ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 19, Number 1, Winter 1989

CONVERSE RESULTS IN THE THEORY OF EQUICONVERGENCE OF INTERPOLATING RATIONAL FUNCTIONS

M.A. BOKHARI

1. Introduction. Since the first extension of Walsh's theorem in 1981 [1], there have been in the last few years a number of direct theorems on the theory of equiconvergence of certain schemes of interpolatory polynomial sequences. A recent paper of Saff and Sharma [3] also gives some direct theorems, but it deals with the equiconvergence of two schemes of rational interpolants. Our object in this paper is to obtain a sort of converse of this theorem on the lines of a corresponding theorem due to Szabados [4] which is related to the Lagrange interpolant and the Taylor sections of an analytic function.

Let $f \in A_\rho$ (the class of functions analytic in $|z| < \rho$ but not in $|z| \leq \rho$, $\rho > 1$). As usual π_s will denote the class of polynomials of degree $\leq s$. For a given $\sigma > 1$ and for a fixed integer $m \geq -1$, let

$$(1.1)\quad r_{n+m,n}(z,f) := B_{n+m,n}(z,f)/(z^n - \sigma^n),\ B_{n+m,n}(z,f) \in \pi_{n+m},$$

interpolate $f \in A_\rho$ in the $n+m+1$ roots of unity. If, for a positive integer l, we set

$$(1.2)\qquad \Delta^{\sigma}_{l,n,m}(z;f) = R_{n+m,n}(z,f) - \sum_{\nu=0}^{l-1} r_{n+m,n}(z,f,\nu),$$

where $r_{n+m,n}(z,f,\nu)$ are certain rational functions given by (2.1) and (2.3), then Saff and Sharma showed that if $\sigma \geq \rho^{l+1}$, then

$$(1.3)\qquad \lim_{n\to\infty} \Delta^{\sigma}_{l,n,m}(z,f) = 0$$

for $|z| < \rho^{l+1}$. And if $\sigma < \rho^{l+1}$, then (1.3) holds for all $z \in \mathbf{C}$ with $|z| \neq \sigma$. Moreover, this result is sharp in the sense that the region of convergence cannot be improved.

Received by the editors on July 31, 1986, and in revised form on December 19, 1986.

When $\sigma \to \infty$, we get an extension of a classic theorem of Walsh [**5**, p. 153]. In this case, we know that

$$(1.4) \qquad \Delta^{\infty}_{l,n,m}(z,f) = L_{n+m}(z,f) - \sum_{\nu=0}^{l-1} P^*_{n+m}(z,f,\nu),$$

where $L_{n+m}(z,f) \in \pi_{n+m}$ is the Lagrange interpolant to $f(z)$ in the $(n+m+1)$-th roots of unity and

$$(1.5) \qquad P^*_{n+m}(z,f,\nu) := \sum_{j=0}^{n+m} a_{\nu(n+m+1)+j} z^j, \quad \nu = 0,1,2,\dots,$$

with $f(z) := \sum_{k=0}^{\infty} a_k z^k$ and $\overline{\lim}_{n\to\infty} |a_n|^{1/n} = \rho^{-1}$.

Let A^*_ρ (or $A^*_\rho C$), $\rho \geq 1$, denote the set of all functions which are analytic in $|z| < \rho$ (or analytic in $|z| < \rho$ and continuous in $|z| \leq \rho$). We shall say that a sequence $\{S_n(z)\}_{n=1}^{\infty}$ is U.B. in $|z| < \gamma^r$ if $\{S_n(z)\}_{n=1}^{\infty}$ is uniformly bounded in every closed subset of $|z| < \gamma^r$.

The following theorem is due to Szabados [**4**]:

THEOREM A. *Let $l \geq 1$ and $m \geq -1$ be fixed integers. If $f \in A^*_1 C$ and if $\{\Delta^{\infty}_{l,n,m}(z,f)\}_{n=1}^{\infty}$ is U.B. in $|z| < \rho^{l+1}$ for some $\rho > 1$, then $f \in A^*_\rho$.*

We shall prove an analogue of the above theorem when $\sigma > 1$ is finite.

