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MULTIPLE POSITIVE SOLUTIONS FOR A SECOND ORDER STURM-LIOUVILLE BOUNDARY VALUE PROBLEM WITH A p-LAPLACIAN VIA VARIATIONAL METHODS

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ABSTRACT. In this paper, we investigate the positive solutions of a second order Sturm-Liouville boundary value problem with a p-Laplacian. By using critical point theory the existence results of multiple positive solutions are obtained.

1. Introduction. In recent years, a great deal of work has been done in the study of the existence of multiple positive solutions for two-point boundary value problems, by which a number of physical and biological phenomena are described. For the background and results, we refer the reader to the monograph by Agarwal, Mawhin, Rabinowitz et al. and some recent contributions such as [**2, 6, 7, 11, 13, 14, 15**].

Various fixed point theorems are applied to get interesting results, see for example, [**6, 7, 11, 13** and the references therein]. Among them, the Krasnosel'skii's fixed point theorem, the Leggett-Williams fixed point theorem, a five functionals fixed point theorem, and the fixed point theorem in cones are very frequently used.

By using the fixed point theorem in cones, Agarwal et al. [**1**], Anuradha, Hai and Shivagi [**3**], Erbe and Wang [**11**] and Ge and Ren [**12**] have studied the existence of positive solutions for the second-order Sturm-Liouville boundary value problem

$$(1.1)\quad \begin{cases} (p(t)x'(t))' + \lambda f(t, x(t)) = 0 & t \in [0,1], \\ \alpha_1 x(0) - \beta_1 p(0)x'(0) = 0 = \alpha_2 x(1) + \beta_2 p(1)x'(1). \end{cases}$$

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By using critical point theory, Averna [**4, 5**], Bonanno [**8, 9**], Ricceri [**16, 17**] have studied the existence of multiple solutions for the equation

$$(\Phi_p(x'(t)))' + \lambda f(t, x(t)) = 0, \quad t \in [0,1],$$

where $\Phi_p(x) = |x|^{p-2}x$, $p > 1$, with the Dirichlet, Neumann and mixed boundary conditions. In [**19**], Tian and Ge have studied the boundary value problem with a p-Laplacian

$$(1.2) \qquad \begin{cases} -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = \lambda f(t, x(t)) & t \in [a,b], \\ \alpha x'(a) - \beta x(a) = A, \quad \gamma x'(b) + \sigma x(b) = B. \end{cases}$$

By using the three-critical-point theorem [**4**], the existence of three solutions was obtained.

On the other hand, to the best of our knowledge, few authors have studied the existence of multiple positive solutions for second order Sturm-Liouville boundary value problems by using variational methods. As a result, the goal of this paper is to fill the gap in this area.

Motivated by the above results, in this paper we study the existence of multiple positive solutions for the following second order Sturm-Liouville boundary value problem (BVP) with a p-Laplacian

$$(1.3) \qquad \begin{cases} -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = f(t, x(t)) & t \in [a,b], \\ \alpha x'(a) - \beta x(a) = A, \quad \gamma x'(b) + \sigma x(b) = B, \end{cases}$$

where $p > 1$, $\Phi_p(x) := |x|^{p-2}x$, $\rho, s \in L^\infty[a,b]$ with $\operatorname{ess\,inf}_{[a,b]} \rho > 0$ and $\operatorname{ess\,inf}_{[a,b]} s > 0$, $0 < \rho(a)$, $\rho(b) < \infty$, $A \leq 0$, $B \geq 0$, $\alpha, \beta, \gamma, \sigma > 0$, $f \in C([a,b] \times [0,+\infty), [0,+\infty))$, $f(t,0) \not\equiv 0$ for $t \in [a,b]$.

Our aim of this paper is to apply critical point theory to problem (1.3) and to prove the existence of two positive solutions. The character of this paper is as follows: we impose some new conditions on the nonlinearity term f, which are different from those in [**1, 3, 11, 12, 19**]. Moreover, the conditions on f are easily verified.

In this paper, we assume that the following conditions hold:

(C1) there exist $\mu > p$, $h \in C([a,b] \times [0,+\infty), [0,+\infty))$, $l \in C([a,b], (0,+\infty))$, $\min_{t \in [a,b]} l(t) > 0$ such that

$$f(t,x) = l(t)\Phi_\mu(x) + h(t,x);$$

(C2) there exist $c \in L^1([a,b],[0,+\infty))$, $d \in C([a,b],[0,+\infty))$ such that

$$h(t,x) \le c(t) + d(t)\Phi_p(x).$$

2. Related lemmas. To begin with, we introduce some notations. Here, and in the sequel, we assume that, $[a,b]$ is a compact real interval, X is a Sobolev space $W^{1,p}([a,b])$ equipped with the norm

$$\|x\| = \left(\int_a^b \rho(t)|x'(t)|^p + s(t)|x(t)|^p \, dt \right)^{1/p},$$

which is clearly equivalent to the usual one; F is the real function

$$F(t,\xi) = \int_0^\xi f(t,x)\, dx.$$

We denote $\|x\|_\infty := \max_{x\in[a,b]} |x(t)|$ to be the norm in $C^0([a,b])$. Moreover, $\|x\|_{L^r}$ stands for the norm in $L^r([a,b])$, $r \in [1,+\infty]$.

