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THE MAHLER MEASURE OF LINEAR FORMS AS SPECIAL VALUES OF SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

R. TOLEDANO

ABSTRACT. We prove that for each $n \geq 4$ there is an analytic function $F_n(x)$ satisfying an algebraic differential equation of degree $n+1$ such that the logarithmic Mahler measure of the linear form $\mathbf{L}_n = x_1 + \cdots + x_n$ can be essentially computed as the evaluation of $F_n(z)$ at $z = n^{-1}$. We show that the coefficients of the series representing $F_n(z)$ can be computed recursively using the nth symmetric power of a second order linear algebraic differential equation, and we give an estimate on the growth of these coefficients.

1. Introduction and definitions. Let I denote the unit interval $[0,1]$. Given a Laurent polynomial in several variables $P \in \mathbf{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ the so called *Mahler's measure* of P is defined as

$$M(P) = \mathrm{e}^{m(P)},$$

where

$$m(P) = \int_{I^n} \log |P(\mathrm{e}^{2\pi i x_1}, \dots, \mathrm{e}^{2\pi i x_n})| \, dx_1 \dots dx_n,$$

is the *logarithmic Mahler's measure* of P.

Let us consider the linear form

$$\mathbf{L}_n = x_1 + \cdots + x_n.$$

We will show here that for each $n \geq 4$ there is an analytic function $F_n(z)$ satisfying an algebraic differential equation such that

$$F_n(n^{-1}) = m(\mathbf{L}_n) - \log \sqrt{n} + \gamma/2. \tag{1.1}$$

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This will be achieved by showing that

$$m(\mathbf{L}_n) = \log \sqrt{n} - \gamma/2 + \sum_{m=2}^{\infty} \frac{c_m(n)}{n^m}, \tag{1.2}$$

where the coefficients $c_m(n)$ are rational numbers satisfying a recurrence relation with integer coefficients.

As was pointed out in [**9**], one of the main interests in formulas like (1.2) lies in the possibility of a numerical computation of the Mahler measure of a linear form in a simple way to a higher degree of accuracy.

This paper will be mainly concerned with the problem of getting a good estimate of the error term in the series appearing in (1.2). We will show that $c_m(n) = O(n^m m^{-5/4})$, see Theorem 2 below, predicting that the series in (1.2) converges rather slowly which is close to what is obtained numerically. On the other hand, the coefficients $c_m(n)$ are easy to compute because, as will be shown in the proof of Corollary 1, they satisfy a recurrence relation that can be explicitly computed for each $n \geq 4$. This is always a desirable property which was not possible to achieve in the method developed in [**9**] to compute numerical approximations of $m(\mathbf{L}_n)$ using Bessel functions. That method, though faster than the present, requires much more complexity in the computations because of the use of numerical integration involving Bessel functions over intervals of increasing length. Following the referee's suggestion, we have included the explicit recurrence formulas for the cases $n = 5$ and $n = 6$ and a small table of values of $c_m(n)$ in the last section.

The numerical computation of $m(\mathbf{L}_n)$ to a high degree of accuracy is of great interest when looking for relations between the Mahler measure of linear forms and special values of L-functions such as the well-known examples found by Smyth (see [**1**] for the cases $n = 3$ and $n = 4$). Recently, Rodríguez-Villegas kindly informed me that, based on numerical evidence, there is a relation between $m(\mathbf{L}_5)$ and $m(\mathbf{L}_6)$ and special values of L-functions associated to certain modular forms.

Finally, we would like to say that, in view of Corollary 1, there seems to be a rather good alternative way of computing $m(\mathbf{L}_n)$ using rational approximations to solutions of algebraic differential equations, but that will be the subject of a forthcoming work.

2. Statements of the main results. Let $L_m(x)$ and $J_0(x)$ denote the mth Laguerre polynomial and the Bessel function of first kind of order zero, respectively.

