ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 39, Number 2, 2009

PERTURBATIONS OF NONASSOCIATIVE BANACH ALGEBRAS

ANAR DOSI

ABSTRACT. In this note we prove that if either $\mathfrak{A}$ is a Banach-Jordan algebra or a Banach-Lie algebra then all perturbations of the multiplication in $\mathfrak{A}$ give algebras topologically isomorphic with $\mathfrak{A}$ whenever certain small-dimension cohomology groups associated with $\mathfrak{A}$ are vanishing.

1. Introduction. The perturbation problems in the associative Banach algebra context are well known [**8**], and serious advancements in that direction were done by Johnson in [**9**] and Raeburn and Taylor in [**10**]. The main question is the following. If we perturb the multiplication in a Banach algebra, do we obtain a topologically isomorphic algebra? One can ask similar questions. Which properties inherit the neighboring multiplication? Is the perturbed Banach algebra homomorphism equivalent to the original one? All these questions were deeply investigated and vividly reflected in the above mentioned papers. The main tools in these investigations are the Hochschild cohomology groups $H^n(\mathfrak{A},\mathfrak{A})$, $n \geq 0$, (see [**7**, 1.3.1]) associated with the original associative Banach algebra $\mathfrak{A}$. It is proved that all these perturbation problems have positive solutions if certain small-dimension cohomologies are vanishing. For instance, if $H^2(\mathfrak{A},\mathfrak{A}) = 0$ and $H^3(\mathfrak{A},\mathfrak{A})$ are Hausdorff, then all Banach algebra multiplications sufficiently close to the original multiplication in $\mathfrak{A}$ define Banach algebras topologically isomorphic with $\mathfrak{A}$. In the latter case, we say that $\mathfrak{A}$ is a stable Banach algebra (see [**7**, 1.2.2] and Definition 2.1 below).

Which nonassociative Banach algebras are stable? That is the main question which we investigate in this paper. Nonassociative structures, especially Jordan and Lie structures, present great interest in analysis.

2000 AMS *Mathematics subject classification.* Primary 46H05, Secondary 46H70.
Keywords and phrases. The inversion constant of an operator, nonassociative stable Banach algebra, cohomologies of Banach-Jordan algebras.
Received by the editors on February 21, 2005, and in revised form on August 2, 2006.
DOI:10.1216/RMJ-2009-39-2-509

On this account we investigate the perturbation problem for Banach-Jordan and Banach-Lie algebras. In a purely algebraic context, the perturbation problem arose in deformation formalism. A deformation of a finite dimensional Lie algebra $\mathfrak{g}$ is a Lie algebra with the same linear space $\mathfrak{g}$ but with perturbed Lie multiplication. That can be expressed in terms of the structure constants $\{C_{ij}^k\}$ of the original Lie algebra with respect to its fixed basis $e = (e_1, \dots, e_n)$ in $\mathfrak{g}$. The structure constants of a deformed Lie algebra with respect to the same basis e can be written as $\{C_{ij}^k(\varepsilon)\}$, which depend upon a family of parameters ε such that

$$\lim_{\varepsilon \to 0} \left\{ C_{ij}^k(\varepsilon) \right\} = \left\{ C_{ij}^k \right\}.$$

Thus the stability of $\mathfrak{g}$ in the above mentioned sense means that all deformed Lie algebras $(\mathfrak{g}, \{C_{ij}^k(\varepsilon)\})$ are isomorphic to $(\mathfrak{g}, \{C_{ij}^k\})$ for small ε. The first results in that direction are due to Gerstenhaber [**5**]. In particular, it was proved that semi-simple Lie algebras are stable (for a geometrical viewpoint, see [**1**]). That result can also be derived (see Remark 6.1 below) from a more general perturbation theorem in the Banach-Lie algebra framework proposed in the present paper. Our approach is based upon the implicit function theorem proposed by Raeburn and Taylor in [**10**]. Roughly speaking, by differentiating the identities of a nonassociative Banach algebra we are raising the cohomology level. By assuming that certain cohomologies vanish, we obtain a positive solution of the perturbation problem on the grounds of the implicit function theorem.

There is a well-developed cohomology theory for Banach-Lie algebras. We prove that if the second cohomology group $H^2(\mathfrak{L})$ of a Banach-Lie algebra $\mathfrak{L}$ is vanishing and the third one $H^3(\mathfrak{L})$ is Hausdorff, then $\mathfrak{L}$ is a stable Banach-Lie algebra. But for Banach-Jordan algebras no explicit (as in the Lie algebra case) cohomology theory exists. We suggest certain construction which defines small (up to third) dimension cohomology groups having similar meaning as in Lie and associative algebras, and we prove under the same conditions on cohomologies of a Banach-Jordan algebra that it is stable; thereby, the perturbation problem has a positive solution.

2. Preliminaries. All linear spaces considered are real or complex. The set of all positive integers is denoted by $\mathbf{N}$ and $\mathbf{Z}_+ = \{0\} \cup \mathbf{N}$. Let $\mathcal{B}(X, Y)$ be a normed space of all bounded linear operators between normed spaces X and Y furnished with the operator norm, and let $\mathcal{B}(X) = \mathcal{B}(X, X)$. The class of all Banach spaces is denoted by **BS**. Let X, $Y \in \mathbf{BS}$, $U \subseteq X$ an open subset, and let $f : U \to Y$ be a map. If f is differentiable on U in the Fréchet sense, then $f' : U \to \mathcal{B}(X, Y)$ denotes the derived map. Moreover, if f' is continuous, then f is said to be a C^1 map. By analogy, it is defined to be a C^∞ map. Let $X \in \mathbf{BS}$. The set of all invertible elements in the Banach algebra $\mathcal{B}(X)$ is denoted by $\mathcal{B}(X)^{-1}$, and let 1_X be the identity operator acting on X. It is well known [**2**, Theorem 5.4.3] that $\mathcal{B}(X)^{-1}$ is an open subset in $\mathcal{B}(X)$ and the map $\mathcal{B}(X)^{-1} \to \mathcal{B}(X)$, $T \mapsto T^{-1}$ is a C^∞ map. The unit ball of a normed space X is denoted by $X_{(1)}$. We use the conventional denotation $X \widehat{\otimes} Y$ for the projective tensor product of $X, Y \in \mathbf{BS}$, and we write $X^{\widehat{\otimes} n}$ instead of n-times projective tensor product of X on itself. The direct sum $X \oplus Y$ is endowed with the sum-norm and X^n denotes the direct sum of n copies of the space X. The bounded linear map $d_n : X \to X^n$, $d_n(x) = (x, \dots, x)$, is called the diagonal map.

2.1. Exterior and symmetric powers of a Banach space. Let X be a Banach space, S_n (herein $n \in \mathbf{N}$) the group of all permutations over the finite set $\{1, \dots, n\}$, and let $\varepsilon(\tau)$ be the sign of a permutation $\tau \in S_n$. Consider an operator $\delta_\tau \in \mathcal{B}(X^{\widehat{\otimes} n})$, $\delta_\tau(x_1 \otimes \cdots \otimes x_n) = x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)}$, $\tau \in S_n$. *The exterior power* $\wedge^n X$ *of* X *was defined* [**3**] *as the image of a bounded linear projector*

$$A_n \in \mathcal{B}\left(X^{\widehat{\otimes} n}\right), \quad A_n = n!^{-1} \sum_{\tau \in S_n} \varepsilon(\tau)\, \delta_\tau,$$

and we write $x_1 \wedge \cdots \wedge x_n$ instead of $A_n(x_1 \otimes \cdots \otimes x_n)$. For brevity, we also use the denotation $\underline{x} = x_1 \wedge \cdots \wedge x_n$, and we write $\underline{x}_i = x_1 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge x_n$ whenever the variable x_i is thrown out from the expression of $\underline{x}$. On the same grounds we write $\underline{x}_{i,j}$ whenever the variables x_i and x_j are thrown out from $\underline{x}$.

