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CONVERGENCE AND GIBBS PHENOMENON OF PERIODIC WAVELET FRAME SERIES

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ABSTRACT. In this paper, we give integral representations of partial sums of the periodic wavelet frame series and then, based on it, we study convergence and the Gibbs phenomenon of the periodic wavelet frame series.

1. Introduction. It is well known that $\{e^{2\pi int}\}$ is an orthonormal basis for $L^2[0,1]$. The convergence of the Fourier series

$$\sum_n c_n e^{2\pi int}, \qquad c_n = \int_0^1 f(t)e^{-2\pi int}\,dt, \quad f \in L^2[0,1],$$

has been systematically studied [**4, 5**].

If $\{\psi_{m,n}\} = \{2^{m/2}\psi(2^m \cdot -n)\}_{m,n\in Z}$, is an orthonormal basis for $L^2(R)$, then $\{\psi_{m,n}\}$ is called a wavelet basis. From 1986 to present, many wavelet bases have been constructed. The convergence of the wavelet series

$$\sum_{m,n} c_{m,n}\psi_{m,n}(t), \qquad c_{m,n} = \int_R f(t)\overline{\psi}_{m,n}(t)\,dt, \quad f \in L^2(R),$$

was studied deeply [**6, 8, 10**]. Meyer [**7**] first constructed periodic wavelet bases. Skopina [**9**] discussed the convergence of the periodic wavelet series.

Wavelet frames are a generalization of wavelet bases. Recently, periodic wavelet frames were constructed [**11**]. In this paper, we will research convergence and the Gibbs phenomenon of the periodic wavelet frame series.

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A common way to research convergence of various series is to express the partial sums of these series in the following form

$$S_n(f;x)=\int f(t)K_n(t,x)\,dt,$$

where the kernel functions $K_n(t,x)$ are estimated easily and then, using these integral representations, discuss whether $S_n(f;x)$ converges to $f(x)$. In this paper, along this line, I am devoted to constructions of kernel functions of the periodic wavelet frame series. From our construction, one sees that it is very difficult to give appropriate integral representations for periodic wavelet frame series. On the other hand, we prove that the kernel function is a reproducing kernel of the space of band-limited functions. Based on this, we show that a kind of periodic wavelet frame series exhibits the Gibbs phenomenon.

2. Frames, wavelet frames, and periodic wavelet frames. As a generalization of orthonormal bases, Duffin and Schaeffer [**3**] introduced first the concept of frames.

Let $\{f_n\}$ be a sequence in Hilbert space H. If there exist $A,B>0$ such that

$$A\|f\|^2\le\sum_n |(f,\, f_n)|^2\le B\|f\|^2\quad\text{for any } f\in H,$$

then $\{f_n\}$ is called a frame for H [**1, 2**]. Hereafter, $\sum_l=\sum_{l\in Z}$ and $\{f_n\}=\{f_n\}_{n\in Z}$.

If $\{f_n\}$ is an orthonormal basis of H, then $\{f_n\}$ is a frame for H. Conversely, it is not true.

Let $\{f_n\}$ and $\{\widetilde{f}_n\}$ be two frames for H. If for any $f\in H$, the following reconstruction formula

$$f=\sum_n(f,\,\widetilde{f}_n)f_n=\sum_n(f,\,f_n)\widetilde{f}_n,$$

holds, then $\{f_n\}$ and $\{\widetilde{f}_n\}$ are called a pair of dual frames for H [**1, 2**], and the series

$$\sum_n(f,\widetilde{f}_n)f_n\quad\text{and}\quad\sum_n(f,f_n)\widetilde{f}_n$$

is called the frame series for $f\in H$.

Definition 2.1. (i) Let $\psi \in L^2(R)$ and $\psi_{m,n} = 2^{m/2}\psi(2^m \cdot -n)$, m, $n \in Z$. If $\{\psi_{m,n}\}$ is a frame for $L^2(R)$, then $\{\psi_{m,n}\}$ is called a wavelet frame for $L^2(R)$. If two wavelet frames are a pair of dual frames for $L^2(R)$, then they are called a pair of dual wavelet frames for $L^2(R)$ [**2**].

(ii) Let $\psi_{m,n}^{per} = \sum_l \psi_{m,n}(\cdot + l)$, $m = 0, 1, \dots ; n = 0, \dots , 2^m - 1$. If $\{1, \psi_{m,n}^{per}\, (m = 0, 1, \dots ; n = 0, \dots , 2^m - 1)\}$ is a frame for $L^2[0, 1]$, then we call it a periodic wavelet frame. If two periodic wavelet frames are a pair of dual frames for $L^2[0, 1]$, then we call them a pair of dual periodic wavelet frames.

Below we state the constructions of the pairs of dual periodic wavelet frames given by the paper [**11**].

Suppose that $\psi \in L^2(R)$ and the Fourier transform $\widehat{\psi}$ satisfies that $\widehat{\psi} \in C^2(R)$ and

$$\operatorname{supp} \widehat{\psi} \subset [-\pi, \pi] \setminus (-\eta, \eta), \quad 0 < \eta < \frac{\pi}{2}, \tag{2.1}$$

$$D(\omega) := \sum_m |\widehat{\psi}(2^m\omega)|^2 > 0, \omega \in R \setminus \{0\}. \tag{2.2}$$

Hereafter, $C^\lambda(R)$ consists of the functions whose derivatives of order λ are continuous on R.

Again suppose that $\widetilde{\psi} \in L^2(R)$ and the Fourier transform $\widehat{\widetilde{\psi}}$ satisfy that

$$\widehat{\widetilde{\psi}}(\omega) = \frac{\widehat{\psi}(\omega)}{D(\omega)}, \quad \omega \neq 0, \qquad \widehat{\widetilde{\psi}}(0) = 0. \tag{2.3}$$

Then we have that [**11**]

$$\widehat{\widetilde{\psi}} \in C^2(R) \quad \text{and} \quad \psi(x), \qquad \widetilde{\psi}(x) = O((1 + |x|)^{-2}). \tag{2.4}$$

Proposition 2.2 [**11**]. *Let ψ and $\widetilde{\psi}$ satisfy* (2.1)–(2.3). *Denote*

$$\psi_{m,n} := 2^{m/2}\psi(2^m \cdot -n), \quad \widetilde{\psi}_{m,n} := 2^{m/2}\widetilde{\psi}(2^m \cdot -n), \quad m,\, n \in Z.$$

Then $\{\psi_{m,n}\}$ and $\{\widetilde{\psi}_{m,n}\}$ are a pair of dual wavelet frames for $L^2(R)$.