Theorem 1. *Let $n \geq 4$ be an integer. Then*

$$m(\mathbf{L}_n) = \log \sqrt{n} - \frac{\gamma}{2} + \sum_{m=2}^{\infty} \frac{c_m(n)}{n^m}, \tag{2.1}$$

where

$$c_m(n) = -\frac{b_m(n)(m-1)!}{2},$$

and $b_m(n)$ is the mth coefficient of the Taylor expansion at the origin of the function $\mathrm{e}^{nx} J_0^n(2\sqrt{x})$. Here γ is the Euler constant. More precisely,

$$b_m(n) = \sum_{\mathbf{m} \in I_m} \frac{L_{m_1}(1) \cdots L_{m_n}(1)}{m_1! \cdots m_n!},$$

where

$$I_m = \Big\{ \mathbf{m} = (m_1, \dots, m_n) \in \mathbf{N}^n \Big/ \sum_{j=1}^{n} m_j = m \Big\},$$

Theorem 2. *For each $n \geq 1$, the following estimate*

$$|c_m(n)| \leq C \, \frac{n^m}{m^{5/4}}, \tag{2.2}$$

holds for $m \geq 2$ where

$$C = \frac{5\sqrt[4]{8}\,|L_{12}(1)|}{4} \approx 1.04315\dots.$$

Corollary 1. *For each $n \geq 1$, the coefficients $c_m(n)$ satisfy a recurrence relation with coefficients given by polynomials with integer coefficients. Also, for each $n \geq 1$, the series*

$$F_n(z) = \sum_{m=2}^{\infty} c_m(n) z^m,$$

defines an analytic function in a disk centered at the origin. Each function F_n satisfies a linear algebraic homogeneous differential equation which is the nth symmetric power of the following second order algebraic differential equation:

$$Dy := zy'' + (1-2z)y' + zy = 0.$$

3. Proof of Theorem 1. We begin with the function

$$\varphi_n(x) = |\{(x_1, \dots, x_n) \in I^n / |\mathrm{e}^{2\pi i x_1} + \cdots + \mathrm{e}^{2\pi i x_n}| < x\}|,$$

which gives the probability that the length of the sum of n unit vectors randomly distributed in the unit circle is less than x. We have that

$$m(\mathbf{L}_n) = \int_0^n \log x \, d\varphi_n(x). \tag{3.1}$$

In his research on the problem of Pearson's random walks Kluyver [**5**] proved that

$$\varphi_n(x) = x \int_0^\infty J_0^n(t) J_1(xt)\, dt,$$

where $J_1(x)$ denotes the Bessel function of first kind of order one.

Using the differential equation satisfied by $J_0(x)$, it is easy to compute the derivative of $\varphi_n(x)$ which is

$$\varphi_n'(x) = x\, \phi_n(x), \tag{3.2}$$

where

$$\phi_n(x) = \int_0^\infty t J_0^n(t) J_0(xt)\, dt.$$

However, differentiation with respect to x of the integral

$$\int_0^\infty x J_0^n(t) J_1(xt)\, dt,$$

is justified only if $n \geq 4$, see [**5**] for details. Therefore, we can use formula (3.2) for $\phi_n(x)$ only when $n \geq 4$.

Using (3.1) we get the following formula which will be used later: For $n \geq 4$, we have

$$m(\mathbf{L}_n) = \int_0^\infty x \log x \, \phi_n(x) \, dx. \tag{3.3}$$

This function $\phi_n(x)$ was an object of considerable research and its behavior for large n was studied, among others, by Pearson and Rayleigh. Pearson, see [**2**] for details, derives an asymptotic expansion of $\phi_n(x)$ in terms of n showing that

$$\int_0^\infty t J_0^n(t) J_0(xt) \, dt = \frac{2 \, \mathrm{e}^{-x^2/n}}{n} \sum_{m=0}^\infty a_m(n) L_m(x^2/n).$$

The above expansion will be very useful for our purposes and we will prove it below giving the coefficients $a_m(n)$ in a more explicit way.