Now let us introduce a *symmetric power* $\vee^n X$ *of* X *as the image of a bounded linear projector*

$$B_n \in \mathcal{B}\left(X^{\widehat{\otimes} n}\right), \quad B_n = n!^{-1} \sum_{\tau \in S_n} \delta_\tau.$$

One can easily observe that, for a Banach space Y, the spaces $\mathcal{B}(\wedge^n X, Y)$ and $\mathcal{B}(\vee^n X, Y)$ are isometrically isomorphic with the spaces of all continuous skew-symmetric and symmetric n-linear maps on X with values in Y, respectively. By analogy, we write $x_1 \vee \cdots \vee x_n$ instead of $B_n(x_1 \otimes \cdots \otimes x_n)$.

2.2. Polynomials. Let X and Y be linear spaces, and let $n \in \mathbf{N}$. A mapping $\varphi : X \to Y$ is said [**2**, 6.1] to be a *homogeneous polynomial of degree* n if φ splits into the superposition $X \overset{d_n}{\to} X^n \overset{f}{\to} Y$ of the diagonal map $d_n : X \to X^n$ and an n-linear map $f : X^n \to Y$. Thus, $\varphi(x) = f(x, \dots, x)$ and thereupon $\varphi(\lambda x) = \lambda^n \varphi(x)$ for all scalars λ. Obviously, one can assume that f is a symmetric n-linear map. In particular, all linear maps $X \to Y$ are homogeneous polynomials of degree 1, and it is convenient to assume constant maps $X \to Y$ to be homogeneous polynomials of degree 0. A mapping $\varphi : X \to Y$ is said to be a *polynomial* if there are a number n and homogeneous polynomials $\varphi_0, \dots, \varphi_n$ (φ_i has degree i) such that $\varphi = \varphi_0 + \cdots + \varphi_n$. In this case we say that φ has degree $\leq n$. The main result [**2**, Theorem 6.3.1] of the algebraic theory of polynomials asserts that the latter expansion for a polynomial φ is unique. To have a more precise formulation of this result, let us introduce a map $\Delta_h \varphi : X \to Y$ by setting

$$(\Delta_h \varphi)(x) = \varphi(x + h) - \varphi(x),$$

where $\varphi : X \to Y$ is an arbitrary mapping and $h \in E$.

Theorem 2.1. *Let $\varphi = \varphi_0 + \cdots + \varphi_n$ be a polynomial of degree $\leq n$, and let $f_n : X^n \to Y$ be a symmetric n-linear map such that $\varphi_n = f_n \cdot d_n$. Then $\Delta_h \varphi$ is a polynomial of degree $\leq n - 1$ and $\Delta_{x_1} \cdots \Delta_{x_n} \varphi$ is a constant map for all $x_1, \dots, x_n \in X$. Moreover,*

$$f_n(x_1, \dots, x_n) = \frac{1}{n!} \Delta_{x_1} \cdots \Delta_{x_n} \varphi.$$

As follows from Theorem 2.1, to each polynomial $\varphi = \varphi_n$ of degree n there uniquely corresponds a symmetric n-linear map $\widetilde{\varphi}(x_1 \vee \cdots \vee x_n) = n!^{-1} \Delta_{x_1} \cdots \Delta_{x_n} \varphi$ such that $\varphi = \widetilde{\varphi} \cdot d_n$. Thus the mapping $\varphi \mapsto \widetilde{\varphi}$ implements a bijection between homogeneous polynomials of degree n and symmetric n-linear maps.

Now assume that X and Y are normed spaces, and let $\mathcal{P}_n(X,Y)$ be a space of all continuous homogeneous polynomials of degree n. We set $\|\varphi\| = \sup\{\|\varphi(x)\| : x \in X_{(1)}\}$ for $\varphi \in \mathcal{P}_n(X,Y)$. It is proved [**2**, Theorem 6.4.1] that $\|\varphi\| < \infty$ and the map $\varphi \mapsto \|\varphi\|$ is a norm on $\mathcal{P}_n(X,Y)$.

Lemma 2.1. *The linear operator $\mathcal{P}_n(X,Y) \to \mathcal{B}(\vee^n X, Y)$, $\varphi \mapsto \widetilde{\varphi}$, is a topological isomorphism. In particular, $\mathcal{P}_n(X,Y) \in$ **BS** whenever $Y \in$ **BS**.*

Proof. By Theorem 2.1 and by the very definition of the function $\Delta_h \varphi$, we conclude that

$$\begin{aligned}\Delta_{x_1} \cdots \Delta_{x_n} \varphi &= \left(\Delta_{x_1} \cdots \Delta_{x_n} \varphi\right)(0) \\ &= \sum_{1 \le i_1 < \cdots < i_p \le n} (-1)^{n-p} \varphi\left(x_{i_1} + \cdots + x_{i_p}\right).\end{aligned}$$

It follows that

$$\begin{aligned}\|\widetilde{\varphi}\| &= n!^{-1} \sup\left\{\|\Delta_{x_1} \cdots \Delta_{x_n} \varphi\| : x_i \in X_{(1)}\right\} \\ &\le n!^{-1} \sum_{1 \le i_1 < \cdots < i_p \le n} \sup\left\{n^n \left\|\varphi\left(n^{-1} x_{i_1} + \cdots + n^{-1} x_{i_p}\right)\right\| : \right. \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. x_i \in X_{(1)}\right\} \\ &\le n!^{-1} (2n)^n \|\varphi\|,\end{aligned}$$

that is, $\|\widetilde{\varphi}\| \le n!^{-1}(2n)^n \|\varphi\|$. Consequently, $\mathcal{P}_n(X,Y) \to \mathcal{B}(\vee^n X, Y)$, $\varphi \mapsto \widetilde{\varphi}$ is a bounded linear operator. It remains to note that the correspondence $\mathcal{B}(\vee^n X, Y) \to \mathcal{P}_n(X,Y)$, $f \mapsto f \cdot d_n$, is a bounded linear operator (namely, $\|f \cdot d_n\| \le \|f\|$), and it is the inverse operator to $\varphi \mapsto \widetilde{\varphi}$, for a symmetric n-linear map related to the polynomial $f \cdot d_n$ is unique (see Theorem 2.1) and therefore it might coincide with mm$f$. □

2.3. Nonassociative stable Banach algebras. Let $\mathfrak{A}$ be a nonassociative algebra with a set of identities I. The algebra $\mathfrak{A}$ is said to be *a nonassociative Banach algebra* if its underlying space is a Banach space and the multiplication $*$ is jointly (or separately) continuous with respect to this norm. Thus we have a bounded multiplication

$*: \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ subjected to a family of identities I. For instance, if I consists of nonassociative polynomials $xy - yx$ and $(xy)x^2 - x(yx^2)$, then a nonassociative Banach algebra $\mathfrak{A}$ with the set of identities I is called *Banach-Jordan*, B-J for short, *algebra*, and if I consists of polynomials x^2 and $(xy)z + (yz)x + (zx)y$ then we say that $\mathfrak{A}$ is *a Banach-Lie*, B-L for short, *algebra*. We shall focus on the latter two classes of nonassociative Banach algebras.

The following definition plays a key role in this note.