Definition 2.1. A function $x \in X$ is said to be a classical solution of BVP (1.3) if x satisfies the equation in (1.3) for all $t \in [a,b]$ and the boundary condition of (1.3). Moreover, x is said to be a positive classical solution of BVP (1.3) if $x(t) \ge 0$, $x(t) \not\equiv 0$, $t \in [a,b]$.

Lemma 2.1. *For $x \in X$, let $x^\pm = \max\{\pm x, 0\}$. Then the following six properties hold*:

(i) $x \in X \Rightarrow x^+, x^- \in X$;

(ii) $x = x^+ - x^-$;

(iii) $\|x^+\|_X \le \|x\|_X$;

(iv) *if (x_n) uniformly converges to x in $C([a,b])$, then (x_n^+) uniformly converges to x^+ in $C([a,b])$*;

(v) $x^+(t)x^-(t) = 0$, $(x^+)'(t)(x^-)'(t) = 0$ *for* $t \in [a,b]$;

(vi) $\Phi_p(x)x^+ = |x^+|^p$, $\Phi_p(x)x^- = -|x^-|^p$.

Lemma 2.2. *If $x \in C([a,b])$ is a classical solution of* BVP

$$\begin{cases} -(\rho(t)\Phi_p(x'(t)))' + s(t)\Phi_p(x(t)) = f(t,x^+(t)) \quad t \in [a,b], \\ \alpha x'(a) - \beta x(a) = A, \quad \gamma x'(b) + \sigma x(b) = B, \end{cases} \tag{2.1}$$

then $x(t) \geq 0$, $x(t) \not\equiv 0$, $t \in [a,b]$ *and hence it is a positive classical solution of* BVP (1.3).

Proof. If $x \in C([a,b])$ is a classical solution of BVP (2.1), by Lemma 2.1 we have

(2.2)
$$\begin{aligned}
0 &= \int_a^b \left[(\rho(t)\Phi_p(x'(t)))' - s(t)\Phi_p(x(t)) + f(t, x^+(t))\right] \times x^-(t)\, dt \\
&= \rho(t)\Phi_p(x'(t))x^-(t)|_a^b \\
&\quad - \int_a^b \left[\rho(t)\Phi_p(x'(t))(x^-)'(t) + s(t)\Phi_p(x(t))x^-(t)\right]\, dt \\
&\quad + \int_a^b f(t, x^+(t))x^-(t)\, dt \\
&\geq \rho(b)\Phi_p\left(\frac{B - \sigma x(b)}{\gamma}\right)x^-(b) - \rho(a)\Phi_p\left(\frac{A + \beta x(a)}{\alpha}\right)x^-(a) \\
&\quad + \int_a^b \left[\rho(t)|(x^-)'(t)|^p + s(t)|x^-(t)|^p\right]\, dt \\
&= \rho(b)\left|\frac{B - \sigma x(b)}{\gamma}\right|^{p-2} \frac{Bx^-(b) + \sigma(x^-(b))^2}{\gamma} \\
&\quad + \rho(a)\left|\frac{A + \beta x(a)}{\alpha}\right|^{p-2} \frac{-Ax^-(a) + \beta(x^-(a))^2}{\alpha} + \|x^-\|_X^p \\
&\geq \|x^-\|_X^p,
\end{aligned}$$

so $x^-(t) = 0$ for $t \in [a,b]$, that is $x(t) \geq 0$ for $t \in [a,b]$. If $x(t) \equiv 0$ for $t \in [a,b]$, the fact $f(t,0) \not\equiv 0$ for $t \in [a,b]$ gives a contradiction. □

Remark 2.1. By Lemma 2.2, in order to find the positive classical solutions of BVP (1.3), it suffices to get classical solutions of (2.1).

For each $x \in X$, put

(2.3)
$$\begin{aligned}
\varphi(x) := \frac{\|x\|_X^p}{p} + \frac{\gamma\rho(b)}{\sigma p}\left|\frac{B - \sigma x(b)}{\gamma}\right|^p + \frac{\alpha\rho(a)}{\beta p}\left|\frac{A + \beta x(a)}{\alpha}\right|^p \\
- \int_a^b \left[F(t, x^+(t)) - f(t,0)x^-(t)\right]\, dt.
\end{aligned}$$

Clearly, φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $x \in X$ is the functional $\varphi'(x) \in X^*$, given by

$$\begin{aligned}\langle \varphi'(x), v\rangle = &\int_a^b [\rho(t)\Phi_p(x'(t))v'(t) + s(t)\Phi_p(x(t))v(t)]\, dt \\ &- \rho(b)\Phi_p\left(\frac{B - \sigma x(b)}{\gamma}\right)v(b) \\ &+ \rho(a)\Phi_p\left(\frac{A + \beta x(a)}{\alpha}\right)v(a) \\ &- \int_a^b f(t, x^+(t))v(t)\, dt\end{aligned} \tag{2.4}$$

for every $v \in X$. By [**19**], $\varphi' : X \to X^*$ is continuous and the critical point of the functional φ is just the solution of BVP (2.1).