Now let us write

$$\mathrm{e}^{nt} J_0^n(2\sqrt{t}) = \sum_{m=0}^\infty b_m(n) \, t^m,$$

and consider the function $\phi_n(x)$. We have

$$\begin{aligned}
\phi_n(x) &= \int_0^\infty t J_0^n(t) J_0(xt) \, dt \\
&= 2 \int_0^\infty J_0^n(2\sqrt{t}) J_0(2x\sqrt{t}) \, dt \\
&= 2 \int_0^\infty J_0^n(2\sqrt{t}) \mathrm{e}^{nt} \mathrm{e}^{-nt} J_0(2x\sqrt{t}) \, dt \\
&= 2 \sum_{m=0}^\infty b_m(n) \int_0^\infty t^m \mathrm{e}^{-nt} J_0(2x\sqrt{t}) \, dt.
\end{aligned}$$

Now, recalling an integral representation of the Laguerre polynomials, [**8**, formula 5.4.1],

$$\mathrm{e}^{-x} L_m(x) = \frac{1}{m!} \int_0^\infty t^m \mathrm{e}^{-t} J_0(2\sqrt{xt}) \, dt,$$

we have that

$$\int_0^\infty t^m \mathrm{e}^{-nt} J_0(2x\sqrt{t}) = \frac{m!\,\mathrm{e}^{-x^2/n}}{n^{m+1}}\, L_m(x^2/n);$$

therefore,

$$\phi_n(x) = 2\mathrm{e}^{-x^2/n} \sum_{m=0}^{\infty} \frac{b_m(n)\, m!}{n^{m+1}}\, L_m(x^2/n), \tag{3.4}$$

which is Pearson's formula. From (3.3) and (3.4) we have

$$\begin{aligned} m(\mathbf{L}_n) &= \int_0^\infty x \log x\, \phi_n(x)\, dx \\ &= \frac{n}{4} \int_0^\infty \log(nx)\, \phi_n(\sqrt{nx})\, dx \\ &= \frac{n \log n}{4} \int_0^\infty \phi_n(\sqrt{nx}) + \frac{n}{4} \int_0^\infty \log x\, \phi_n(\sqrt{nx})\, dx \\ &= \frac{1}{2} \log n + \frac{n}{4} \int_0^\infty \log x\, \phi_n(\sqrt{nx})\, dx \\ &= \frac{1}{2} \log n + \frac{1}{2} \sum_{m=0}^{\infty} \frac{b_m(n)\, m!}{n^m} \int_0^\infty \mathrm{e}^{-x} \log x\, L_m(x)\, dx \\ &= \frac{1}{2} \log n - \frac{\gamma}{2} + \sum_{m=2}^{\infty} \frac{c_m(n)}{n^m}, \end{aligned} \tag{3.5}$$

where

$$c_m(n) = -\frac{b_m(n)\,(m-1)!}{2} \quad \text{for } m \geq 2. \tag{3.6}$$

Here we have used that [**9**]

$$\int_0^\infty \mathrm{e}^{-x} \log x\, L_m(x)\, dx = \begin{cases} -\gamma & \text{if } m = 0 \\ -1/m & \text{if } m \geq 1, \end{cases}$$

and that

$$b_0(n) = 1 \quad \text{and} \quad b_1(n) = 0.$$

The last assertion follows from the formula

$$b_m(n) = \sum_{\mathbf{m}\in I_m} \frac{L_{m_1}(1)\cdots L_{m_n}(1)}{m_1!\cdots m_n!},$$

where

$$I_m = \Big\{\mathbf{m} = (m_1,\dots,m_n)\in \mathbf{N}^n \Big/ \sum_{j=1}^{n} m_j = m\Big\},$$

which is obtained using the generating series

$$\mathrm{e}^t J_0(2\sqrt{t}) = \sum_{m=0}^{\infty} \frac{L_m(1)}{m!}\, t^m.$$

4. Proof of Theorem 2. For the functions $J_0(x)$ and $Y_0(x)$, we have [**8**, Theorem 7.31.2] the following estimates

$$\sup_{x\ge 0}\{x^{1/2}\,|J_0(x)|\},\quad \sup_{x\ge 0}\{x^{1/2}\,|Y_0(x)|\} \le (2/\pi)^{1/2}. \tag{4.1}$$