Definition 2.1. Let $\mathfrak{A}$ be a nonassociative Banach algebra with a set of identities I and let $*: \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ be relevant bounded multiplication. We say that $\mathfrak{A}$ is stable whenever there is a constant $\varepsilon > 0$ such that if $m: \mathfrak{A}\widehat{\otimes}\mathfrak{A} \to \mathfrak{A}$ is any bounded multiplication satisfied to all identities from I with $\|m - *\| < \varepsilon$ then there exists $T \in \mathcal{B}(\mathfrak{A})^{-1}$ such that $T(m(a \otimes b)) = T(a) * T(b)$. Further, $\mathfrak{A}$ is said to be strongly stable if additionally we have an estimation $\|T - 1_{\mathfrak{A}}\| \leq C\|m - *\|$, where $C > 0$ is a constant does not depend upon m and T.

Thus the stability of an algebra $\mathfrak{A}$ means that all multiplications sufficiently close to the original one give algebras topologically isomorphic with $\mathfrak{A}$. As we mentioned above, so are all associative Banach algebras and associative Banach $*$-algebras whenever $H^2(\mathfrak{A}, \mathfrak{A}) = 0$ and $H^3(\mathfrak{A}, \mathfrak{A})$ is Hausdorff [**9, 10**].

2.4. Johnson lemma and implicit function theorem. Our approach is strongly based on the implicit function theorem proposed in [**10**]. Here we briefly remind the reader about this theorem and demonstrate its connection with the perturbation problem of Banach space complexes which were solved by Johnson in [**9**, Lemma 6.1].

Let $T: X \to Y$ be a Banach space operator with the closed range, and let

$$T^{\sim}: X/\ker(T) \longrightarrow \operatorname{im}(T), \quad T^{\sim}(x^{\sim} \operatorname{mod} \ker(T)) = Tx,$$

be the induced operator. The latter has a bounded inverse, and the norm of this inverse operator is called the *inversion constant of* T, and it is denoted by $\operatorname{ic}(T)$. Further, let

$$S = (S_1, S_2), \quad S_1 \in \mathcal{B}(X, Y), \quad S_2 \in \mathcal{B}(Y, Z)$$

be Banach space operators. We say that S is *a differential pair* if $S_2S_1 = 0$, and a differential pair S is said to be *exact* if $\operatorname{im}(S_1) = \ker(S_2)$ and $\operatorname{im}(S_2)$ is closed. The following assertion belongs to Johnson [**9**].

Lemma 2.2. *Let $X, Y, Z \in \mathbf{BS}$, and let $S = (S_1, S_2)$, $S_1 \in \mathcal{B}(X, Y)$, $S_2 \in \mathcal{B}(Y, Z)$ be an exact differential pair. Then so is a differential pair sufficiently close to S. Namely, if $T = (T_1, T_2)$, $T_1 \in \mathcal{B}(X, Y)$, $T_2 \in \mathcal{B}(Y, Z)$ is a differential pair with $k_T < 1$, then T is exact, where*

$$k_T = c_1 \|S_1 - T_1\| + c_2 \|S_2 - T_2\| + c_1c_2 \|S_1 - T_1\| \|S_2 - T_2\|,$$

$c_i > \operatorname{ic}(S_i)$, $i = 1, 2$. Moreover,

$$\operatorname{ic}(T_1) \leq (1 - k_T)^{-1} c_1 (1 + c_2 \|S_2 - T_2\|),$$
$$\operatorname{ic}(T_2) \leq (1 - k_T)^{-1} c_2 (1 + c_1 \|S_1 - T_1\|).$$

Remark 2.1. The latter assertion plays an important role in multivariable spectral theory. Namely, the Johnson lemma 2.2 automatically involves the following well-known assertion that all π-type Slodkowski spectra $\sigma_{\pi,n}(\mathfrak{X}, \mathfrak{d})$ [**3, 4**] of a nonnegative parametrized Banach space complex $(\mathfrak{X}, \mathfrak{d})$ are closed sets.

Now let us formulate the implicit function theorem.

Theorem 2.2. *Let $X, Y, Z \in \mathbf{BS}$, $U \subseteq X$ and $V \subseteq Y$ be open subsets. Let $f : U \to V$ and $g : V \to Z$ be C^2 maps and $u_0 \in U$, $v_0 \in V$ fixed points with $f(u_0) = v_0$. If $g \cdot f$ is a constant map, $\operatorname{im}(f'(u_0)) = \ker(g'(v_0))$ and $\operatorname{im}(g'(v_0))$ is closed, then there exist $\varepsilon > 0$ and $C > 0$ such that for each $v \in V$ with $\|v - v_0\| < \varepsilon$ and $g(v) = g(v_0)$ there is a $u \in U$ with $\|u - u_0\| \leq C\|v - v_0\|$ and $f(u) = v$.*

Note that differential pairs in the assertion appear naturally. Namely, since $g \cdot f$ is a constant map, one follows $g'(f(u)) \cdot f'(u) = 0$, $u \in U$, by virtue of the chain rule. Thereby, $S(u) = (f'(u), g'(f(u)))$, $u \in U$, are differential pairs and $S(u_0)$ is an exact one by assumption. Since

f and g are C^2 maps, one can conclude that $k_{S(u)} < 1$ uniformly on a certain neighborhood of u_0. By the Johnson lemma, all $S(u)$ are exact differential pairs on a sufficiently small neighborhood of u_0 and

$$\begin{aligned} \operatorname{ic}\left(f'\left(u\right)\right) &\leq \left(1-k_{S(u)}\right)^{-1} c_1\left(1+c_2\left\|g'\left(f\left(u\right)\right)-g'\left(v_0\right)\right\|\right), \\ \operatorname{ic}\left(g'\left(f\left(u\right)\right)\right) &\leq \left(1-k_{S(u)}\right)^{-1} c_2\left(1+c_1\left\|f'\left(u\right)-f'\left(u_0\right)\right\|\right). \end{aligned}$$

Thus, by shrinking U if necessary, one may assume that $\operatorname{ic}(f'(u))$ and $\operatorname{ic}(g'(f(u)))$ are bounded uniformly on U. The latter are essential to proceed as in Newton's method and end the proof [**10**, Theorem 1].

3. Small cohomologies of nonassociative Banach algebras. In this section we briefly review the main cochain complex associated with a B-L algebra, and we define small (up to third) dimension cohomologies of a Banach-Jordan algebra having similar meaning as in Lie and associative algebras.