2. Preliminaries and statement of main result. Let $f \in A^*_1 C$ and let

(2.1)
$$r_{n+m,n}(z,f,0) = P_{n+m,n}(z,f,0)/(z^n - \sigma^n), \quad P_{n+m,n}(z,f,0) \in \pi_{n+m},$$

be the rational function which interpolates $f(z)$ in the zeros of $z^{m+1}(z^n - \sigma^{-n})$. Set

$$(2.2) \qquad \alpha_{n,m}(z) := 1 - z^{m+1}\sigma^{-n}, \quad \beta_{n,m}(z) := z^{m+1}(z^n - \sigma^{-n}).$$

Let $N(\nu) := (\nu+1)(n+m+1)$, $\nu = 0,1,2,\dots$, and let $S_{N(\nu)}(z)$ denote the unique polynomial in $\pi_{N(\nu)}$ which interpolates the function $\{\alpha_{n,m}(z)\}^\nu (z^n - \sigma^n) f(z)$ in Hermite sense at $N(\nu)+1$ zeros of

$\{\beta_{n,m}(z)\}^{\nu+1}$. Then Saff and Sharma (cf. [**3**, (3.6)]) established the following relation:

$$S_{N(\nu)}(z) - \alpha_{n,m}(z)S_{N(\nu-1)}(z) = \{\beta_{n,m}(z)\}^{\nu}P_{n+m,n}(z,f,\nu),$$

where $P_{n+m,n}(z,f,\nu)$ is a polynomial in $\pi_{n+m}, \nu = 1,2,3,\dots$. This enables us to define rational functions $r_{n+m,n}(z,f,\nu), \nu = 1,2,\dots,$ of the form

(2.3)

$$r_{n+m,n}(z,f,\nu) = P_{n+m,n}(z,f,\nu)/(z^n - \sigma^n), \quad P_{n+m,n}(z,f,\nu) \in \pi_{n+m}.$$

We state our main result:

THEOREM 2.1. *Let $m \geq -1$ and $l \geq 1$ be fixed integers, and let $f \in A_1^*C$. If, for some $\rho > 1$ and for some $\sigma \geq \rho^{l+1}$, the sequence $\{\Delta^{\sigma}_{l,m,n}(z,f)\}_{n=1}^{\infty}$ given by* (1.2) *is U.B. in $|z| < \rho^{l+1}$, then $f \in A_\rho^*$.*

REMARK 2.1. Theorem 2.1 may be looked upon as a partial converse of the statement (1.3). A natural question which arises at this point is the following: If $1 < \sigma < \rho^{l+1}$ and if $\{\Delta^{\sigma}_{l,n,m}(z,f)\}_{n=1}^{\infty}$ is uniformly bounded on every compact subset of the domain $\{z : |z| \neq \sigma\}$, is $f \in A_\rho^*$? We assert that, in general, the answer is in the negative. This is easily seen on taking $\hat{f}(z) = (z-\eta)^{-1}$ where we choose $\alpha \in (0,1)$ such that $\sigma < \rho^{(l+1)\alpha} =: \eta^{l+1}$. Then, $\hat{f} \in A_\eta$ and, from the Saff-Sharma Theorem (cf. (1.3)), we have $\Delta^{\sigma}_{l,n,m}(a,\hat{f}) \to 0$ on every compact subset of $\{z : |z| \neq \sigma\}$. But $\hat{f} \notin A_\rho$.

REMARK 2.2. Theorem 2.1 is also valid if we consider $m < -1$. (See [**3**] for the construction of the rational functions $r_{n+m,n}(z,f,\nu), \nu = 0,1,2,\dots,$ when $m < -1$.)

3. Some lemmas. In this section, we shall compare some polynomial interpolatory processes with some rational ones, and then show that the sequences $\{\Delta^{\infty}_{l,n+m}(z,f)\}_{n=1}^{\infty}$ and $\{\Delta^{\sigma}_{l,n,m}(z,f)\}$, $\sigma \geq \rho^{l+1}$, given by (1.4) and (1.2) respectively, are either both *bounded* or both *unbounded* in the region $|z| < \sqrt{\sigma}$. It will enable us to show that f is analytic in $|z| < \min(\rho, \frac{1}{\sigma 2(l+1)})$, which is the main idea that underlies the proof of Theorem 2.1.

