1+4a^2)^{1/2})/2a$ (if $a \neq 0$) using a modified truncation of the series given by the function H (for $k = 1$). We mention that some other estimates are due to Bernstein [**3**] and Reddy [**11**], see also [**7**].

Motivated by the observations in §3, estimates can be given for the following generalized Zolotarev polynomials:

Let $a \in \mathbf{R}, k \in \mathbf{N}$, then consider the minimum problem

$$||aT_{m+1+k} + T_{m+1} + p^*||_\infty \leq ||aT_{m+1+k} + T_{m+1} + p||_\infty$$

for all $p \in \mathbf{P}_m$ (polynomials of degree $\leq m$).

The unique solution $aT_{m+1+k} + T_{m+1} + p^*$ (with $p^* \in \mathbf{P}_m$) is called a *generalized Zolotarev polynomial.* We have the following estimates:

PROPOSITION 4. *Let* $-1 < b < 1, a := -b/(1-b^2)$, *and* $k \in \mathbf{N}$. *Then*

$$\frac{1-|b|^{s+1}}{1-b^2} \leq ||aT_{m+1+k} + T_{m+1} + p^*||_\infty \leq \frac{1+|b|^{s+1}}{1-b^2}$$

for $s \in \mathbf{N}$ *such that* $2m+1-k < ks \leq 2m+1$.

PROOF . Determine $s \geq 1$ such that $2m+1-k < ks \leq 2m+1$, and consider

$$\begin{aligned} P_{s+1}(z) &= H(z) - b^{s+1} z^{-k(s+1)} H(z) \\ &= az^{m+1+k} + \sum_{n=0}^{s} b^n z^{m+1-kn} - ab^{s+1} z^{m+1-ks} \end{aligned}$$

with $||P_{s+1}|| \leq (1+|b|^{s+1})/(1-b^2)$.

Now

$$\begin{aligned} &||aT_{m+1+k} + T_{m+1} + p^*||_\infty \\ &\leq \sup_{|z|=1} \left| az^{m+1+k} + z^{m+1} + \sum_{n=1}^{s} b^n z^{m+1-kn} - ab^{s+1} z^{m+1-ks} \right| \\ &\leq ||P_{s+1}|| \leq \frac{1+|b|^{s+1}}{1-b^2}, \end{aligned}$$

hence the upper bound is settled.

(Note that $\mathrm{Re}(\sum_{n=1}^{s} b^n z^{m+1-kn} - ab^{s+1} z^{m+1-ks})$ is in $\mathbf{P}_m$.)

In order to get the lower bound, we observe that the graph of H has winding number $m+1$ with respect to the origin; further we have $H(1) = (1-b^2)^{-1}$ and $H(-1) = (-1)^{m+1}(1-b^2)^{-1}$. Hence there exist $m+2$ points $-1 = x_0 < x_1 < \cdots < x_{m+1} = 1$ such that for $h = \mathrm{Re} H$ we have

$$h(x_\mu) = (-1)^{m+1-\mu} ||h||_\infty = (-1)^{m+1-\mu}(1-b^2)^{-1} \quad (0 \leq \mu \leq m+1).$$

Proposition 2 yields $||h - \mathrm{Re} P_{s+1}||_\infty \leq ||H - P_{s+1}|| = |b|^{s+1}/(1-b^2)$, hence (with $g = \mathrm{Re}\ P_{s+1}$)

$$|g(x_\mu)| \geq |h(x_\mu)| - ||h-g||_\infty \geq (1-|b|^{s+1})/(1-b^2)$$

for $0 \leq \mu \leq m+1$. Now the de la Vallée-Poussin principle completes the proof. □

REMARKS.

(i) Obviously, (2) is achieved from Proposition 4 for $k = 1$.

(ii) With the aid of strong unicity constants one can get bounds for $||(aT_{m+1+k} + T_{m+1} + p^*) - \mathrm{Re} P_{s+1}||_\infty$.

(iii) Modified considerations lead to the best approximation of (real) functions like $x \to (x-c)^{-1} (c > 1)$ as well as $x \to (c-x)^{-s}$, see Meinardus [**10**], Achieser [**1**] and Rivlin [**12**].

REFERENCES

1. N.I. Achieser, *Vorlesungen über Approximationstheorie*, Akademie-Verlag, Berlin, 1967.

2. S.J. Al'per, *Asymptotic values of best approximation of analytic functions in a complex domain*, Uspehi Mat. Nauk **14** No. 1 (85) (1959), 131-134.

3. S.N. Bernstein, *Collected works*, Vol. 1., Akad. Nauk SSSR, Moscow, 1952.

4. C. Carathéodory and L. Fejér, *Über den Zusammenhang der Extremen von harmonischen Funktionen mit ihren Koeffizienten und über den Picard-Landau'schen Satz*, Rend. Circ. Mat. Palermo **32** (1911), 218-239.

5. B.C. Carlson and J. Todd, *Zolotarev's first problem - the best approximation by polynomials of degree* $\leq n-2$ to $x^n - n\sigma x^{n-1}$ *in* [-1,1], Aequationes math. **26** (1983), 1-33.

6. M.H. Gutknecht and L.N. Trefethen, *Real polynomial Chebyshev approximation by the Carathéodory-Fejér method*, SIAM J. Numer. Anal. **19** (1982), 358-371, and **20** (1983), 420-436.

7. W. Haussmann and K. Zeller, *Approximate Zolotarev polynomials*, Comp. Math. Appl., **12B** (1986), 1133-1140.

8. P. Henrici, *Applied and computational complex analysis*, Vol. 1, Wiley, New York, 1974.

9. V. Klotz, *Gewisse rationale Tschebyscheff-Approximationen in der komplexen Ebene*, J. Approx. Theory **19** (1977), 51-60.

10. G. Meinardus, *Approximation von Funktionen und ihre numerische Behandlung*, Springer, Berlin, 1964.

11. A.R. Reddy, *A note on a result of Zolotarev and Bernstein*, Manuscripta math. **20** (1977), 95-97.

12. T.J. Rivlin, *Polynomials of best uniform approximation to certain rational functions*, Numer. Math. **4** (1962), 345-349.

13. ———, *Some explicit polynomial approximations in the complex domain*, Bull. Amer. Math. Soc. **73** (1967), 467-469.

14. S. Saks and A. Zygmund, *Analytic functions*, Polish Scientific Publ., Warsaw 1965.

15. I. Schur, *Über Potenzreihen, die im Inneren des Einheitskreises beschränkt sind*, J. Reine Angew. Math. **147** (1917), 205-232, and **148** (1918), 122-145.

16. L.N. Trefethen, *Near-circularity of the error curve in complex Chebyshev approximation*, J. Approx. Theory **31** (1981), 344-367.

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