Proposition 2.3 [**11**]. *Let* ψ *and* $\widetilde{\psi}$ *satisfy* (2.1)–(2.3). *Denote*

$$(2.5)\quad \psi_{m,n}^{per} := \sum_l \psi_{m,n}(\cdot + l), \qquad \widetilde{\psi}_{m,n}^{per} := \sum_l \widetilde{\psi}_{m,n}(\cdot + l), \quad m,\, n \in Z$$

and

$$(2.6)\quad \begin{gathered} g_0 = 1, \quad \widetilde{g}_0 = 1, \quad g_{2^m+n} = \psi_{m,n}^{per}, \quad \widetilde{g}_{2^m+n} = \widetilde{\psi}_{m,n}^{per} \\ m = 0, 1, \dots ;\ n = 0, 1, \dots, 2^m - 1. \end{gathered}$$

Then $\{g_k\}_0^\infty$ and $\{\widetilde{g}_k\}_0^\infty$ are a pair of dual periodic wavelet frames where both g_k and $\widetilde{g}_k$ are trigonometric polynomials.

Let $f \in L[0,1]$ and $\{g_k\}$, $\{\widetilde{g}_k\}$ be stated as in (2.6). In this paper, we research the convergence and Gibbs phenomenon of the periodic wavelet frame series

$$(2.7)\qquad \sum_{k=0}^{\infty} c_k g_k(x), \quad \text{where } c_k = \int_0^1 f(t)\overline{\widetilde{g}}_k(t)\, dt.$$

Similarly, we can consider the other periodic wavelet frame series $\sum_{k=0}^{\infty} d_k \widetilde{g}_k(x)$, where $d_k = \int_0^1 f(t)\overline{g}_k(t)\, dt$.

3. Integral representations of partial sums. Now we give integral representations of the partial sums of the periodic wavelet frame series (2.7). Integral representations play a central role in the research of convergence and the Gibbs phenomenon.

Let $f \in L[0,1]$. Denote the partial sums of the series (2.7) of f:

$$S_\nu(f;x) = \sum_{k=0}^{\nu} c_k g_k(x), \quad \text{where } c_k = \int_0^1 f(t)\overline{\widetilde{g}}_k(t)\, dt.$$

Then, for $\nu = 2^M + j$, $M \ge 0$; $0 \le j \le 2^M - 1$, we have

(3.1)

$$\begin{aligned} S_\nu(f;x) &= c_0 g_0(x) + \sum_{m=0}^{M-1} \sum_{n=0}^{2^m-1} c_{2^m+n} g_{2^m+n}(x) + \sum_{n=0}^{j} c_{2^M+n} g_{2^M+n}(x) \\ &=: A_1 + A_2 + A_3, \quad \nu \ge 1. \end{aligned}$$

Define f^* as a 1-periodic function satisfying $f^*(x) = f(x)$, $x \in [0, 1)$. So, for any $l \in Z$,

$$f^*(t) = f(t - l), \quad l \le t \le l + 1.$$

Below we give integral representations of A_1, A_2 and A_3, respectively.

Lemma 3.1. *For $x \in R$, $M \ge 0$, and $j = 0, 1, \dots, 2^M - 1$, we have*

(i) $A_2 = \sum_{m=0}^{M-1} \int_R f^*(t) \sum_n \overline{\widetilde{\psi}}_{m,n}(t)\, \psi_{m,n}(x)\, dt$, *and*

(ii) $A_3 = \int_R f^*(t) \sum_{n \in \sigma_M(j)} \overline{\widetilde{\psi}}_{M,n}(t)\, \psi_{M,n}(x)\, dt$, *where* $\sigma_M(j) = \{2^M l + k : k = 0, 1, \dots, j;\ l \in Z\}$.

Proof. Since $\psi(x)$, $\widetilde{\psi}(x) = O((1 + |x|)^{-2})$, by (2.5)–(2.7), we have

$$\begin{aligned}
&\sum_{n=0}^{2^m-1} c_{2^m+n} g_{2^m+n}(x) \\
&= 2^m \sum_{n=0}^{2^m-1} \int_0^1 f^*(t) \Big(\sum_l \overline{\widetilde{\psi}}(2^m(t + l) - n) \Big) \Big(\sum_{l'} \psi(2^m(x + l') - n) \Big) dt \\
&= 2^m \sum_{n=0}^{2^m-1} \sum_{l'} \int_0^1 f^*(t) \Big(\sum_l \overline{\widetilde{\psi}}(2^m(t + l' + l) - n) \psi(2^m(x + l') - n) \Big) dt \\
&= 2^m \sum_{n=0}^{2^m-1} \sum_{l'} \int_R f^*(t) \overline{\widetilde{\psi}}(2^m(t + l') - n) \psi(2^m(x + l') - n)\, dt \\
&= 2^m \int_R f^*(t) \Big(\sum_n \overline{\widetilde{\psi}}(2^m t - n) \psi(2^m x - n) \Big) dt.
\end{aligned}$$

From this and (3.1), we get (i). Similarly, we can obtain (ii). Lemma 3.1 is proved. □

In order to give an appropriate integral representation of A_1, we need to define two functions of h and $\widetilde{h}$.

Definition 3.2. Define $h \in L^2(R)$ to be such that

$$\widehat{h}(\omega) = \frac{1}{D(\omega)} \sum_{m>0} |\widehat{\psi}(2^m \omega)|^2, \quad \omega \ne 0, \quad \text{and } \widehat{h}(0) = 1,$$

where $D(\omega)$ is stated in (2.2). Define $\widetilde{h} \in L^2(R)$ as

$$\widehat{\widetilde{h}}(\omega) = \frac{1}{c}\int_R \alpha(\omega - u)\mathcal{X}_{[-(3\pi/4),(3\pi/4)]}(u)\,du,$$

where $\mathcal{X}_{[a,b]}$ is the characteristic function of $[a,b]$ and

$$\alpha(t) = \begin{cases} e^{-\pi^2/(\pi^2-16t^2)} & |t| < \pi/4 \\ 0 & |t| \geq \pi/4, \end{cases} \qquad c = \int_R \alpha(\omega)\,d\omega.$$

Lemma 3.3. *The functions h and $\widetilde{h}$ satisfy the following*:

(i) $\operatorname{supp}\widehat{h} \subset [-(\pi/2),(\pi/2)]$, $\widehat{h}(\omega) = 1$, $\omega \in (-\eta,\eta)$, $\widehat{h} \in C^2(R)$.

(ii) $\operatorname{supp}\widehat{\widetilde{h}} \subset [-\pi,\pi]$, $\widehat{\widetilde{h}}(\omega) = 1$, $\omega \in [-(\pi/2),(\pi/2)]$, $\widehat{\widetilde{h}} \in C^\infty(R)$.

(iii) $h(t) = O((1+|t|)^{-2})$, $\widetilde{h}(t) = O((1+|t|)^{-s})$ for any $s > 0$.

Here η is stated in (2.1).