LEMMA 3.1. *Let $m \geq -1$ and $\sigma > 1$. If $f \in A_1^* C$, then*

$$\lim_{n\to\infty} \{L_{n+m}(z,f) - r_{n+m,n}(z,f)\} = 0, \quad \textit{for } |z| < \sqrt{\sigma}, \tag{3.1}$$

where $L_{n+m}(z,f)$ and $R_{n+m,n}(z,f) = B_{n+m,n}(z,f)/(z^n - \sigma^n)$ are defined by (1.4) and (1.1) respectively. Moreover, the convergence in (3.1) is uniform and geometric on every closed subset of the region $|z| < \sqrt{\sigma}$.

PROOF. Let ω be a primitive $(n+m+1)$th root of unity. From the definition of Lagrange interpolating polynomial, we have

$$L_{n+m}(z,f) = \sum_{k=0}^{n+m} \frac{z^{n+m+1}-1}{z-\omega^k} \cdot \frac{\omega^k}{n+m+1} f(\omega^k)$$

and

$$B_{n+m,n}(z,f) = \sum_{k=0}^{n+m} \frac{z^{n+m+1}-1}{z-\omega^k} \cdot \frac{\omega^k}{n+m+1} (\omega^{kn} - \sigma^n) f(\omega^k).$$

This gives us

$$R_{n+m,n}(z,f) - L_{n+m}(z,f) = \sum_{k=0}^{n+m} \frac{z^{n+m+1}-1}{z-\omega^k} \cdot \frac{\omega^k f(\omega^k)}{n+m+1} \cdot \frac{\omega^{kn} - z^n}{z^n - \sigma^n}$$

$$= \sum_{k=0}^{n+m} \sum_{j=0}^{n+m} z^{n+m-j} \omega^{k(j+1)} \frac{f(\omega^k)}{n+m+1} \cdot \frac{\omega^{kn} - z^n}{z^n - \sigma^n}.$$

Since $f \in A_1^* C$, there is an $M > 0$ so that $|f(t)| \leq M$ for every $|t| \leq 1$. Let $|z| = \tau$, $\tau \geq 1$. Then, from the above relation, we have

$$|L_{n+m}(z,f) - R_{n+m,n}(z,f)| \leq M(n+m+1)\tau^{n+m} \frac{\tau^n + 1}{|\tau^n - \sigma^n|}.$$

If $\sigma > \tau$, we obtain

$$\overline{\lim_{n\to\infty}} \left\{ \max_{|z|=\tau} |R_{n+m,n}(z,f) - L_{n+m}(z,f)| \right\}^{1/n} \le \frac{\tau^2}{\sigma}$$

which establishes Lemma 3.1.

LEMMA 3.2. *Let $m \ge -1$ be a fixed integer and $\sigma > 1$. If $f \in A_1^* C$, then the conclusion of Lemma 3.1 remains valid if $L_{n+m}(z,f)$ and $R_{n+m,n}(z,f)$ are replaced by $P_{n+m}^*(z,f,0)$ and $r_{n+m,n}(z,f,0)$* (cf. (1.5), (2.1)) *respectively.*

PROOF. It is easy to see that $r_{n+m}(z,f,0)$ has the integral representation

$$r_{n+m,n}(z,f,0) = \frac{1}{2\pi i} \int_{|t|=\delta} \frac{t^n - \sigma^n}{z^n - \sigma^n} \cdot \frac{f(t)}{t-z} \cdot \frac{t^{m+1}(t^n - \sigma^{-n}) - z^{m+1}(z^n - \sigma^{-n})}{t^{m+1}(t^n - \sigma^{-n})} dt,$$

where $\sigma^{-1} < \delta < 1$. Also, we can write

$$P_{n+m}^*(z,f,0) = \frac{1}{2\pi i} \int_{|t|=\delta} \frac{f(t)}{t-z} \cdot \frac{t^{n+m+1} - z^{n+n+1}}{t^{n+m+1}} dt.$$

An elementary calculation now shows that

$$(3.2) \qquad r_{n+m,n}(z,f,0) - P_{n+m}^*(z,f,0) = \frac{1}{2\pi} \int_{|t|=\delta} \frac{f(t) K_n(t,z) dt}{(t-z)(t^n - \sigma^{-n})(z^n - \sigma^n) t^{n+m+1}},$$

where

$$(3.3) \qquad K_n(t,z) := (t^{n+m+1} - z^{n+m+1})(t^{2n} - t^n z^n - 1) - t^n(t^{m+1} - z^{m+1}) - \sigma^{-n}(t^n - z^n)(t^{n+m+1} - t^n z^{m+1} - z^{n+m+1}).$$