3.1. Banach-Lie algebras. Let $\mathfrak{L}$ be a B-L algebra. Then $\mathfrak{L}$ turns into a Banach $\mathfrak{L}$-module by dint of the adjoint representation $\operatorname{ad}: \mathfrak{L} \to \mathcal{B}(\mathfrak{L})$. The latter involves [**3**], the following cochain Banach space complex

(3.1)

$$0 \longrightarrow \mathfrak{L} \xrightarrow{d^0} \mathcal{B}\left(\mathfrak{L}\right) \xrightarrow{d^1} \mathcal{B}\left(\wedge^2\mathfrak{L},\mathfrak{L}\right) \xrightarrow{d^2} \cdots \xrightarrow{d^{n-1}} \mathcal{B}\left(\wedge^n\mathfrak{L},\mathfrak{L}\right) \xrightarrow{d^n} \cdots$$

with the coboundary operators

$$d^n\omega\left(\underline{a}\right) = \sum_{i=1}^{n+1}\left(-1\right)^{i+1}\left[a_i,\omega\left(\underline{a}_i\right)\right] + \sum_{i<j}\left(-1\right)^{i+j}\omega\left(\left[a_i,a_j\right]\wedge\underline{a}_{i,j}\right),$$

where $\omega \in \mathcal{B}(\wedge^n\mathfrak{L},\mathfrak{L})$, $\underline{a} = a_1 \wedge \cdots \wedge a_{n+1}$. The cohomology spaces $\ker(d^n)/\operatorname{im}(d^{n-1})$, $n \in \mathbf{Z}_+$, of the cochain complex (3.1) are called *cohomology groups of a* B-L *algebra* $\mathfrak{L}$ and are denoted by $H^n(\mathfrak{L})$. Since we are interested in small dimension cohomologies, it is reasonable to have a more explicit description of the relevant coboundary operators. So, $d^0x(a) = [x,a]$, $x \in \mathfrak{L}$, thereby $d^0x = \operatorname{ad}(x)$. For d^1, we have

$$\begin{aligned} d^1T\left(a_1\wedge a_2\right) &= \left[a_1,Ta_2\right]-\left[a_2,Ta_1\right]-T\left[a_1,a_2\right] \\ &= -T\left[a_1,a_2\right]+\left[a_1,Ta_2\right]+\left[Ta_1,a_2\right], \end{aligned}$$

whence $\ker(d^1)$ is the subspace in $\mathcal{B}(\mathfrak{L})$ of all Lie derivations. Further,

$$\begin{aligned} d^2\omega\,(a_1 \wedge a_2 \wedge a_3) &= [a_1, \omega\,(a_2 \wedge a_3)] - [a_2, \omega\,(a_1 \wedge a_3)] + [a_3, \omega\,(a_1 \wedge a_2)] \\ &\quad - \omega\,([a_1, a_2] \wedge a_3) + \omega\,([a_1, a_3] \wedge a_2) - \omega\,([a_2, a_3] \wedge a_1)\,. \end{aligned}$$

Thus, $H^0(\mathfrak{L})$ is the center of the B-L algebra $\mathfrak{L}$, $H^1(\mathfrak{L})$ is the quotient of all bounded Lie derivations modulo the inner derivations of $\mathfrak{L}$.

3.2. Banach-Jordan algebras. Now let $\mathfrak{A}$ be a commutative nonassociative Banach algebra (that is, $I = \{xy - yx\}$ is the set of identities, see subsection 2.3), and let $\varphi_z : \mathfrak{A} \to \mathfrak{A}$ be a mapping given by the rule $\varphi_z(x) = (x^2 z)x - x^2(zx)$, where $z \in \mathfrak{A}$. Note that $\{\varphi_z : z \in \mathfrak{A}\} \subseteq \mathcal{P}_3(\mathfrak{A}, \mathfrak{A})$. Indeed, it is beyond a doubt $f_z : \mathfrak{A}^3 \to \mathfrak{A}$, $f_z(x, y, t) = ((xy)z)t - (xy)(zt)$, is a bounded 3-linear map and $f_z \cdot d_3 = \varphi_z$, thereby φ_z is a homogeneous polynomial of degree 3 (see subsection 2.2). By Lemma 2.1, the map $\varphi \mapsto \widetilde{\varphi}$ implements a topological isomorphism $\mathcal{P}_3(\mathfrak{A}, \mathfrak{A}) \to \mathcal{B}(\vee^3\mathfrak{A}, \mathfrak{A})$ between the homogeneous polynomials of degree 3 and bounded 3-linear maps. Let us calculate $\widetilde{\varphi}_z \in \mathcal{B}(\vee^3\mathfrak{A}, \mathfrak{A})$. Taking into account that $f_z(x, y, t)$ is symmetric with respect to the first two variables x and y, one follows that

$$\begin{aligned} 3\widetilde{\varphi}_z\,(x \vee y \vee t) &= f_z\,(x, y, t) + f_z\,(t, y, x) + f_z\,(x, t, y) \\ &= ((xy)\,z)\,t + ((ty)\,z)\,x + ((xt)\,z)\,y \\ &\quad - (xy)\,(zt) - (ty)\,(zx) - (xt)\,(zy)\,. \end{aligned}$$

Now assume that $\mathfrak{A} = \mathfrak{J}$ is a B-J algebra. Then all homogeneous polynomials φ_z, $z \in \mathfrak{J}$, are vanishing on $\mathfrak{J}$. It follows that $\widetilde{\varphi}_z(x \vee y \vee t) = 0$ for all $x, y, x, t \in \mathfrak{J}$. Consequently,

$$((ab)\,c)\,d + ((ad)\,c)\,b + a\,((bd)\,c) = (ab)\,(cd) + (ac)\,(bd) + (ad)\,(bc)\,, \tag{3.2}$$

for all $a, b, c, d \in \mathfrak{J}$. Note that all variables on the righthand side are symmetrically situated, thereby so might be on the left one. By permuting the variables a and c, and then b and c, we deduce the following relations:

(3.3)

$$\begin{aligned} ((ab)\,c)\,d + ((ad)\,c)\,b + a\,((bd)\,c) &= (a\,(bc))\,d + (a\,(bd))\,c + (a\,(cd))\,b \\ &= a\,(b\,(cd)) + ((ac)\,b)\,d + ((ad)\,b)\,c, \end{aligned}$$

$$(3.4)\quad (a(bc))d+(a(bd))c+(a(cd))b=(ab)(cd)+(ac)(bd)+(ad)(bc)\,.$$

The identities (3.2)–(3.4) are well known in the theory of Jordan algebras.

The latter equality in (3.3) is equivalent to the following

$$a(b(cd))-(a(cd))b=(a(bc))d-((ac)b)d+(a(bd))c-((ad)b)c$$

or in an operator form

$$(3.5)\qquad [L_a,L_b](cd)=([L_a,L_b]c)d+c([L_a,L_b]d)\,.$$

Thus, all operators $[L_a,L_b]$ are derivations of the Jordan algebra $\mathfrak{J}$. Moreover, from the identity (3.4) we derive (by setting $c=d$) the following formulae

$$(3.6)\qquad 2(a(bc))c+\left(ac^2\right)b=(ab)c^2+2(ac)(bc)\,.$$

The latter will be used below in Lemma 3.1. It is also useful to have an operator version of the identity (3.4) itself. It is the following

$$(3.7)\qquad [L_d,L_{bc}]+[L_c,L_{bd}]+[L_b,L_{cd}]=0.$$

Now we introduce a cochain complex associated with a B-J algebra. Let $\mathfrak{J}$ be a B-J algebra. Consider the following sequence of Banach space operators:
(3.8)

$$0\longrightarrow\mathfrak{J}^{\widehat{\otimes}3}\stackrel{\delta^0}{\longrightarrow}\mathfrak{J}^{\widehat{\otimes}2}\stackrel{\delta^1}{\longrightarrow}\mathcal{B}(\mathfrak{J})\stackrel{\delta^2}{\longrightarrow}\mathcal{B}\left(\vee^2\mathfrak{J},\mathfrak{J}\right)\stackrel{\delta^3}{\longrightarrow}\mathcal{B}(\mathfrak{J},\mathcal{P}_3(\mathfrak{J},\mathfrak{J}))\,,$$

where

$$\begin{aligned}
\delta^0(a\otimes b\otimes c)&=a\otimes bc+b\otimes ac+c\otimes ab,\\
\delta^1(a\otimes b)&=[L_a,L_b]\,,\\
\left(\delta^2T\right)(a\vee b)&=aTb+bTa-T(ab)\,,\\
\left(\delta^3\omega\right)(a)b&=\omega\left(ab\vee b^2\right)+\omega(a\vee b)b^2+(ab)\omega(b\vee b)\\
&\quad-\omega\left(ab^2\vee b\right)-\omega\left(a\vee b^2\right)b-(a\omega(b\vee b))b.
\end{aligned}$$