Proof. By (2.1) and (2.2), we have $\sum_{m\leq 0} |\widehat{\psi}(2^m\omega)|^2 = 0$, $\omega \in (-\eta,\eta)$ and

$$\sum_{m>0} |\widehat{\psi}(2^m\omega)|^2 = \sum_m |\widehat{\psi}(2^m\omega)|^2 = D(\omega), \omega \in (-\eta,\eta)\setminus\{0\}.$$

From this and Definition 3.2, we get $\widehat{h}(\omega) = 1$, $\omega \in (-\eta,\eta)$. Since $\operatorname{supp}\widehat{\psi} \subset [-\pi,\pi]$, we have $\operatorname{supp}\widehat{h} \subset [-(\pi/2),(\pi/2)]$, and for $0 < \eta/2 \leq |\omega| \leq r$, the following formulas hold.

$$\sum_{m>0} |\widehat{\psi}(2^m\omega)|^2 = \sum_{0<m\leq \log_2(2\pi/\eta)} |\widehat{\psi}(2^m\omega)|^2$$

and

$$D(\omega) = \sum_{\log_2(\eta/r)\leq m\leq \log_2(2\pi/\eta)} |\widehat{\psi}(2^m\omega)|^2.$$

Noticing that r can be an arbitrarily large number, by $\widehat{\psi} \in C^2(R)$, it follows from (2.2) and Definition 3.2 that $\widehat{h} \in C^2\left(R\setminus[-(\eta/2),(\eta/2)]\right)$. Noticing that $\widehat{h}(\omega) = 1$, $\omega \in (-\eta,\eta)$, we have $\widehat{h} \in C^2(R)$. (i) follows.

By the definition of $\widetilde{h}$, we get (ii). By $\widehat{h} \in C^2(R)$ and $\widehat{\widetilde{h}} \in C^\infty(R)$, we get (iii). Lemma 3.3 is proved. $\square$

Lemma 3.4. $A_1 = \int_R f^*(t) \sum_n \overline{\widetilde{h}}(t-n) h(x-n)\, dt$.

Proof. By Poisson's summation formula [**7**], we have

$$\widetilde{h}^{per}(x) = \sum_l \widetilde{h}(x+l) = \sum_k \widehat{\widetilde{h}}(2k\pi) e^{2k\pi i x}.$$

But by Lemma 3.3 (ii), we have $\widehat{\widetilde{h}}(2k\pi) = 0$, $k \neq 0$, and $\widehat{\widetilde{h}}(0) = 1$. So $\widetilde{h}^{per}(x) = 1$. Similarly, by Lemma 3.3 (i), we have $h^{per}(x) = 1$. From this and $g_0(x) = \widetilde{g}_0(x) = 1$, we have

$$A_1 = c_0 g_0(x) = c_0 h^{per}(x), \quad c_0 = \int_0^1 f(t) \overline{\widetilde{h}^{per}}(t)\, dt.$$

Since $h(x)$, $\widetilde{h}(x) = O((1+|x|)^{-2})$, similar to the argument of Lemma 3.1, we have

$$\begin{aligned} A_1 &= \int_0^1 f^*(t) \bigg(\sum_l \overline{\widetilde{h}}(t+l) \bigg) \bigg(\sum_{l'} h(x+l') \bigg) dt \\ &= \int_R f^*(t) \sum_n \overline{\widetilde{h}}(t-n) h(x-n)\, dt. \end{aligned}$$

Lemma 3.4 is proved. $\square$

Denote $h_{m,n} := 2^{m/2} h(2^m \cdot -n)$, $\widetilde{h}_{m,n} := 2^{m/2} \widetilde{h}(2^m \cdot -n)$, $m, n \in Z$. Define kernel functions as

$$K_M(t,x) = \sum_n \overline{\widetilde{h}}_{M,n}(t) h_{M,n}(x), \quad M \geq 0;\ t,\ x \in R, \tag{3.2}$$

$$\begin{gathered} K_{M,j}(t,x) = K_M(t,x) + \sum_{n \in \sigma_M(j)} \overline{\widetilde{\psi}}_{M,n}(t) \psi_{M,n}(x) \\ M \geq 0;\ j = 0, 1, \dots, 2^M - 1;\ t,\ x \in R \end{gathered} \tag{3.3}$$

where $\sigma_M(j) = \{2^M l + k : k = 0, 1, \dots, j;\ l \in Z\}$.

With the help of the above kernel functions, we will give simple and clear integral representations of partial sums of the periodic wavelet frame series (2.7).

Theorem 3.5. *Let $f \in L[0,1]$, and let f^* be a 1-periodic function and $f^*(x) = f(x)$, $x \in [0,1]$. Then, for $\nu = 2^M + j$, $M \geq 0$; $j = 0, \dots, 2^M - 1$,*

$$S_\nu(f;x) = \int_R f^*(t) K_{M,j}(t,x)\,dt \quad \text{and} \quad \int_R K_{M,j}(t,x)\,dt = 1, \quad x \in R.$$

Proof. By (3.1), $S_\nu(f;x) = A_1 + A_2 + A_3$, $\nu = 2^M + j$. From Lemmas 3.1 and 3.4, it follows that

$$A_1 + A_2 = \int_R f^*(t) L_M(t,x)\,dt \tag{3.4}$$

where

$$L_M(t,x) = \sum_n \overline{\widetilde{h}}_{0,n}(t)\, h_{0,n}(x) + \sum_{m=0}^{M-1} \sum_n \overline{\widetilde{\psi}}_{m,n}(t) \psi_{m,n}(x). \tag{3.5}$$

Below we prove a

Claim. $L_M(t,x) = K_M(t,x)$ $(M \geq 0)$ *almost everywhere.*

For this purpose, we only need to prove that for any $\gamma \in L^2(R)$ and $x \in R$,

$$(\gamma, \overline{L}_M(\cdot, x)) = (\gamma, \overline{K}_M(\cdot, x)). \tag{3.6}$$

Hereafter, $(\cdot,\cdot)$ is the inner product of the space $L^2(R)$.