Since $\sup_{|t|\le 1} |f(t)| \le M$ for some $M > 0$, from (3.2) we obtain

$$(3.4) \qquad |r_{n+m,n}(z,f,0) - P_{n+m}^*(z,f,0)| \le \frac{M}{2\pi(\delta^n - \sigma^{-n})|z^n - \sigma^n|} \int_{|t|=\delta} \left| \frac{K_n(t,z)}{t-z} \right| |dt|,$$

whereas

$$\begin{aligned}\frac{K_n(t,z)}{t-z} =&(t^{2n} - t^n z^n - 1)\sum_{j=0}^{n+m} t^j z^{n+m-j} - t^n \sum_{j=0}^{m} t^j z^{m-j} \\ &- \sigma^{-n}(t^{n+m+1} - t^n z^{m+1} - z^{n+m+1})\sum_{j=0}^{n-1} t^j z^{n-j-1}.\end{aligned} \tag{3.5}$$

If $|z| = \tau \geq 1$, and $|t| = \delta < 1$, then

$$\begin{aligned}\left|\frac{K_n(t,z)}{t-z}\right| \leq &(\delta^{2n} + \delta^n \tau^n + 1)(n+m+1)\tau^{n+m} + \delta^n(m+1)\tau^m \\ &+ \sigma^{-n}(\delta^{n+m+1} + \delta^n \tau^{m+1} + \tau^{n+m+1})n\tau^{n-1}.\end{aligned}$$

Notice that the relation (3.4) holds for all $\delta \in (\sigma^{-1}, 1)$, which, upon using (3.5) and then letting $\delta \to 1$, gives us

$$|r_{n+m,n}(z,f,0) - P^*_{n+m}(z,f,0)| \leq \frac{CM(n+m+1)|z|^{2n}}{(1-\sigma^{-n})|z^n - \sigma^n|}.$$

Here C is constant independent of n. If $\sigma > |z| = \tau$, then it is easy to see that

$$\lim_{n\to\infty}\left\{\sup_{|z|=\tau}|r_{n+m,n}(z,f,0) - P^*_{n+m}(z,f,0)|\right\}^{1/n} \leq \frac{\tau^2}{\sigma}$$

which proves the lemma.

LEMMA 3.3. *Let $m \geq -1$ be a fixed integer and $\sigma > 1$. If $f \in A_1^*C$, then the conclusion of Lemma 3.1 remains valid if $L_{n+m}(z,f)$ and $R_{n+m,n}(z,f)$ are replaced by $P^*_{n+m}(z,f,v)$ and $r_{n+m,n}(z,f,\nu)$, $\nu =$ 1, 2, 3, ..., (cf. (1.5) and (2.3)) respectively.*

PROOF. An integral representation of $r_{n+m,n}(z,f,\nu), \nu \geq 1$, is given by (cf. [3], (3.13))

(3.6)

$$r_{n+m,n}(z,f,\nu)\frac{1}{2\pi i}\int_{|t|=\delta}\frac{t^n - \sigma^n}{z^n - \sigma^n}\cdot\frac{(\alpha_{n,m}(z))^{\nu-1}}{(\beta_{n,m}(z))^{\nu+1}}\frac{H_n(t,z,\nu)}{t-z}f(t)dt,$$

where $\sigma^{-1} < \delta < 1$ and

$$
\begin{aligned}
H_n(t,z,\nu) :&= \alpha_{n,m}(z)\beta_{n,m}(t) - \alpha_{n,m}(t)\beta_{n,m}(z) \\
&= t^{n+m+1} - z^{n+m+1} - \sigma^{-n}(tz)^{m+1}(t^n - z^n) \\
&\quad - \sigma^{-n}(t^{m+1} - z^{m+1}).
\end{aligned} \tag{3.7}
$$

Also, from (1.5), we have
(3.8)

$$
P^*_{n+m}(z,f,\nu) = \frac{1}{2\pi i}\int_{|t|=\delta} \frac{f(t)}{(\nu+1)(n+m+1)} \frac{t^{n+m+1} - z^{n+m+1}}{t-z} dt.
$$