To be correct, first note that δ^2T is a bounded symmetric bilinear map, for $ab=ba$. Further, note that $(\delta^3\omega)(a)\in\mathcal{P}_3(\mathfrak{J},\mathfrak{J})$, that is,

$(\delta^3\omega)(a)$ is a homogeneous polynomial of degree 3. Indeed, one needs (see subsection 2.2) to prove that $(\delta^3\omega)(a)(b) = \zeta_a(b,b,b)$ for a certain $\zeta_a \in \mathcal{B}(\mathfrak{J}^{\widehat{\otimes}3}, \mathfrak{J})$. We set

$$\begin{aligned}\zeta_a\left(x_1,x_2,x_3\right) &= \omega\left(ax_1 \vee x_2x_3\right) + \omega\left(a \vee x_1\right)\left(x_2x_3\right) + \left(ax_1\right)\omega\left(x_2 \vee x_3\right) \\ &\quad - \omega\left(a\left(x_1x_2\right) \vee x_3\right) - \omega\left(a \vee \left(x_1x_2\right)\right)x_3 - \left(a\omega\left(x_1 \vee x_2\right)\right)x_3.\end{aligned}$$

It is beyond a doubt ζ_a is a bounded 3-linear map and $\zeta_a(x,x,x) = (\delta^3\omega)(a)(x)$; therefore, $(\delta^3\omega)(a)$ is a continuous polynomial of degree 3.

Lemma 3.1. $\delta^n\delta^{n-1} = 0$ *for all* n, $1 \le n \le 3$.

Proof. First, note that the equality $\delta^1\delta^0 = 0$ follows from (3.7). Moreover, using (3.5), we conclude that $\delta^2\delta^1(a \otimes b) = 0$ for all $a, b \in \mathfrak{J}$, whence $\delta^2\delta^1 = 0$.

Now take $T \in \mathcal{B}(\mathfrak{J})$, and let $\omega = \delta^2 T \in \mathcal{B}(\vee^2\mathfrak{J}, \mathfrak{J})$. We have to prove that $\delta^3\omega = 0$. Take $a, b \in \mathfrak{J}$. Then

$$\begin{aligned}\omega\left(ab \vee b^2\right) &= (ab)\,Tb^2 + b^2T\,(ab) - T\left((ab)\,b^2\right), \\ \omega\,(a \vee b)\,b^2 &= (aTb)\,b^2 + (bTa)\,b^2 - b^2T\,(ab), \\ (ab)\,\omega\,(b \vee b) &= 2\,(ab)\,(bTb) - (ab)\,Tb^2, \\ -\omega\left(ab^2 \vee b\right) &= -\left(ab^2\right)Tb - bT\left(ab^2\right) + T\left(\left(ab^2\right)b\right), \\ -\omega\left(a \vee b^2\right)b &= -\left(aTb^2\right)b - \left(b^2Ta\right)b + bT\left(ab^2\right), \\ -\left(a\omega\,(b \vee b)\right)b &= -2\left(a\,(bTb)\right)b + \left(aTb^2\right)b.\end{aligned}$$

Taking into account that $(ab)b^2 = (ab^2)b$, $(bTa)b^2 = (b^2Ta)b$, we derive from the above equalities that $(\delta^3\omega)(a)b = (aTb)b^2 + 2(ab)(bTb) - (ab^2)Tb - 2(a(bTb))b$. But the latter is vanishing out of (3.6), that is, $\delta^3\omega = 0$. Consequently, $\delta^3\delta^2 = 0$. $\square$

Thus, the sequence (3.8) is a cochain Banach space complex by virtue of Lemma 3.1. Their cohomology spaces are called small cohomologies of B-J algebra $\mathfrak{J}$ and denoted by $H^n(\mathfrak{J})$, $n = 0,1,2,3$. Thus $H^0(\mathfrak{J})$ is the set of all absolutely convergent series $\sum_{n\in\mathbf{N}} a_n \otimes b_n \otimes c_n \in \mathfrak{J}^{\widehat{\otimes}3}$ such that $\sum_{n\in\mathbf{N}} a_n \otimes b_nc_n = -\sum_{n\in\mathbf{N}} b_n \otimes a_nc_n - \sum_{n\in\mathbf{N}} c_n \otimes a_nb_n$. In particular,

$a\otimes a\otimes a\in H^0(\mathfrak{J})$ whenever $a^2=0$. The first cohomology $H^1(\mathfrak{J})$ is the quotient space of all absolutely convergent series $\sum_{n\in\mathbf{N}} a_n\otimes b_n\in\mathfrak{J}\widehat{\otimes}\mathfrak{J}$ such that $\sum_{n\in\mathbf{N}} a_n(xb_n)=\sum_{n\in\mathbf{N}}(a_nx)b_n$, $x\in\mathfrak{J}$, modulo all series $\sum_{n\in\mathbf{N}} a_n\otimes b_nc_n+b_n\otimes a_nc_n+c_n\otimes a_nb_n$ with $\sum_{n\in\mathbf{N}}\|a_n\|\|b_n\|\|c_n\|<\infty$. Further, $H^2(\mathfrak{J})$ is the quotient of all derivations of $\mathfrak{J}$ modulo derivations of the form $\sum_{n\in\mathbf{N}}[L_{a_n},L_{b_n}]$ with $\sum_{n\in\mathbf{N}}\|a_n\|\|b_n\|<\infty$.

4. The function f**.** We intend to apply the implicit function theorem to the perturbation problem for nonassociative Banach algebras. Therefore, we might suggest the main functions f and g (see Theorem 2.2) within that problem. The latter is the main goal of this and the next sections.

Let $\mathfrak{X}\in\mathbf{BS}$, and let $*:\mathfrak{X}\widehat{\otimes}\mathfrak{X}\to\mathfrak{X}$, $x\otimes y\mapsto x*y$, be a bounded linear operator. Let us introduce the following function

$$f:\mathcal{B}(\mathfrak{X})^{-1}\longrightarrow\mathcal{B}\left(\mathfrak{X}\widehat{\otimes}\mathfrak{X},\mathfrak{X}\right),\quad f(T)(x\otimes y)=T^{-1}(Tx*Ty),$$

where $x,y\in\mathfrak{X}$.