Using the Parseval equality of the Fourier transform, we have

$$\overline{(\gamma, \widetilde{\psi}_{m,n})} = \frac{1}{2\pi} \overline{(\widehat{\gamma}, \widehat{\widetilde{\psi}}_{m,n})} = \frac{1}{2\pi} \int_R 2^{-m/2} \overline{\widehat{\gamma}}(\omega) \widehat{\widetilde{\psi}}(2^{-m}\omega) e^{-i2^{-m}n\omega}\,d\omega.$$

Using the inversion formula of the Fourier transform, we have

$$\psi_{m,n}(x) = \frac{1}{2\pi} \int_R 2^{-m/2} e^{ix\omega} \widehat{\psi}(2^{-m}\omega) e^{-i2^{-m}n\omega}\,d\omega.$$

Since $\operatorname{supp} \widehat{\psi}$, $\operatorname{supp} \widehat{\widetilde{\psi}} \subset [-\pi, \pi]$, the above formulas can be rewritten in the forms

$$\overline{(\gamma, \widetilde{\psi}_{m,n})} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_1(\omega) e^{-in\omega} \, d\omega$$

and

$$\psi_{m,n}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_2(\omega) e^{-in\omega} \, d\omega,$$

where

$$\Phi_1(\omega) = 2^{m/2} \overline{\widehat{\gamma}}(2^m \omega) \widehat{\widetilde{\psi}}(\omega) \quad \text{and} \quad \Phi_2(\omega) = 2^{m/2} e^{i2^m x\omega} \widehat{\psi}(\omega).$$

Again, by the Parseval equality of the Fourier series, we obtain that for $x \in R$,

$$\begin{aligned} \sum_n (\gamma, \, \widetilde{\psi}_{m,n}) \psi_{m,n}(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\Phi}_1(\omega) \Phi_2(\omega) \, d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} 2^m e^{i2^m x\omega} \widehat{\gamma}(2^m \omega) \overline{\widehat{\widetilde{\psi}}}(\omega) \widehat{\psi}(\omega) \, d\omega \\ &= \frac{1}{2\pi} \int_R e^{ix\omega} \widehat{\gamma}(\omega) \widehat{\psi}(2^{-m}\omega) \overline{\widehat{\widetilde{\psi}}}(2^{-m}\omega) \, d\omega. \end{aligned} \tag{3.7}$$

Take $n_2 > n_1 > 1$. Since $\|\psi_{m,n}\|_{L^2(R)} = \|\psi\|_{L^2(R)}$, we have

$$\left\| \sum_{n=n_1}^{n_2} \overline{\psi}_{m,n}(x) \widetilde{\psi}_{m,n} \right\|_{L^2(R)} \le \|\psi\|_{L^2(R)} \left(\sum_{n=n_1}^{n_2} |\overline{\psi}_{m,n}(x)| \right).$$

Again, by (2.4), we see that for $x \in R$, the series $\sum_n \overline{\psi}_{m,n}(x) \widetilde{\psi}_{m,n}$ is convergent in the norm $L^2(R)$. From this and (3.7),

$$\begin{aligned} \left(\gamma, \sum_n \overline{\psi}_{m,n}(x) \widetilde{\psi}_{m,n}\right) &= \sum_n (\gamma, \widetilde{\psi}_{m,n}) \psi_{m,n}(x) \\ &= \frac{1}{2\pi} \int_R e^{ix\omega} \widehat{\gamma}(\omega) (\widehat{\psi}(2^{-m}\omega) \overline{\widehat{\widetilde{\psi}}}(2^{-m}\omega)) \, d\omega. \end{aligned} \tag{3.8}$$

Similar to the argument of (3.8), we obtain by Lemma 3.3 (iii) that

$$\left(\gamma, \sum_n \overline{h}_{m,n}(x)\widetilde{h}_{m,n}\right) = \frac{1}{2\pi}\int_R e^{ix\omega}\widehat{\gamma}(\omega)\widehat{h}(2^{-m}\omega)\overline{\widehat{\widetilde{h}}}(2^{-m}\omega)\,d\omega. \tag{3.9}$$

By Lemma 3.3 (i)–(ii), we know that $\widehat{\widetilde{h}}(\omega) = 1$, $\omega \in [-(\pi/2), (\pi/2)]$ and $\operatorname{supp}\widehat{h} \subset [-(\pi/2), (\pi/2)]$, so we have

$$\widehat{h}(\omega)\overline{\widehat{\widetilde{h}}}(\omega) = \widehat{h}(\omega), \quad \omega \in R. \tag{3.10}$$

By (2.3) and (2.2), we have

$$\widehat{\psi}(\omega)\overline{\widehat{\widetilde{\psi}}}(\omega) = \frac{|\widehat{\psi}(\omega)|^2}{\overline{D}(\omega)} = \frac{|\widehat{\psi}(\omega)|^2}{D(\omega)}, \quad \omega \neq 0.$$

Again, noticing that $D(2^m\omega) = D(\omega)$ (by (2.2)), it follows from Definition 3.2 and (2.3) that

$$\widehat{h}(\omega) = \sum_{m>0}\frac{|\widehat{\psi}(2^m\omega)|^2}{D(2^m\omega)} = \sum_{m>0}\widehat{\psi}(2^m\omega)\overline{\widehat{\widetilde{\psi}}}(2^m\omega), \quad \omega \neq 0.$$

By (3.10), we have

$$\widehat{h}(\omega)\overline{\widehat{\widetilde{h}}}(\omega) = \sum_{m<0}\widehat{\psi}(2^{-m}\omega)\overline{\widehat{\widetilde{\psi}}}(2^{-m}\omega), \quad \omega \neq 0. \tag{3.11}$$

From this and (3.9), we get

$$\left(\gamma, \sum_n \overline{h}_{0,n}(x)\widetilde{h}_{0,n}\right) = \frac{1}{2\pi}\int_R e^{ix\omega}\widehat{\gamma}(\omega)\left(\sum_{m<0}\widehat{\psi}(2^{-m}\omega)\overline{\widehat{\widetilde{\psi}}}(2^{-m}\omega)\right)d\omega. \tag{3.12}$$

By (3.8), (3.12), and (3.5), we have

$$(\gamma, \overline{L}_M(\cdot, x)) = \frac{1}{2\pi}\int_R e^{ix\omega}\widehat{\gamma}(\omega)\left(\sum_{m<M}\widehat{\psi}(2^{-m}\omega)\overline{\widehat{\widetilde{\psi}}}(2^{-m}\omega)\right)d\omega.$$

Again, by (3.11), we get

$$(\gamma, \overline{L}_M(\cdot, x)) = \frac{1}{2\pi}\int_R e^{ix\omega}\widehat{\gamma}(\omega)\widehat{h}(2^{-M}\omega)\overline{\widehat{\widetilde{h}}}(2^{-M}\omega)\, d\omega.$$

From this, (3.9) and (3.2), we have

$$(\gamma, \overline{K}_M(\cdot, x)) = \Big(\gamma, \sum_n \overline{h}_{M,n}(x)\widetilde{h}_{M,n}\Big) = (\gamma, \overline{L}_M(\cdot, x)),$$

i.e., (3.6) holds. The claim follows, i.e.,

$$K_M(t, x) = \sum_n \overline{\widetilde{h}}_{0,n}(t)h_{0,n}(x) + \sum_{m=0}^{M-1}\sum_n \overline{\widetilde{\psi}}_{m,n}(t)\psi_{m,n}(x). \tag{3.13}$$

From this claim and (3.4), we get

$$A_1 + A_2 = \int_R f^*(t)K_M(t, x)\, dt, \quad M \geq 0.$$

By Lemma 3.1 (ii):

$$A_3 = \int_R f^*(t)\Big(\sum_{n\in\sigma_M(j)} \overline{\widetilde{\psi}}_{M,n}(t)\psi_{M,n}(x)\Big)\, dt.$$