Since $\{\alpha_{n,m}(t)\}^{\nu-1} = 1 + \sum_{j=1}^{\nu-1}(-1)^j\binom{\nu-1}{j}(t^{m+1}\sigma^{-n})^j$, from (3.6) and (3.7) $r_{n+m,n}(a,f,\nu)$ can be rewritten as

$$
r_{n+m,n}(z,f,\nu) = Q_{n+m,n}(z,f,\nu) + T_{n+m,n}(z,f,\nu) \tag{3.9}
$$

with

$$
\begin{cases}
Q_{n+m,n}(z,f,\nu) &= \frac{1}{2\pi i}\int_{|t|=\delta} \frac{t^n-\sigma^n}{z^n-\sigma^n} \cdot \frac{f(t)}{(t^{m+1}(t^n-\sigma^{-n}))^{\nu+1}} \\
&\quad \cdot \frac{t^{n+m+1}-z^{n+m+1}}{t-z} dt, \\
T_{n+m,n}(z,f,\nu) &= \frac{1}{2\pi i}\int_{|t|=\delta} \frac{t^n-\sigma^n}{z^n-\sigma^n} \cdot \frac{f(t)J_n(t,z)}{(t^{m+1}(t^n-\sigma^{-n}))^{\nu+1}} dt,
\end{cases} \tag{3.10}
$$

where

$$
\begin{aligned}
J_n(t,z) &:= H_n(t,z,\nu)\sum_{j=1}^{\nu-1}\binom{\nu-1}{j}(-t^{m+1}\sigma^{-n})^j \\
&\quad - \sigma^{-n}((tz)^{m+1}(t^n - z^n) + t^{m+1} - z^{m+1}).
\end{aligned}
$$

Now one can easily see after some computation that

$$
T_{n+m,n}(z,f,\nu) = O\left(\frac{1+|z|^n}{|z^n - \sigma^n|}\right). \tag{3.11}
$$

Also,

$$
\overline{\lim_{n\to\infty}} \left\{ \sup_{\substack{|z|=\tau \\ \tau<\sqrt{\sigma}}} |Q_{n+m,n}(z,f,\nu) - P^*_{n+m,n}(z,f,\nu)| \right\}^{1/n} \le \frac{\tau^2}{\sigma} \tag{3.12}
$$

which follows from (3.10) and (3.8) on mimicking the procedure starting at (3.1) in Lemma 3.2. Therefore, from (3.9)-(3.12), we conclude that

$$\overline{\lim_{n\to\infty}}\left\{\sup_{\substack{|z|=\tau\\ \tau<\sqrt{\sigma}}}|r_{n+m,n}(z,f,\nu)-P^*_{n+m}(z,f,\nu)|\right\}^{1/n}\le\frac{\tau^2}{\sigma}.$$

REMARK 3.1. If l is a fixed positive integer then it follows directly from Lemma 3.3 that

$$\overline{\lim_{n\to\infty}}\left\{\sup_{\substack{|z|=\tau\\ \tau<\sqrt{\sigma}}}\left|\sum_{\nu=0}^{l-1}r_{n+m,n}(z,f,\nu)-\sum_{\nu=0}^{l-1}P^*_{n+m}(z,f,\nu)\right|\right\}^{1/n}\le\frac{\tau^2}{\sigma}.$$

Next, we prove

LEMMA 3.4. *Let $l\ge 1$ and $m\ge -1$ be fixed integers and $\sigma>1$. If $f\in A_1^*C$, then $\{\Delta^{\infty}_{l,n,m}(z,f,)\}_{n=1}^{\infty}$ is U.B. in $|z|<\sqrt{\sigma}$ if and only if the sequence $\{\Delta^{\sigma}_{l,n,m}(z,f)\}_{n=1}^{\infty}$ is also where $\Delta^{\infty}_{l,n,m}(z,f)$ and $\Delta^{\sigma}_{l,n,m}(z,f,)$ are given by (1.4) and (1.2).*

PROOF. From the triangle inequality and the definition of $\Delta^{\infty}_{l,n,m}(z,f)$ and $\Delta^{\sigma}_{l,n,m}(z,f)$, we note that

$$\begin{aligned}\Big||\Delta^{\sigma}_{l,n,m}(z,f,)|-|\Delta^{\infty}_{l,n,m}(z,f)|\Big| &\le |\Delta^{\sigma}_{l,n,m}(z,f)-\Delta^{\infty}_{l,n,m}(z,f)|\\ &\le |R_{n+m,n}(z,f)-L_{n+m}(z,f)|\\ &\quad+\left|\sum_{\nu=0}^{l-1}r_{n+m,n}(z,f,\nu)-\sum_{\nu=0}^{l-1}P^*_{n+m}(z,f,\nu)\right|.\end{aligned}$$

An application of Lemma 3.1 and Remark 3.1 now gives the desired result.