For the Laguerre polynomials, the following estimate [**8**, formula 7.21.3]

$$\mathrm{e}^{-x/2}\,|L_m(x)| \le 1, \tag{4.2}$$

holds for all $x\ge 0$ and $m\in\mathbf{N}$.

Lemma 1. *Let $-1\le r<0$ and $m\in\mathbf{N}$. Let M_m^r defined by*

$$M_m^r = \sup\Big\{\mathrm{e}^{-t^2/2}\,\big|L_m(t^2)\big| : m^r \le t\le 1\Big\}.$$

Then the estimate

$$M_m^r < \bigg(\frac{7}{5\sqrt{\pi}} + \frac{2}{5}\, m^{(14r-1)/4}\bigg) m^{-(2r+1)/4},$$

holds for all $-1\le r<0$ and $m\in\mathbf{N}$.

Proof. Let $x>0$. We have ([**8**, formula 8.64.3] for $\alpha=0$)

$$\mathrm{e}^{-t^2/2} L_m(t^2) = J_0(2N^{1/2}t) - \frac{\pi}{2}\int_0^t F(x,t)\mathrm{e}^{-x^2/2}x^3 L_m(x^2)\,dx \tag{4.3}$$

where

$$F(x,t) = J_0(2N^{1/2}t)Y_0(2N^{1/2}x) - Y_0(2N^{1/2}t)J_0(2N^{1/2}x),$$

and $N = m + 1/2$.

Let us consider

$$\int_0^t F(x,t)\mathrm{e}^{-x^2/2}x^3 L_m(x^2)\,dx = \int_0^{m^r} + \int_{m^r}^t F(x,t)\mathrm{e}^{-x^2/2}x^3 L_m(x^2)\,dx \\ = I_1 + I_2. \tag{4.4}$$

From (4.1) we have that

$$x^{1/2}|F(x,t)| \le \frac{2}{\pi(Nt)^{1/2}},$$

and since $N > m$ and $m^r \le t \le 1$, we see that $(Nt)^{-1/2} < m^{-(r+1)/2}$.

Using the above estimates and (4.2), we have that

$$|I_1| \le \frac{2}{\pi(Nt)^{1/2}} \int_0^{m^r} x^{5/2}\,dx < \frac{4}{7\pi} m^{(6r-1)/2}.$$

Similarly,

$$|I_2| \le \frac{2M_m^r}{\pi(Nt)^{1/2}} \int_{m^r}^t x^{5/2}\,dx < \frac{4M_m^r}{7\pi} m^{-(r+1)/2} \le \frac{4M_m^r}{7\pi},$$

because $r \ge -1$. From (4.1), (4.3) and the above estimates, we have that the following inequality

$$\begin{aligned} \mathrm{e}^{-t^2/2}\left|L_m(t^2)\right| &< \frac{1}{\pi^{1/2}N^{1/4}t^{1/2}} + \frac{2}{7}\,m^{(6r-1)/2} + \frac{2}{7}\,M_m^r \\ &< \frac{1}{\pi^{1/2}}\,m^{-(2r+1)/4} + \frac{2}{7}\,m^{(6r-1)/2} + \frac{2}{7}\,M_m^r, \end{aligned}$$

holds for $m^r \le t \le 1$. Therefore,

$$M_m^r < \frac{1}{\pi^{1/2}}\,m^{-(2r+1)/4} + \frac{2}{7}\,m^{(6r-1)/2} + \frac{2}{7}\,M_m^r,$$

and the conclusion of the lemma readily follows. □

We now use the above lemma to prove an estimate for the alternating sequence

$$L_m(1) = \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{k!}, \quad m = 0, 1, 2 \dots .$$

This sequence has been an object of some research, see [**3, 4**] for details. In [**3**] the authors show that

$$L_m(1) = c_1\, m^{-1/4} \sin(2m^{1/2} + c_2) + o(m^{-1/4}) \quad \text{as } m \to \infty$$

for some (unspecified) positive constants c_1 and c_2. Here we get the following estimate.