Lemma 4.1. *The map* f *is a* C^∞ *map, and its derived map*

$$f':\mathcal{B}(\mathfrak{X})^{-1}\longrightarrow\mathcal{B}\left(\mathcal{B}(\mathfrak{X}),\mathcal{B}\left(\mathfrak{X}\widehat{\otimes}\mathfrak{X},\mathfrak{X}\right)\right)$$

is acting by the rule

$$\begin{aligned}&(f'(T)G)(x\otimes y)\\&\qquad=-T^{-1}GT^{-1}(Tx*Ty)+T^{-1}(Gx*Ty)+T^{-1}(Tx*Gy)\end{aligned}$$

for all $G\in\mathcal{B}(\mathfrak{X})$ *and* $x,y\in\mathfrak{X}$*. In particular,*

$$(f'(1_{\mathfrak{X}})G)(x\otimes y)=-G(x*y)+Gx*y+x*Gy.$$

Proof. Consider the following maps

$$\begin{aligned}&\zeta_1:\mathcal{B}(\mathfrak{X})^{-1}\longrightarrow\mathcal{B}(\mathfrak{X}),\quad\zeta_1(T)=T^{-1};\\&\zeta_2:\mathcal{B}(\mathfrak{X})\times\mathcal{B}(\mathfrak{X})\longrightarrow\mathcal{B}\left(\mathfrak{X}\widehat{\otimes}\mathfrak{X},\mathfrak{X}\right),\quad\zeta_2(T,G)(a\otimes b)=Ta*Gb;\\&\zeta_3:\mathcal{B}(\mathfrak{X})\times\mathcal{B}\left(\mathfrak{X}\widehat{\otimes}\mathfrak{X},\mathfrak{X}\right)\longrightarrow\mathcal{B}\left(\mathfrak{X}\widehat{\otimes}\mathfrak{X},\mathfrak{X}\right),\quad\zeta_3(T,\omega)=T\cdot\omega.\end{aligned}$$

It is well known [**2**, Theorem 5.4.3] that ζ_1 is a C^∞ map. Further, ζ_2 and ζ_3 are continuous bilinear maps; therefore [**2**, Proposition 5.4.1], they are C^∞ maps too.

Further, note that $f = \zeta_3 \cdot (\zeta_1 \times (\zeta_2 \cdot d_2))$, where $d_2 : \mathcal{B}(\mathfrak{X}) \to \mathcal{B}(\mathfrak{X})^2$ is the diagonal map, and $\zeta_1 \times (\zeta_2 \cdot d_2) : \mathcal{B}(\mathfrak{X})^{-1} \to \mathcal{B}(\mathfrak{X}) \times \mathcal{B}(\mathfrak{X}\widehat{\otimes}\mathfrak{X}, \mathfrak{X})$, $T \mapsto (\zeta_1(T), \zeta_2(T,T))$. Now, since f is a composition of C^∞ maps, it is automatically a C^∞ map. Moreover,

$$\begin{aligned} f'(T)\,G &= \zeta_3'\left(\zeta_1(T), (\zeta_2 \cdot d_2)(T)\right) \cdot \left(\zeta_1'(T) \cdot G, \zeta_2'(d_2(T)) \cdot d_2(G)\right) \\ &= \zeta_3\left(\zeta_1'(T) \cdot G, (\zeta_2 \cdot d_2)(T)\right) + \zeta_3\left(\zeta_1(T), \zeta_2'(d_2(T)) \cdot d_2(G)\right) \\ &= -\zeta_3\left(T^{-1} \cdot G \cdot T^{-1}, \zeta_2(T,T)\right) + \zeta_3\left(T^{-1}, \zeta_2'(T,T) \cdot (G,G)\right) \\ &= -T^{-1} \cdot G \cdot T^{-1} \cdot \zeta_2(T,T) + T^{-1} \cdot \zeta_2(G,T) + T^{-1} \cdot \zeta_2(T,G), \end{aligned}$$

which in turn implies that

$$\begin{aligned} &(f'(T)\,G)(x \otimes y) \\ &\qquad = -T^{-1}GT^{-1}(Tx * Ty) + T^{-1}(Gx * Ty) + T^{-1}(Tx * Gy) \end{aligned}$$

for all $G \in \mathcal{B}(\mathfrak{X})$ and $x, y \in \mathfrak{X}$. ◻

Assume that $\mathfrak{X} = \mathfrak{L}$ is a B-L algebra and $*$ is the Lie multiplication or Lie brackets $[\cdot,\cdot]$. Then $f(\mathcal{B}(\mathfrak{L})^{-1}) \subseteq \mathcal{B}(\wedge^2\mathfrak{L}, \mathfrak{L})$ and $f(T)(x \wedge y) = T^{-1}[Tx, Ty]$. As follows from Lemma 4.1, $f'(1_{\mathfrak{L}}) : \mathcal{B}(\mathfrak{L}) \to \mathcal{B}(\wedge^2\mathfrak{L}, \mathfrak{L})$ is merely the first coboundary operator d^1 of the main cochain complex (3.8) defining the cohomologies of the B-L algebra $\mathfrak{L}$.

Further, assume that $\mathfrak{X} = \mathfrak{J}$ is a B-J algebra and $*$ is the Jordan multiplication $\cdot$. Then $f(\mathcal{B}(\mathfrak{J})^{-1}) \subseteq \mathcal{B}(\vee^2\mathfrak{J}, \mathfrak{J})$ and $f(T)(x \vee y) = T^{-1}(Tx \cdot Ty)$. By Lemma 4.1, $f'(1_{\mathfrak{J}}) : \mathcal{B}(\mathfrak{J}) \to \mathcal{B}(\vee^2\mathfrak{J}, \mathfrak{J})$, $(f'(1_{\mathfrak{J}})G)(x \vee y) = -G(x \cdot y) + Gx \cdot y + x \cdot Gy$, is the second coboundary operator δ^2 of the main complex (3.8) defining the small cohomologies of the B-J algebra $\mathfrak{J}$.

5. The function g. Now we introduce the next required function g. The function g strongly depends on the choice of a nonassociative Banach algebra. Therefore, we consider the B-J algebra and B-L algebra cases separately.

5.1. The function g for a B-J algebra. Now let $\mathfrak{J}$ be a B-J algebra. We define a map

$$g : \mathcal{B}\left(\vee^2\mathfrak{J}, \mathfrak{J}\right) \longrightarrow \mathcal{B}\left(\mathfrak{J}, \mathcal{P}_3\left(\mathfrak{J}, \mathfrak{J}\right)\right),$$
$$g\left(\omega\right)\left(a\right)\left(b\right) = \omega\left(\omega\left(a \vee b\right) \vee \omega\left(b \vee b\right)\right) - \omega\left(\omega\left(a \vee \omega\left(b \vee b\right)\right) \vee b\right),$$

where $a, b \in \mathfrak{J}$. Undoubtedly, $g^{-1}(0) \subseteq \mathcal{B}(\vee^2\mathfrak{J}, \mathfrak{J})$ is a subset of all (jointly) continuous Jordan multiplications on $\mathfrak{J}$.

Lemma 5.1. *The equality $g \cdot f = 0$ holds.*

Proof. Take $T \in \mathcal{B}(\mathfrak{J})^{-1}$. Then $f(T)(f(T)(a \vee b) \vee f(T)(b \vee b)) = T^{-1}((TaTb)(Tb)^2)$ and $f(T)(f(T)(a \vee f(T)(b \vee b)) \vee b) = T^{-1}((Ta(Tb)^2)Tb)$. It follows that

$$Tg\left(f\left(T\right)\right)\left(a\right)\left(b\right) = \left(TaTb\right)\left(Tb\right)^2 - \left(Ta\left(Tb\right)^2\right)Tb = 0$$

due to the Jordan identity. Consequently, $g(f(T)) = 0$. □

By Lemma 5.1, $\mathrm{im}(f) \subseteq g^{-1}(0)$ and $\mathrm{im}(f)$ consists of those continuous Jordan multiplications on $\mathfrak{J}$ topologically isomorphic with the original B-J algebra $\mathfrak{J}$ (with respect to the multiplication $f(1_{\mathfrak{J}})$). Indeed, if $T \in \mathcal{B}(\mathfrak{J})^{-1}$, then $\mathfrak{J}$ furnished with the Jordan multiplication $a * b = T^{-1}(TaTb)$ is topologically isomorphic with $\mathfrak{J}$ by means of the invertible operator T.