REMARK 3.2. If $\sigma\ge\rho^{2(l+1)}$, then lemma 3.4 also holds if $|z|<\sqrt{\sigma}$ is replaced by $|z|<\rho^{l+1}$. For this, it is enough to note that the lemmas 3.1-3.3 are valid for the region $|z|<\rho^{l+1}<\sqrt{\sigma}$.

4. Proof of Theorem 2.1.

4. Proof of Theorem 2.1. First, assume that $\sigma \geq \rho^{2(l+1)}$. By the hypothesis of Theorem 2.1, $\{\Delta^{\sigma}_{l,n,m}(z,f)\}_{n=1}^{\infty}$ is U.B. in $|z| < \rho^{l+1}$. From Remark 3.2, it follows that $\{\Delta^{\infty}_{l,n,m}(z,f)\}_{n=1}^{\infty}$ is U.B. in $|z| < \rho^{l+1}$, too. Thus, $f \in A^*_{\rho}$ by Theorem A.

Next, consider $\rho^{l+1} \leq \sigma < \rho^{2(l+1)}$. Then $\{\Delta^{\sigma}_{l,n,m}(z,f)\}_{n+1}^{\infty}$, being a U.B. sequence in $|z| < \rho^{l+1}$ is also U.B. in $|z| < \sqrt{\sigma}$. Now from Lemma (3.4), it implies that the sequence $\{\Delta^{\infty}_{l,n,m}(z,f)\}_{n=1}^{\infty}$ is U.B. in $|z|, \sqrt{\sigma}$. If we let $\xi^{l+1} := \sqrt{\sigma}$, then $f \in A^*_{\xi}$ (cf. Theorem A). Notice that $\xi > 1$. Let $\rho_1 := \sup\{\eta : f \in A^*_{\eta}\}$. Then $\rho_1 > 1$, $f \in A^*_{\rho_1}$ and f has a singularity on $|z| = \rho_1$.

The proof will be completed by showing that $\rho_1 \geq \rho$. Assume that $\rho_1 < \rho$. Then the set $D^* = \{z : \rho_1^{l+1} < |z| < \rho^{l+1}\}$ contains infinitely many points, and $\{\Delta^{\sigma}_{l,n,m}(z,f)\}_{n=1}^{\infty})$, being U.B. in $|z| < \rho^{l+1}$, is bounded at each point of D^*. On the other hand, $\sigma \geq \rho^{l+1} > \rho_1^{l+1}$. Thus, $\{\Delta^{\sigma}_{l,n,m}(z,f)\}_{n=1}^{\infty}$ can not be bounded at more than l points in the region $|z| > \rho_1^{l+1}$ (cf. [**2**, Remark 2.2]). This contradicts the boundedness of $\{\Delta_{l,n,m}(z,f)\}_{n=1}^{\infty}$ at each point of D^*. Therefore, $\rho_1 \geq \rho$.

Acknowledgement. The author wishes to thank Professor A. Sharma for his kind support and advice.

REFERENCES

1. A.S. Cavaretta Jr., A. Sharma and R.S. Varga, *Interpolation in the roots of unity: an extension of a Theorem of J.L. Walsh,* Resultate der Mathematik **3** (1981), 155-191.

2. M.A. Bokhari, *Equiconvergence of some sequences of complex interpolating rational functions (Quantitative estimates and sharpness)*, To appear in J.A.T. (1988).

3. E.B. Saff and A. Sharma, *On equiconvergence of certain sequence of rational Interpolants,* in *Rational Approximation and Interpolation,* Lecture Notes in Math., 1105, Springer, Berlin-New York, 1984, 256-271.

4. J. Szabados, *Converse results in the theory of overconvergence of complex interpolating polynomials,* Analysis, **2** (1982), 267-280.

5. J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain,* 5th ed. Vol XX AMS Colloq. Publication.