Lemma 2. *Let $L_m(1)$ be the mth Laguerre polynomial evaluated at 1. Then*

$$\sup_{m \geq 0} \Big\{ (m+1/2)^{1/4} |L_m(1)| \Big\} = (25/2)^{1/4} |\, L_{12}(1)|.$$

Proof. If $m = 0$, it is easy to check that $(1/2)^{1/4}|L_0(1)| < (25/2)^{1/4}|\, L_{12}(1)|$. Let us suppose now that $m \in \mathbf{N}$. Taking $t = 1$ in (4.3), we have that

$$\mathrm{e}^{-1/2} L_m(1) = J_0(2N^{1/2}) - \frac{\pi}{2} \int_0^1 F(x,1) \mathrm{e}^{-x^2/2} x^3 L_m(x^2)\, dx,$$

where $N = m + 1/2$ and $F(x,t)$ is the same function defined in the previous lemma. Let $-1 \leq r < 0$. Splitting the above integral into two parts, one from 0 to m^r and the other from m^r to 1, and using the estimates that we found in the proof of the previous lemma, we get

$$\begin{aligned}
\mathrm{e}^{-1/2}\, |L_m(1)| &\leq \frac{1}{\sqrt{\pi} N^{1/4}} + \frac{2}{7N^{1/2}}\, m^{7r/2} + \frac{2}{7N^{1/2}}\, M_m^r \\
&< \frac{1}{\sqrt{\pi} N^{1/4}} + \frac{2}{7N^{1/2}}\, m^{7r/2} \\
&\quad + \frac{2}{7N^{1/2}} \left(\frac{7}{5\sqrt{\pi}} + \frac{2}{5}\, m^{(14r-1)/4} \right) m^{-(2r+1)/4}.
\end{aligned}$$

Choosing $r = -1/14$ we have that

$$(m+1/2)^{1/4}\,|L_m(1)| < \sigma(m),$$

where

$$\begin{aligned}\sigma(m) = (\mathrm{e}/\pi)^{1/2} &+ \frac{2\mathrm{e}^{1/2}}{7(m+1/2)^{1/4}m^{1/4}} \\ &+ \left(\frac{2}{5\sqrt{\pi}} + \frac{4}{35m^{1/2}}\right)\frac{\mathrm{e}^{1/2}}{(m+1/2)^{1/4}m^{3/14}}.\end{aligned}$$

Now $\sigma(m)$ is a decreasing function of m so that it is easy to verify that $\sigma(m) \le 0.933027\ldots$ if $m \ge 49000$. For $0 \le m < 49000$ we compute the values of $(m+1/2)^{1/4}|L_m(1)|$ and find that its highest value is $0.933028\ldots$, and it is reached at $m = 12$. $\square$

Now we are in a position to prove Theorem 2. Recall that

$$b_m(n) = \sum_{\mathbf{m}\in I_m} \frac{\prod_{j=1}^n L_{m_j}(1)}{m_1!\cdots m_n!},$$

where

$$I_m = \Big\{\mathbf{m} = (m_1,\ldots,m_n) \in \mathbf{N}^n \Big/ \sum_{j=1}^n m_j = m\Big\}.$$