Finally by (3.1) and (3.3), we get

$$S_\nu(f; x) = \int_R f^*(t)K_{M,j}(t, x)\, dt, \quad \nu = 2^M + j. \tag{3.14}$$

Now let $f(t) = 1$, $0 \leq t \leq 1$. By Lemma 3.1 and (2.3), we get

$$\begin{aligned} A_2 &= \sum_{m=0}^{M-1}\sum_n \Big(\int_R \overline{\widetilde{\psi}}_{m,n}(t)\, dt\Big)\,\psi_{m,n}(x) \\ &= 2^{-m/2}\sum_{m=0}^{M-1}\sum_n \overline{\widehat{\widetilde{\psi}}}(0)\psi_{m,n}(x) = 0. \end{aligned}$$

Similarly, we have $A_3 = 0$. By (3.1) and $g_0 = \widetilde{g}_0 = 1$, we get

$$S_\nu(1;x) = A_1 = c_0 g_0(x) = \left(\int_0^1 \overline{\widetilde{g}}_0(t)\, dt \right) g_0(x) = 1.$$

From this and (3.14), we have $\int_R K_{M,j}(t,x)\, dt = 1$, $x \in R$. Theorem 3.5 is proved. $\square$

We give the following corollary which is used in the discussion of the Gibbs phenomenon.

Corollary 3.6. *Under the conditions of Theorem* 3.5*, for* $M \geq 0$*,*

$$S_{2^M-1}(f;x) = \int_R f^*(t) K_M(t,x)\, dt \quad x \in R \tag{3.15}$$

and

$$\int_R K_M(t,x)\, dt = 1, \quad x \in R. \tag{3.16}$$

Proof. By Theorem 3.5, we know that, for $M \geq 0$,

$$S_{2^{M+1}-1}(f;x) = S_{2^M+(2^M-1)}(f;x) = \int_R f^*(t) K_{M,2^M-1}(t,x)\, dt. \tag{3.17}$$

By (3.3) and $\sigma_M(2^M-1) = Z$, we have $K_{M,2^M-1}(t,x) = K_M(t,x) + \sum_n \overline{\widetilde{\psi}}_{M,n}(t)\psi_{M,n}(x)$. Again, by (3.13), we obtain that

$$\begin{aligned} K_{M,2^M-1}(t,x) &= \sum_n \overline{\widetilde{h}}_{0,n}(t) h_{0,n}(x) + \sum_{m=0}^{M} \sum_n \overline{\widetilde{\psi}}_{m,n}(t)\psi_{m,n}(x) \\ &= K_{M+1}(t,x). \end{aligned}$$

From this and (3.17), we obtain that for $M \geq 1$, (3.15) holds. By (3.1) and Lemma 3.4,

$$S_0(f;x) = c_0 g_0(x) = A_1 = \int_R f^*(t) \sum_n \overline{\widetilde{h}}(t-n) h(x-n)\, dt.$$

Again, by (3.2), we know that for $M = 0$, (3.15) holds. Taking $f = 1$ in (3.15), we get (3.16).

4. Convergence and Gibbs phenomenon. Based on integral representations of the kernel functions, we will discuss convergence and the Gibbs phenomenon of periodic wavelet frame series.

4.1 Convergence of the periodic wavelet frame series. First, we give estimates of the kernel functions.

Lemma 4.1. *Let the kernel functions $K_M(t,x)$ and $K_{M,j}(t,x)$ be stated in* (3.2) *and* (3.3), *respectively. Then, for $M \geq 0$, $j = 0, \ldots, 2^M - 1$, we have*

(i) $K_M(t,x) = O(2^M)(1+2^M|t-x|)^{-2}$ *and*

(ii) $K_{M,j}(t,x) = O(2^M)(1+2^M|t-x|)^{-2}$.

Proof. We first prove that

$$P_M(t,x) := \sum_{n\in\sigma_M(j)} \overline{\widetilde{\psi}}_{M,n}(t)\psi_{M,n}(x) = O(2^M)(1+2^M|t-x|)^{-2}, \tag{4.1}$$

where $\sigma_M(j)$ is stated in Lemma 3.1.

From $|P_M(t,x)| \leq \sum_n |\widetilde{\psi}_{M,n}(t)\psi_{M,n}(x)| =: Q_M(t,x)$ and $\psi(x)$, $\widetilde{\psi}(x) = O((1+|x|)^{-2})$, we have

$$|Q_0(t,x)| \leq \sum_n |\widetilde{\psi}(t-n)\psi(x-n)| = O(1)\sum_n (1+|t-n|)^{-2}(1+|x-n|)^{-2}. \tag{4.2}$$

For $|t+x| \leq 1$ and $n \in Z$, by a known inequality [**8**, page 79],

$$(1+|t-n|)(1+|x-n|) \geq \frac{1}{4}(1+|t-x|)(1+||t-x|-2|n||)$$

and (4.2), we know that, for $|t+x| \leq 1$,

$$|Q_0(t,x)| = O((1+|t-x|)^{-2})\sum_n (1+||t-x|-2|n||)^{-2} = O((1+|t-x|)^{-2}).$$

Since $Q_0(t+l, x+l) = Q_0(t,x)$, t, $x \in R$, $l \in Z$, we obtain that for t, $x \in R$,

$$Q_0(t,x) = O((1+|t-x|)^{-2}).$$

Again, by $Q_M(t,x) = 2^M Q_0(2^M t, 2^M x)$, we get (4.1).

Since $K_M(t,x) = \sum_n \widetilde{\widetilde{h}}_{M,n}(t) h_{M,n}(x)$ and $h(x), \widetilde{h}(x) = O((1+|x|)^{-2})$, similar to the argument of (4.1), we get (i). Combining (i) with (4.1), by (3.3), we get (ii). Lemma 4.1 is proved. $\square$

Now we discuss the uniform convergence and convergence in the norm of $L^p[0,1]$ as well as the almost everywhere convergence of the periodic wavelet frame series (2.7).