Lemma 5.2. *The map g is a C^∞ map and its derived map*

$$g' : \mathcal{B}\left(\vee^2\mathfrak{J}, \mathfrak{J}\right) \longrightarrow \mathcal{B}\left(\mathcal{B}\left(\vee^2\mathfrak{J}, \mathfrak{J}\right), \mathcal{B}\left(\mathfrak{J}, \mathcal{P}_3\left(\mathfrak{J}, \mathfrak{J}\right)\right)\right)$$

is acting by the rule

$$\begin{aligned}\left(g'\left(\omega\right)\tau\right)\left(a\right)\left(b\right) = {} & \tau\left(\omega\left(a \vee b\right) \vee \omega\left(b \vee b\right)\right) + \omega\left(\tau\left(a \vee b\right) \vee \omega\left(b \vee b\right)\right) \\ & + \omega\left(\omega\left(a \vee b\right) \vee \tau\left(b \vee b\right)\right) - \tau\left(\omega\left(a \vee \omega\left(b \vee b\right)\right) \vee b\right) \\ & - \omega\left(\tau\left(a \vee \omega\left(b \vee b\right)\right) \vee b\right) - \omega\left(\omega\left(a \vee \tau\left(b \vee b\right)\right) \vee b\right),\end{aligned}$$

where $a, b \in \mathfrak{J}$. In particular, if ω is the Jordan multiplication $\cdot$, then

$$\begin{aligned}\left(g'\left(\cdot\right)\tau\right)\left(a\right)\left(b\right) = {} & \tau\left(\left(ab\right) \vee b^2\right) + \tau\left(a \vee b\right)b^2 + \left(ab\right)\tau\left(b \vee b\right) \\ & - \tau\left(ab^2 \vee b\right) - \tau\left(a \vee b^2\right)b - \left(a\tau\left(b \vee b\right)\right)b.\end{aligned}$$

Proof. Let $g_1 : \mathcal{B}(\vee^2\mathfrak{J}, \mathfrak{J}) \to \mathcal{B}(\mathfrak{J}, \mathcal{P}_3(\mathfrak{J}, \mathfrak{J}))$ be a mapping given by the rule $g_1(\omega)(a)(b) = \omega(\omega(a \vee b) \vee \omega(b \vee b))$. Obviously, g_1 splits into the decomposition $\eta \cdot t$, where

$$t : \mathcal{B}\left(\vee^2\mathfrak{J}, \mathfrak{J}\right) \longrightarrow \mathcal{B}\left(\mathfrak{J}^{\widehat{\otimes} 4}, \mathfrak{J}\right),$$
$$t\left(\omega\right)\left(a \otimes b \otimes c \otimes d\right) = \omega\left(\omega\left(a \vee b\right) \vee \omega\left(c \vee d\right)\right),$$

and $\eta : \mathcal{B}(\mathfrak{J}^{\widehat{\otimes} 4}, \mathfrak{J}) \to \mathcal{B}(\mathfrak{J}, \mathcal{P}_3(\mathfrak{J}, \mathfrak{J}))$, $\eta(\beta)(a)(b) = \beta(a \otimes b \otimes b \otimes b)$. We also introduce a map $\zeta : \mathcal{B}(\vee^2\mathfrak{J}, \mathfrak{J})^3 \to \mathcal{B}(\mathfrak{J}^{\widehat{\otimes} 4}, \mathfrak{J})$ by setting $\zeta(\omega_1, \omega_2, \omega_3)(a \otimes b \otimes c \otimes d) = \omega_1(\omega_2(a \vee b) \vee \omega_3(c \vee d))$. Then ζ is a C^∞ map, for ζ is a bounded 3-linear map. Moreover, $t = \zeta \cdot d_3$, where $d_3 : \mathcal{B}(\vee^2\mathfrak{J}, \mathfrak{J}) \to \mathcal{B}(\vee^2\mathfrak{J}, \mathfrak{J})^3$ is the diagonal map. Using the chain rule, we assert that t is a C^∞ map and $t'(\omega)\tau = \zeta'(\omega, \omega, \omega)(\tau, \tau, \tau) = \zeta(\tau, \omega, \omega) + \zeta(\omega, \tau, \omega) + \zeta(\omega, \omega, \tau)$. Further, taking into account that η is a bounded linear map, we conclude that g_1 is a C^∞ map and

$$\begin{aligned}
\left(g_1'\left(\omega\right)\tau\right)\left(a\right)\left(b\right) &= \eta\left(t'\left(\omega\right)\tau\right)\left(a\right)\left(b\right) \\
&= \left(t'\left(\omega\right)\tau\right)\left(a \otimes b \otimes b \otimes b\right) \\
&= \zeta\left(\tau, \omega, \omega\right)\left(a \otimes b \otimes b \otimes b\right) \\
&\quad + \zeta\left(\omega, \tau, \omega\right)\left(a \otimes b \otimes b \otimes b\right) \\
&\quad + \zeta\left(\omega, \omega, \tau\right)\left(a \otimes b \otimes b \otimes b\right) \\
&= \tau\left(\omega\left(a \vee b\right) \vee \omega\left(b \vee b\right)\right) \\
&\quad + \omega\left(\tau\left(a \vee b\right) \vee \omega\left(b \vee b\right)\right) \\
&\quad + \omega\left(\omega\left(a \vee b\right) \vee \tau\left(b \vee b\right)\right).
\end{aligned}$$

Using the same argument, one can prove that

$$g_2 : \mathcal{B}\left(\vee^2\mathfrak{J}, \mathfrak{J}\right) \longrightarrow \mathcal{B}\left(\mathfrak{J}, \mathcal{P}_3\left(\mathfrak{J}, \mathfrak{J}\right)\right),$$
$$g_2\left(\omega\right)\left(a\right)\left(b\right) = \omega\left(\omega\left(a \vee \omega\left(b \vee b\right)\right) \vee b\right),$$

is a C^∞ map and

$$\begin{aligned}
\left(g_2'\left(\omega\right)\tau\right)\left(a\right)\left(b\right) = \tau\left(\omega\left(a \vee \omega\left(b \vee b\right)\right) \vee b\right) + \omega\left(\tau\left(a \vee \omega\left(b \vee b\right)\right) \vee b\right) \\
+ \omega\left(\omega\left(a \vee \tau\left(b \vee b\right)\right) \vee b\right).
\end{aligned}$$

Consequently, $g = g_1 - g_2$ is a C^∞ map and $(g'(\omega)\tau)(a)(b) = (g_1'(\omega)\tau)(a)(b) - (g_2'(\omega)\tau)(a)(b)$. □

As follows from Lemma 5.2, the derived map $g'(\cdot) : \mathcal{B}(\vee^2\mathfrak{J},\mathfrak{J}) \to \mathcal{B}(\mathfrak{J},\mathcal{P}_3(\mathfrak{J},\mathfrak{J}))$ is merely the coboundary operator δ^3 of the main complex (3.8) associated by the B-J algebra $\mathfrak{J}$.