For each vector $\mathbf{m}$ appearing in the above formula for $b_m(n)$, the values of $L_{m_j}(1)$ are either 1 or 0 for $m_j = 0, 1$, respectively, so that we just need to estimate the product

$$\prod_{j=1}^k |L_{m_j}(1)|,$$

where k is the number of components m_j in $\mathbf{m}$ such that $m_j \ge 2$. If $k = 1$, then the only nonzero component in $\mathbf{m}$ has value m so that, from Lemma 2 we have

$$\begin{aligned}|L_m(1)| &\le \sqrt[4]{25/2}\,|L_{12}(1)|\,(m+1/2)^{-1/4} \\ &< \sqrt[4]{25/2}\,|L_{12}(1)|\,m^{-1/4}.\end{aligned} \tag{4.5}$$

Assume now that $k \geq 2$ and consider the following elementary inequality: let $a_1, \dots, a_k$ be k positive integers such that each $a_j \geq b > 0$. Then

$$\prod_{j=1}^{k}(a_j+1) \geq \frac{b^{k-1}-1}{b-1} \sum_{j=1}^{k} a_j. \tag{4.6}$$

Applying (4.6) to the case $a_j = 2m_j$ with $m_j \geq 2$ such that $m_1 + \cdots + m_k = m$ we have that for $k \geq 2$,

$$\begin{aligned}\prod_{j=1}^{k}(m_j+1/2) &= \frac{1}{2^k}\prod_{j=1}^{k}(2m_j+1)\\ &\geq \frac{4^{k-1}-1}{2^k 3}\sum_{j=1}^{k} 2m_j \geq \frac{2^k m}{8}.\end{aligned}$$

Therefore,

$$\prod_{j=1}^{k}(m_j+1/2)^{-1/4} \leq \sqrt[4]{8}\, 2^{-k/4} m^{-1/4}. \tag{4.7}$$

On the other hand, from Lemma 2 we have that

$$|L_{m_j}(1)| \leq \sqrt[4]{25/2}\,|L_{12}(1)|\,(m_j+1/2)^{-1/4}.$$

The above estimate, together with (4.7), implies that for $k \geq 2$,

$$\begin{aligned}\prod_{j=1}^{k}|L_{m_j}(1)| &\leq \left(\sqrt[4]{25/2}\,|L_{12}(1)|\right)^k \prod_{j=1}^{k}(m_j+1/2)^{-1/4}\\ &\leq \sqrt[4]{8}\, a^k\, m^{-1/4},\end{aligned} \tag{4.8}$$

where

$$a = \sqrt{5/2}\,|L_{12}(1)| \approx 0.7845\dots.$$

Since $0 < a < 1$ and $k \geq 2$, we can say that

$$\prod_{j=1}^{k}|L_{m_j}(1)| \leq \sqrt[4]{8}\, a^2\, m^{-1/4}. \tag{4.9}$$

It is easy to check that $\sqrt[4]{8}\,a^2 > \sqrt[4]{25/2}\,|L_{12}(1)|$, so that using (4.5) and (4.9) we have, for any vector $\mathbf{m} \in I_m$, that

$$\prod_{j=1}^{n} |L_{m_j}(1)| \le \frac{c}{m^{1/4}},$$

where $c = \sqrt[4]{8}\,a^2 = 5/2\,\sqrt[4]{8}\,|L_{12}(1)|^2$. Hence,

$$\begin{aligned} |b_m(n)| &\le \sum_{\mathbf{m}\in I_m} \frac{\prod_{j=1}^{n} |L_{m_j}(1)|}{m_1! \cdots m_n!} \\ &\le \frac{c}{m^{1/4}} \sum_{\mathbf{m}\in I_m} \frac{1}{m_1! \cdots m_n!} = \frac{c\,n^m}{m^{1/4} m!}, \end{aligned}$$

and (2.2) follows because

$$c_m(n) = -\frac{b_m(n)(m-1)!}{2}.$$

5. Proof of Corollary 1. Estimate (2.2) implies that for $n \ge 1$ the series $F_n(z)$ defines an analytic function in a disk with center the origin and radius at least n^{-1}. In order to show that F_n satisfies a linear homogeneous differential equation with coefficients in $\mathbf{Z}[z]$, we will show that for each $n \ge 2$ and $m \ge 2$ arbitrary but fixed we can find an integer $k = k(n) \ge 1$ and a linear form $L(z_0, \ldots, z_k)$ with coefficients in $\mathbf{Z}[m]$ such that $L(c_m(n), \ldots, c_{m+k}(n)) = 0$. This implies that we can compute the coefficients $c_m(n)$ in a recursive way using the form L from which we can read off an algebraic linear differential equation satisfied by $F_n(z)$.