Theorem 4.2. *Let $f \in L[0,1]$, and let f^* be a 1-periodic function and $f^*(x) = f(x)$, $x \in [0,1)$.*

(i) *If $f^* \in C(a,b)$, then the series (2.7) converges to f^* uniformly on every closed interval in (a,b).*

(ii) *If $f \in L^p[0,1]$, $1 \le p < \infty$, then the series (2.7) converges to f in the norm of $L^p[0,1]$.*

(iii) *If $f \in L^p[0,1]$, $1 \le p < \infty$, then the series (2.7) converges to f^* almost everywhere on R.*

Proof. (i) By the assumption, f^* is continuous in (a,b). Take $[a_1,b_1] \subset (a,b)$. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f^*(t) - f^*(x)| < \varepsilon$, $x \in [a_1,b_1]$, $|t-x| \le \delta$. By Theorem 3.5, we obtain that for $x \in [a_1,b_1]$, $\nu = 2^M + j$,

$$\begin{aligned} S_\nu(f;x) &= f^*(x) \int_{x-\delta}^{x+\delta} K_{M,j}(t,x)\,dt \\ &\quad + \int_{x-\delta}^{x+\delta} K_{M,j}(t,x)(f^*(t) - f^*(x))\,dt \\ &\quad + \int_{|t-x|\ge\delta} f^*(t) K_{M,j}(t,x)\,dt =: I_1 + I_2 + I_3. \end{aligned} \tag{4.3}$$

For $x \in [a_1, b_1]$, we have

$$|I_2| \le \int_{x-\delta}^{x+\delta} |f^*(t) - f^*(x)||K_{M,j}(t,x)|\,dt \le \varepsilon \int_R |K_{M,j}(t,x)|\,dt.$$

By Lemma 4.1,

$$I_2 = O(2^M\varepsilon) \int_R (1 + 2^M|t-x|)^{-2}\,dt = O(\varepsilon) \int_R (1+|t|)^{-2}\,dt = O(\varepsilon).$$

Hereafter, the bounds in the terms "O" are independent of x, M, j.

By Lemma 4.1 and Theorem 3.5, for $x \in [a_1, b_1]$, we have

$$\begin{aligned} I_1 &= f^*(x) \int_R K_{M,j}(t,x)\,dt - f^*(x) \int_{|t-x|\ge\delta} K_{M,j}(t,x)\,dt \\ &= f^*(x) + O\Big(2^M \max_{x\in[a_1,b_1]} |f^*(x)|\Big) \int_{|t-x|\ge\delta} (1 + 2^M|t-x|)^{-2}\,dt, \end{aligned}$$

where

$$\begin{aligned} 2^M \int_{|t-x|\ge\delta} (1+2^M|t-x|)^{-2}\,dt &= O(1) \int_{|t|\ge 2^M\delta} (1+|t|)^{-2}\,dt \\ &= o(1),\ M \to \infty. \end{aligned}$$

Hence, $I_1 = f^*(x) + o(1)$, $M \to \infty$, uniformly for $x \in [a_1, b_1]$ and $j = 0, \dots, 2^M - 1$.

By the hypothesis $f \in L[0,1]$, for any $x \in R$, we have

(4.4)

$$\begin{aligned} I_3 &= O(2^M) \int\limits_{|t|\ge\delta} |f^*(x+t)|(1+2^M|t|)^{-2}\,dt \\ &= O\left(2^M\right) \left(\int_{\delta\le|t|\le1} + \sum_{l=1}^{\infty} \int_{l\le|t|\le l+1} \right) |f^*(x+t)|(1+2^M|t|)^{-2}\,dt \\ &= O\left(2^M\right) \left(\int_0^1 |f^*(t)|\,dt \right) \left(\frac{1}{(2^M\delta)^2} + \frac{1}{2^{2M}} \sum_{l=1}^{\infty} \frac{1}{l^2} \right) = o(1),\ M \to \infty. \end{aligned}$$

Summarizing the above results, by (4.3), we conclude that the series (2.7) converges uniformly to f^* on $[a_1, b_1]$, so we get (i).

(ii) By Theorem 3.5 and Lemma 4.1, we get

$$S_\nu(f;x) - f(x) = O(1)\int_R |f^*(x+2^{-M}t) - f^*(x)|(1+|t|)^{-2}\,dt.$$

Using Minkowski's inequality, we get

(4.5)
$$\|S_\nu(f) - f\|_{L^p[0,1]} = O(1)\int_R \|f^*(\cdot+2^{-M}t) - f^*\|_{L^p[0,1]}(1+|t|)^{-2}\,dt.$$

Since $f \in L^p[0,1]$ and f^* is 1-periodic and $f^*(t) = f(t)$, $t \in [0,1)$, we have

$$\lim_{M\to\infty} \|f^*(\cdot+2^{-M}t) - f^*\|_{L^p[0,1]} = 0$$

and

$$\|f^*(\cdot+2^{-M}t) - f^*\|_{L^p[0,1]} \le 2\|f\|_{L^p[0,1]}.$$

From this and (4.5), we get $\|S_\nu(f) - f\|_{L^p[0,1]} = o(1)$, $\nu \to \infty$. (ii) follows.

(iii) Let x be a Lebesgue point of f^*, i.e., $\lim_{\delta\to 0}(1/\delta)\int_{-\delta}^{\delta}|f^*(x+t) - f^*(x)|\,dt = 0$. Denote

$$q(t) = |f^*(x+t) - f^*(x)| \quad \text{and} \quad Q(t) = \int_0^t q(s)\,ds.$$

Then for $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|Q(t)| \le \varepsilon|t|, \quad 0 \le |t| \le \delta. \tag{4.6}$$

By Theorem 3.5 and Lemma 4.1, we obtain that, for $\nu = 2^M + j$,

$$\begin{aligned}|S_\nu(f;x) - f(x)| &\le \int_R |(f^*(t) - f^*(x))K_{M,j}(t,x)|\,dt \\ &= O(2^M)\left(\int_{|t|\le\delta} + \int_{|t|\ge\delta}\right) \\ &\quad |f^*(x+t) - f^*(x)|(1+2^M|t|)^{-2}\,dt \\ &=: J_1 + J_2.\end{aligned}$$

Using integration by parts, we obtain by (4.6) that, for $M > \log_2 1/\delta$,

$$J_1 = O(\varepsilon) + O(2^{2M})\int_{|t|\le\delta} Q(t)(1+2^M t)^{-3}\,dt = O(\varepsilon).$$

For J_2, we have

$$J_2 = O(2^M)\int_{|t|\geq\delta} |f^*(x+t)|(1+2^M|t|)^{-2}\,dt$$
$$+ O(2^M)|f^*(x)|\int_{|t|\geq\delta}(1+2^M|t|)^{-2}\,dt = J_{21}+J_{22}.$$

Similar to the argument of (4.4), we have $J_{21} = o(1)$. On the other hand,

$$J_{22} = O(1)\int_{|t|\geq 2^M\delta}(1+|t|)^{-2}\,dt = o(1), \quad M\to\infty.$$

So $J_2 = o(1)$. Hence, series (2.7) converges to f at x. Since $f\in L^p[0,1]$, $1\leq p<\infty$, almost all points are the Lebesgue points of f^*, so we get (iii). Theorem 4.2 is proved. □

Corollary 4.3. *If $f\in C[0,1]$ and $f(0)=f(1)$, then series* (2.7) *converges uniformly to f on* $[0,1]$.