5.2. The function g for a B-L **algebra.** The same argument can be carried out for a B-L algebra $\mathfrak{L}$. Namely, we introduce a function

$$g : \mathcal{B}\left(\wedge^2\mathfrak{L},\mathfrak{L}\right) \longrightarrow \mathcal{B}\left(\wedge^3\mathfrak{L},\mathfrak{L}\right),$$
$$g(\omega)(a\wedge b\wedge c) = \omega(a\wedge\omega(b\wedge c)) - \omega(b\wedge\omega(a\wedge c)) + \omega(c\wedge\omega(a\wedge b)),$$

where $a,b,c \in \mathfrak{L}$. Note that $g^{-1}(0) \subseteq \mathcal{B}(\wedge^2\mathfrak{L},\mathfrak{L})$ is a subset of all (jointly) continuous Lie multiplications on $\mathfrak{L}$. Obviously, $g\cdot f = 0$, thereby $\mathrm{im}(f) \subseteq g^{-1}(0)$ and $\mathrm{im}(f)$ consist of those continuous Lie multiplications on $\mathfrak{L}$ topologically isomorphic with the original B-L algebra $\mathfrak{L}$ (with respect to the multiplication $f(1_{\mathfrak{L}})$). Moreover, the map g is a C^∞ map and its derived map

$$g' : \mathcal{B}\left(\wedge^2\mathfrak{L},\mathfrak{L}\right) \longrightarrow \mathcal{B}\left(\mathcal{B}\left(\wedge^2\mathfrak{L},\mathfrak{L}\right),\mathcal{B}\left(\wedge^3\mathfrak{L},\mathfrak{L}\right)\right)$$

is acting by the rule

$$\begin{aligned}&(g'(\omega)\tau)(a\wedge b\wedge c)\\&\qquad= \tau(a\wedge\omega(b\wedge c)) - \tau(b\wedge\omega(a\wedge c)) + \tau(c\wedge\omega(a\wedge b))\\&\qquad\quad + \omega(a\wedge\tau(b\wedge c)) - \omega(b\wedge\tau(a\wedge c)) + \omega(c\wedge\tau(a\wedge b)),\end{aligned}$$

where $a,b,c \in \mathfrak{L}$. In particular, if ω is the Lie multiplication $[\cdot,\cdot]$, then

$$\begin{aligned}&(g'([\cdot,\cdot])\tau)(a\wedge b\wedge c)\\&\qquad= [a,\tau(b\wedge c)] - [b,\tau(a\wedge c)] + [c,\tau(a\wedge b)] - \tau([a,b]\wedge c)\\&\qquad\quad + \tau([a,c]\wedge b) - \tau([b,c]\wedge a).\end{aligned}$$

Thus, the derived map $g'([\cdot,\cdot]) : \mathcal{B}(\wedge^2\mathfrak{L},\mathfrak{L}) \to \mathcal{B}(\wedge^3\mathfrak{L},\mathfrak{L})$ coincides with the coboundary operator d^2 of the main cochain complex (3.1) associated by the B-L algebra $\mathfrak{L}$.

6. The main result. Now we are in a position to formulate and prove the main result of this paper.

Theorem 6.1. *If either* $\mathfrak{A} = \mathfrak{J}$ *is a Banach-Jordan algebra with* $H^3(\mathfrak{J}) = 0$ *and the closed image* $\mathrm{im}(\delta^3)$ *or* $\mathfrak{A} = \mathfrak{L}$ *is a Banach-Lie algebra with* $H^2(\mathfrak{L}) = 0$ *and the closed image* $\mathrm{im}(d^2)$*, then* $\mathfrak{A}$ *is a strongly stable algebra.*

Proof. We prove the assertion for a B-J algebra $\mathfrak{J}$. The same argument can be applied to the B-L algebra case. Let $X = \mathcal{B}(\mathfrak{J})$, $Y = \mathcal{B}(\vee^2\mathfrak{J}, \mathfrak{J})$, $Z = \mathcal{B}(\mathfrak{J}, \mathcal{P}_3(\mathfrak{J}, \mathfrak{J}))$ be Banach spaces, and let $U = \mathcal{B}(\mathfrak{J})^{-1}$ be an open subset in X. Consider functions $f : U \to Y$ and $g : Y \to Z$ proposed in Sections 4 and 5. By Lemma 5.1, $g \cdot f$ is a constant map. Moreover, $f(1_{\mathfrak{J}})$ is the Jordan multiplication $\cdot$ in $\mathfrak{J}$. By Lemma 4.1, the derived map $f'(1_{\mathfrak{J}}) : X \to Y$ is the second coboundary operator δ^2. Moreover, $g'(\cdot) : Y \to Z$ coincides with the coboundary operator δ^3 by virtue of Lemma 5.2. By assumption, $H^3(\mathfrak{J}) = 0$ and the image $\mathrm{im}(\delta^3)$ is closed, that is, $(f'(1_{\mathfrak{J}}), g'(\cdot))$ is an exact differential pair. Using the implicit function theorem, we infer that there exist $\varepsilon > 0$ and $C > 0$ such that for each $\tau \in \mathcal{B}(\vee^2\mathfrak{J}, \mathfrak{J})$ with $\|\tau - \cdot\| < \varepsilon$ and $g(\tau) = g(\cdot) = 0$ there is a $T \in \mathcal{B}(\mathfrak{J})^{-1}$ with $\|T - 1_{\mathfrak{J}}\| \leq C\|\tau - \cdot\|$ and $f(T) = \tau$. The latter means that a sufficiently close to $\cdot$ Jordan multiplication τ is represented as $\tau(a \vee b) = T^{-1}(Ta \cdot Tb)$ for a certain invertible operator T. Appealing to Definition 2.1, we conclude that $\mathfrak{J}$ is a strongly stable B-J algebra. □

Remark 6.1. Let $\mathfrak{g}$ be a finite-dimensional normed nonassociative algebra. If either $\mathfrak{g}$ is a Jordan algebra with $H^3(\mathfrak{g}) = 0$ or $\mathfrak{g}$ is a Lie algebra with $H^2(\mathfrak{g}) = 0$, then $\mathfrak{g}$ is a stable algebra. In particular, a semi-simple Lie algebra is stable. Indeed, taking into account that $\dim(\mathfrak{g}) < \infty$, we infer that all members of the main complexes (3.1) and (3.8) are finite-dimensional linear spaces; therefore, $\mathrm{im}(d^2)$ and $\mathrm{im}(\delta^3)$ are automatically closed. Finally, it is well known [**6**, 3.5] that $H^2(\mathfrak{g}) = 0$ whenever $\mathfrak{g}$ is a semi-simple Lie algebra. By Theorem 6.1, $\mathfrak{g}$ is a stable B-L algebra (see also [**5**]).

Acknowledgments. I wish to thank Yu. V. Turovskii for useful discussions of results of the paper.

REFERENCES

1. H. Bjar and O.A. Laudal, *Deformation of Lie algebras and Lie algebras of deformations*, Comp. Math. **75** (1990), 69–111.

2. H. Cartan, *Calcul differentiel, formes differentiel*, Hermann, Paris, 1967.

3. A.A. Dosiev, *Spectra of infinite parametrized Banach complexes*, J. Operator Theory **48** (2002), 585–614.

4. ———, *Spectral mapping framework*, Banach Center Publ. **67** (2005), 161–169.

5. M. Gerstenhaber, *On the deformations of rings and algebras*, Annals Math. **79** (1964), 59–103.

6. M. Goto and F.D. Grosshans, *Semisimple Lie algebras*, Lecture Notes Pure Appl. Math. **38**, Springer Verlag, New York, 1978.

7. A.Ya. Helemskii, *The homology of Banach and topological algebras*, Kluwer Acad. Publ. **41**, 1989.

8. K. Jarosz, *Perturbations of Banach algebras*, Springer-Verlag, New York, 1985.

9. B.E. Johnson, *Perturbations of Banach algebras*, Proc. London Math. Soc. **34** (1977), 439–458.

10. I. Raeburn and J.L. Taylor, *Hochschild cohomology and perturbations of Banach algebras*, J. Funct. Anal. **25** (1977), 258–266.

Middle East Technical University NCC, Guzelyurt KKTC, Mersin 10, Turkey
Email address: dosiev@yahoo.com