Now we show how to construct a linear form $L(z_1, \ldots, z_k)$ with coefficients in $\mathbf{Z}[m]$ with the above property. Notice that the function $\mathrm{e}^z J_0(2\sqrt{z})$ satisfies a homogeneous linear differential equation with coefficients in $\mathbf{Z}[z]$ because $J_0(2\sqrt{z})$ does. More precisely, $y = \mathrm{e}^z J_0(2\sqrt{z})$ satisfies

$$Dy := zy'' + (1-2z)y' + zy = 0. \tag{5.1}$$

Now we can find, in a finite number of steps, the ordinary differential equation (with coefficients in $\mathbf{Z}[z]$) satisfied by the nth power of

$e^z J_0(2\sqrt{z})$, from which we can read off the recurrence satisfied by the coefficients $b_m(n)$. Such a differential equation is simply the nth symmetric power of D denoted by $\mathrm{Sym}^n(D)$ and can be computed using, for instance, the DEtools package in Maple. It is known [**6**] that the nth symmetric power of a second order linear differential equation always produces a linear differential equation of order $n+1$ so that we can say that for each $n \geq 2$ the recurrence satisfied by the coefficients $b_m(n)$ is given by an expression (of at most $n+1$ terms) of the form

$$\sum_{j=0}^{k} P_j(m)\, b_{m+j}(n) = 0,$$

where each $P_j \in \mathbf{Z}[z]$, $k = k(n) \leq n+1$, is a positive integer and $m \geq 0$ is an integer arbitrary but fixed. Then, for any $0 \leq j \leq n$ and any $m \geq 2$ fixed, we have

$$P_j(m)\frac{(m+k-1)!}{(m+j-1)!}\, c_{m+j}(n) = -\frac{1}{2}\, P_j(m)\,(m+k-1)!\, b_{m+j}(n).$$

Therefore,

$$\begin{aligned}\sum_{j=0}^{k} P_j(m)\frac{(m+k-1)!}{(m+j-1)!}\, c_{m+j}(n) &= \frac{-(m+k-1)!}{2} \sum_{j=0}^{n} P_j(m) b_{m+j}(n) \\ &= 0,\end{aligned}$$

so that the recurrence satisfied by the coefficients $c_m(n)$ with $m \geq 2$ arbitrary but fixed is

$$\sum_{j=0}^{k} Q_j(m) c_{m+j}(n) = 0, \tag{5.2}$$

where

$$Q_j(z) = P_j(z)\frac{(m+k-1)!}{(m+j-1)!} \in \mathbf{Z}[z]. \tag{5.3}$$

This shows that the required linear form is

$$L(z_0, \dots, z_k) = \sum_{j=0}^{k} Q_j(m) z_j,$$

and the corollary is proved. $\square$

6. Recurrence formulas for $n = 5$ and $n = 6$. For $n = 5$,

$$c_{m+1} = \sum_{j=0}^{5} \frac{q_j}{(m+1)^6} c_{m-j},$$

$$\begin{aligned}
q_0 &= 30m^6 + 40m^5 + 30m^4 + 12m^3 + 2m^2,\\
q_1 &= -375m^6 + 1075m^5 - 1334m^4 + 914m^3 - 331m^2 + 51m,\\
q_2 &= 2500m^6 - 16500m^5 + 43590m^4 - 57990m^3 + 38840m^2\\
&\quad - 10440m,\\
q_3 &= -9375m^6 + 92500m^5 - 357100m^4 + 673850m^3 - 618725m^2\\
&\quad + 218850m\\
q_4 &= 18750m^6 - 237500m^5 + 1156250m^4 - 2687500m^3 + 2950000m^2\\
&\quad - 1200000m,\\
q_5 &= -15625m^6 + 234375m^5 - 1328125m^4 + 3515625m^3 - 4281250m^2\\
&\quad + 1875000m.
\end{aligned}$$