4.2. Gibbs phenomenon of periodic wavelet frame series. Now we discuss the Gibbs phenomenon of periodic wavelet frame series (2.7). For convenience, we assume that ψ is a real-valued even function on R. From Definition 3.2, we know that h, $\widehat{h}$ and $\widetilde{h}$, $\widehat{\widetilde{h}}$ are all real-valued even functions, so the corresponding kernel function

$$K_0(t,x) = \sum_n \widetilde{h}(t-n)h(x-n) \tag{4.7}$$

is a real-valued function.

Theorem 4.4. *Let a real-valued function $f\in L[0,1]$ and f^* be a 1-periodic function on R and $f^*(t)=f(t)$, $t\in[0,1]$. Let f^* be continuous in $0<|x|<\varepsilon$ and $f^*(0+)-f^*(0-)=2d>0$.*

(i) *If, for some $a>0$, $\int_{-\infty}^0 K_0(t,a)\,dt<0$, then*

$$\limsup_{\nu\to\infty} S_\nu(f;(a/\nu+1)) > f^*(0+).$$

(ii) *If, for some* $a < 0$, $\int_0^\infty K_0(t,a)\,dt < 0$, *then*

$$\liminf_{\nu\to\infty} S_\nu(f^*;(a/\nu+1)) < f^*(0-).$$

Namely, in the two cases, the periodic wavelet frame series (2.7) exhibits the Gibbs phenomenon at the origin.

Proof. Let τ^* be a 1-periodic function on R and $\tau^*(x) = \operatorname{sgn} x\,(|x| \le (1/2))$,

$$F(x) = f^*(x) - d\tau^*(x) \quad x \ne 0, \tag{4.8}$$

and $F(0) = 1/2(f^*(0+) + f^*(0-))$. Then F is continuous in $(-\varepsilon,\varepsilon)$. Hence, by Theorem 4.2 (i), we know that $\lim_{\nu\to\infty} S_\nu(F;x) = F(x)$ uniformly on $[-(\varepsilon/2),(\varepsilon/2)]$. Specially, for any $a \in R$,

$$\lim_{M\to\infty} S_{2^M-1}(F;2^{-M}a) = F(0). \tag{4.9}$$

By Corollary 3.6, we have

$$\begin{aligned} S_{2^M-1}(\tau^*;2^{-M}a) &= \left(\int_{|t|\le 1/2} + \int_{|t|\ge 1/2}\right)\tau^*(t)K_M(t,2^{-M}a)\,dt \\ &=: T_1 + T_2. \end{aligned}$$

Since $|\tau^*(t)| = 1$ $(t \ne 0)$, by Lemma 4.1,

$$T_2 = O(1)\int_{|t|\ge 2^{M-1}} (1+|t-a|)^{-2}\,dt = o(1).$$

By (3.2),

$$K_M(t,2^{-M}a) = 2^M \sum_n \widetilde{h}(2^M t - n)h(a-n) = 2^M K_0(2^M t, a).$$

So we have

$$T_1 = \int_0^{2^{M-1}} K_0(t,a)\,dt - \int_{-2^{M-1}}^0 K_0(t,a)\,dt.$$

By Corollary 3.6, we get $\int_R K_0(t,a)\,dt = 1$. So we have

$$T_1 = \int_R K_0(t,a)\,dt - 2\int_{-\infty}^{0} K_0(t,a)\,dt + o(1)$$
$$= 1 - 2\int_{-\infty}^{0} K_0(t,a)\,dt + o(1).$$

Now if, for some $a > 0$, $\int_{-\infty}^{0} K_0(t,a)\,dt = -\delta < 0$, then we have $T_1 = 1 + 2\delta + o(1)$. So we get

$$S_{2^M-1}(\tau^*; 2^{-M}a) = 1 + 2\delta + o(1).$$

Again by (4.8) and (4.9),

$$\begin{aligned} S_{2^M-1}(f; 2^{-M}a) &= S_{2^M-1}(F; 2^{-M}a) + dS_{2^M-1}(\tau^*; 2^{-M}a) \\ &= F(0) + d(1+2\delta) + o(1). \end{aligned}$$

Hence $\lim_{M\to\infty} S_{2^M-1}(f; 2^{-M}a) > F(0) + d = f^*(0+)$, (i) follows. Similarly, we get (ii). Theorem 4.4 is proved. $\square$

Below we discuss when ψ belongs to which class of functions, hypothesis (i) or (ii) in Theorem 4.4 is fulfilled. First, we prove that $K_0(t,x)$ is a reproducing kernel of a space of band-limited functions.

Lemma 4.5. *Let $u \in L^2(R)$ be a real-valued function. If* $\operatorname{supp}\widehat{u} \subset [-\eta,\eta]$ *(η is stated in (2.1)), then*

$$\int_R u(x)K_0(t,x)\,dx = u(t) \text{ almost everywhere } t \in R.$$

Proof. Denote $v(t) := \int_R u(x)K_0(t,x)\,dx$. Since the function $K_0(t,x) \le \sum_n |\widetilde{h}(t-n)h(x-n)|$ and

$$\sum_n |\widetilde{h}(t-n)| \int_R |h(x-n)u(x)|\,dx \le C\|u\|_{L^2(R)}\|h\|_{L^2(R)}$$
$$\cdot \sum_n (1+|t-n|)^{-2} < \infty \quad (C \text{ is a constant}),$$

we obtain that, for almost every $t \in R$,

$$v(t) = \sum_n a_n \widetilde{h}(t-n), \quad \left(a_n = \int_R u(x) h(x-n)\, dx \right). \tag{4.10}$$

Since $\overline{u} = u$, by Plancherel's theorem and $\operatorname{supp} \widehat{u} \subset [-\pi, \pi]$, we get

$$a_n = \int_R \overline{u}(x) h(x-n)\, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} p(\omega) e^{-in\omega}\, d\omega,$$
$$\text{where } p(\omega) = \overline{\widehat{u}}(\omega) \widehat{h}(\omega) \in L^2[-\pi, \pi].$$

On the other hand, by the inversion formula of the Fourier transform and $\operatorname{supp} \widehat{\widetilde{h}} \subset [-\pi, \pi]$, we obtain that, for almost every $t \in R$,

$$\widetilde{h}(t-n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} q(\omega) e^{-in\omega}\, d\omega, \quad \text{where } q(\omega) = \widehat{\widetilde{h}}(\omega) e^{it\omega} \in L^2[-\pi, \pi].$$

So a_n and $\widetilde{h}(t-n)$ are Fourier coefficients of $p(\omega)$ and $q(\omega)$, respectively.

Using the Parseval equality in the Fourier series, we have

$$\sum_n \overline{a}_n \widetilde{h}(t-n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{p}(\omega) q(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{u}(\omega) \widehat{h}(\omega) \widehat{\widetilde{h}}(\omega) e^{it\omega}\, d\omega.$$

Since u and h are both real-valued functions, we know that a_n is real. By (4.10) and $\operatorname{supp} \widehat{u} \subset [-\eta, \eta]$, we have

$$v(t) = \frac{1}{2\pi} \int_{-\eta}^{\eta} \widehat{u}(\omega) \widehat{h}(\omega) \widehat{\widetilde{h}}(\omega) e^{it\omega}\, d\omega.$$

By Lemma 3.3, we have $\widehat{h}(\omega) \widehat{\widetilde{h}}(\omega) = 1$, $\omega \in (-\eta, \eta)$. Again, by $\operatorname{supp} \widehat{u} \subset [-\eta, \eta]$, we get

$$v(t) = \frac{1}{2\pi} \int_{-\eta}^{\eta} \widehat{u}(\omega) e^{it\omega}\, d\omega = \frac{1}{2\pi} \int_R \widehat{u}(\omega) e^{it\omega}\, d\omega = u(t) \text{ a.e. } t \in R.$$

Lemma 4.5 is proved. □

Lemma 4.6. *$\zeta(x) := \int_R K_0(t, x) \operatorname{sgn} t\, dt$ is a bounded continuous function on R.*

Proof. By (4.7) and $h(t), \widetilde{h}(t) = O((1+|t|)^{-2})$, we have

$$\zeta(x) = \sum_n b_n h(x-n), \quad \text{where } b_n = \int_R \widetilde{h}(t-n)\,\mathrm{sgn}\, t\, dt$$

and

$$|b_n| \le \int_R |\widetilde{h}(t)|\, dt = O(1), \qquad |h(x-n)| = O((1+|x-n|)^{-2}).$$

So $\zeta(x) = O(1)$ and the series $\sum_n b_n h(x-n)$ uniformly converges on each closed interval in R. Again, since $\mathrm{supp}\,\widehat{h}$ is bounded, we get $h \in C(R)$. Therefore, the sum $\zeta(x)$ of the series $\sum_n b_n g(x-n)$ is continuous on R. Lemma 4.6 is proved.

So far, we always assume that $\widehat{\psi} \in C^2(R)$, and we know that $K_0(t,x) = O((1+|t-x|)^{-2})$ (by Lemma 4.1). Now we assume that $\widehat{\psi} \in C^4(R)$. Similarly, we can obtain that

$$K_0(t,x) = O((1+|t-x|)^{-4}). \tag{4.11}$$

Lemma 4.7. *Let ψ be a real-valued even function on R and $\widehat{\psi} \in C^4(R)$. Then either there exists an $a>0$ such that $\int_{-\infty}^0 K_0(t,a)\,dt < 0$ or there exists an $a<0$ such that $\int_0^\infty K_0(t,a)\,dt < 0$.*

Proof. We take a real-valued differentiable function $g \in L \cap L^\infty(R)$ such that $\mathrm{supp}\,\widehat{g}$ is bounded and $g' \in L^\infty(R)$, $g(0) \ne 0$, $g'(0) \ne 0$. For example, we may take $g(t) = (\sin(t-(\pi/2))/(t-(\pi/2)))^2$.

Here $g \in L \cap L^\infty(R)$ implies $g \in L^2(R)$.

Let $g_M(t) = g(2^{-M}t)$. Then $g_M \in L^2(R)$, and for a large $M>0$, $\mathrm{supp}\,\widehat{g}_M \subset [-\eta,\eta]$ (η is stated in (2.1)). By Lemma 4.5,

$$\int_R g_M(x) K_0(t,x)\,dx = g_M(t) \text{ almost everywhere } t \in R. \tag{4.12}$$

Let

$$\beta(x) = \mathrm{sgn}\, x - \zeta(x), \tag{4.13}$$

where $\zeta(x)$ is stated in Lemma 4.6. Then $\beta(x)$ is a bounded function and is continuous at $x \neq 0$.

Since $g_M \in L(R)$ and Lemma 4.1, using the Fubini theorem, we get

$$\int_R g_M(x)\beta(x)\,dx = \int_R g_M(x)\operatorname{sgn} x\,dx - \int_R \left(\int_R g_M(x)K_0(t,x)\,dx\right)\operatorname{sgn} t\,dt.$$

Again by (4.12), we obtain that, for a large M,

$$\int_R g_M(x)\beta(x)\,dx = 0. \tag{4.14}$$

By (4.13) and $\int_R K_0(t,x)\,dt = 1$, we get

$$\begin{aligned} \beta(x) &= 2\int_{-\infty}^{0} K_0(t,x)\,dt, \quad x > 0, \\ \beta(x) &= -2\int_{0}^{\infty} K_0(t,x)\,dt, \quad x < 0. \end{aligned} \tag{4.15}$$

Since $\widehat{\psi} \in C^4(R)$, by (4.11) and (4.15), we have

$$\beta(x) = O((1+|x|)^{-3}), \quad x \neq 0. \tag{4.16}$$

Again since $\lim_{M\to\infty} g_M(x) = g(0) \neq 0$ and $g \in L^\infty(R)$, in (4.14), letting $M \to \infty$, we get $\int_R \beta(x)\,dx = 0$. From this and (4.14), we have

$$\int_R \left(\frac{g_M(x) - g_M(0)}{2^{-M}x}\right) x\beta(x)\,dx = 0. \tag{4.17}$$

Here, the absolute value of the integrand $\leq \|g'\|_{L^\infty}|x\beta(x)|$, $x \neq 0$. By (4.16), we have $x\beta(x) \in L(R)$. Since

$$\lim_{M\to\infty} \frac{g_M(x) - g_M(0)}{2^{-M}x} = g'(0) \neq 0,$$

letting $M \to \infty$ in (4.17), we get

$$\int_R x\beta(x)\,dx = 0. \tag{4.18}$$

Below we prove that there exists a point $a \in R$ such that $a\beta(a) < 0$.

If it is not true, then we have $x\beta(x) \geq 0$ for all $x \in R$. From this and (4.18), it follows that $\beta(x) = 0$ for all $x \neq 0$ since $\beta(x)$ is continuous at $x \neq 0$. Again, by (4.13), we have $\zeta(x) = \operatorname{sgn} x$, $x \neq 0$. However, by Lemma 4.6, $\zeta(x)$ is continuous at $x = 0$. This is a contradiction.

From this, we see that either there exists an $a > 0$ such that $\beta(a) < 0$ or there exists an $a < 0$ such that $\beta(a) > 0$. Again, by (4.15), we get Lemma 4.7.

Combining Theorem 4.4 with Lemma 4.7, we get the following

Theorem 4.8. *Suppose that ψ is a real-valued even function on R and $\widehat{\psi} \in C^4(R)$. Let a real-valued function $f \in L[0,1]$ and f^* be a 1-periodic function on R and $f^*(t) = f(t)$, $t \in [0,1]$. If f^* is continuous in $0 < |x| < \varepsilon$ and $f^*(0+) - f^*(0-) = 2d > 0$, then the periodic wavelet frame series* (2.7) *exhibits the Gibbs phenomenon at the origin.*

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