For $n = 6$,

$$c_{m+1} = \sum_{j=0}^{6} \frac{q_j}{(m+1)^7} c_{m-j},$$

$$\begin{aligned}
q_0 &= 42m^7 + 70m^6 + 70m^5 + 42m^4 + 14m^3 + 2m^2,\\
q_1 &= -756m^7 + 2436m^6 - 3724m^5 + 3388m^4 - 1832m^3 + 560m^2\\
&\quad - 72m,\\
q_2 &= 7560m^7 - 57960m^6 + 187992m^5 - 331416m^4 + 334224m^3\\
&\quad - 182160m^2 + 41760m,\\
q_3 &= -45360m^7 + 529200m^6 - 2549232m^5 + 6494256m^4 - 9209376m^3\\
&\quad + 6859296m^2 - 2078784m,
\end{aligned}$$

$$\begin{aligned}
q_4 &= 163296m^7 - 2485728m^6 + 15392160m^5 - 49502880m^4 \\
&\quad + 86776704m^3 - 77922432m^2 + 27578880m, \\
q_5 &= -326592m^7 + 5987520m^6 - 44089920m^5 + 166017600m^4 \\
&\quad - 334430208m^3 + 337478400m^2 - 130636800m, \\
q_6 &= 279936m^7 - 5878656m^6 + 48988800m^5 - 205752960m^4 \\
&\quad + 454616064m^3 - 493807104m^2 + 201553920m.
\end{aligned}$$

TABLE 1. Values of the coefficients $c_m(n)$ for $2 \le m \le 10$ and $n = 5, 6, 7, 8$.

$\mathbf{n} = 5$	$\mathbf{n} = 6$	$\mathbf{n} = 7$	$\mathbf{n} = 8$
$c_2 = \frac{5}{8}$	$c_2 = \frac{3}{4}$	$c_2 = \frac{7}{8}$	$c_2 = 1$
$c_3 = \frac{5}{9}$	$c_3 = \frac{2}{3}$	$c_3 = \frac{7}{9}$	$c_3 = \frac{8}{9}$
$c_4 = -\frac{95}{64}$	$c_4 = -\frac{75}{32}$	$c_4 = -\frac{217}{64}$	$c_4 = -\frac{37}{8}$
$c_5 = -\frac{193}{30}$	$c_5 = -\frac{243}{25}$	$c_5 = -\frac{2051}{150}$	$c_5 = -\frac{1372}{75}$
$c_6 = -\frac{3305}{576}$	$c_6 = -\frac{569}{144}$	$c_6 = \frac{1729}{1728}$	$c_6 = \frac{181}{18}$
$c_7 = \frac{18495}{392}$	$c_7 = \frac{53051}{490}$	$c_7 = \frac{171419}{840}$	$c_7 = \frac{83812}{245}$
$c_8 = \frac{1620475}{6144}$	$c_8 = \frac{2560001}{5120}$	$c_8 = \frac{74803267}{92160}$	$c_8 = \frac{4551053}{3840}$
$c_9 = \frac{44157415}{81648}$	$c_9 = \frac{5586593}{13608}$	$c_9 = -\frac{27774509}{58320}$	$c_9 = -\frac{137253407}{51030}$
$c_{10} = -\frac{2564608651}{1612800}$	$c_{10} = -\frac{5455012667}{672000}$	$c_{10} = -\frac{25628788717}{1152000}$	$c_{10} = -\frac{59833271}{1260}$

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Departamento de Matemática, Facultad de Ingeniería Química, Universidad Nacional del Litoral, Argentina
Email address: ridatole@gmail.com