Lemma 2.3. *For $x \in X$, then $\|x\|_\infty \leq \Delta \|x\|_X$, where*

$$\Delta = 2^{1/q} \times \max\left\{\frac{1}{(b-a)^{1/p}\left(\operatorname{ess\,inf}_{[a,b]} s\right)^{1/p}}, \frac{(b-a)^{1/q}}{\left(\operatorname{ess\,inf}_{[a,b]} \rho\right)^{1/p}}\right\};$$

here $1/p + 1/q = 1$.

Proof. For $x \in X$, it follows from the mean value theorem that

$$x(\tau) = \frac{1}{b-a}\int_a^b x(\theta)\, d\theta$$

for some $\tau \in [a, b]$. Hence, for $t \in [a, b]$, using the Hölder inequality,

$$\begin{aligned}|x(t)| &= \left|x(\tau) + \int_\tau^t x'(\theta)\, d\theta\right| \\ &\leq \frac{1}{b-a}\int_a^b |x(\theta)|\, d\theta + \int_a^b |x'(\theta)|\, d\theta \\ &\leq (b-a)^{-1/p}\left(\int_a^b |x(\theta)|^p\, d\theta\right)^{1/p}\end{aligned}$$

$$
\begin{aligned}
&+ (b-a)^{1/q}\left(\int_a^b |x'(\theta)|^p d\theta\right)^{1/p} \\
&\le \frac{1}{(b-a)^{1/p}\left(\operatorname{ess\,inf}_{[a,b]} s\right)^{1/p}}\left(\int_a^b s(\theta)|x(\theta)|^p\, d\theta\right)^{1/p} \\
&\quad + \frac{(b-a)^{1/q}}{\left(\operatorname{ess\,inf}_{[a,b]} \rho\right)^{1/p}}\left(\int_a^b \rho(\theta)|x'(\theta)|^p\, d\theta\right)^{1/p} \\
&\le 2^{1/q}\max\left\{\frac{1}{(b-a)^{1/p}\left(\operatorname{ess\,inf}_{[a,b]} s\right)^{1/p}},\right. \\
&\qquad\qquad\qquad \left.\frac{(b-a)^{1/q}}{\left(\operatorname{ess\,inf}_{[a,b]} \rho\right)^{1/p}}\right\}\|x\|_X,
\end{aligned}
$$

which completes the proof. □

Lemma 2.4 [**20**, Theorem 38.A]. *For the functional $F : M \subseteq X \to [-\infty, +\infty]$ with $M \neq \varnothing$, $\min_{u\in M} F(u) = \alpha$ has a solution in case the following hold*:

(i) *X is a real reflexive Banach space*;

(ii) *M is bounded and weak sequentially closed, i.e., by definition, for each sequence (u_n) in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we always have $u \in M$*;

(iii) *F is weak sequentially lower semi-continuous on M.*

Lemma 2.5 [**10**]. *Let E be a Banach space and $\varphi \in C^1(E, R)$ satisfy the Palais-Smale condition. Assume there exist $x_0, x_1 \in E$ and a bounded open neighborhood Ω of x_0 such that $x_1 \in E \setminus \overline{\Omega}$ and*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf_{x\in\partial\Omega} \varphi(x).$$

Let

$$\Gamma = \{h \mid h : [0,1] \longrightarrow E \text{ is continuous and } h(0) = x_0, h(1) = x_1\}$$

and

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} \varphi(h(s)).$$

Then c is a critical value of φ, that is, there exists $x^* \in E$ such that $\varphi'(x^*) = \Theta$ and $\varphi(x^*) = c$, where $c > \max\{\varphi(x_0), \varphi(x_1)\}$.

Lemma 2.6. *Suppose that* (C1) *and* (C2) *hold. Furthermore, we assume*

(C3)

$$\frac{\|d\|_\infty}{\operatorname{ess\,inf}_{[a,b]} s} < \frac{\mu - p}{\mu}.$$

Then the functional φ satisfies the Palais-Smale condition, i.e., every sequence $\{x_n\}$ in X satisfying $\varphi(x_n)$ is bounded and $\varphi'(x_n) \to 0$ has a convergent subsequence.

Proof. First we prove that (x_n) is a bounded sequence in X. By Lemma 2.1 and (2.4), we have

(2.5)

$$\begin{aligned}
&\langle \varphi'(x_n), x_n^- \rangle \\
&= \int_a^b \left[\rho(t)\Phi_p(x_n'(t))(x_n^-)'(t) + s(t)\Phi_p(x_n(t))x_n^-(t) - f(t, x_n^+(t))x_n^-(t)\right] dt \\
&\quad - \rho(b)\Phi_p\left(\frac{B - \sigma x_n(b)}{\gamma}\right)x_n^-(b) \\
&\quad + \rho(a)\Phi_p\left(\frac{A + \beta x_n(a)}{\alpha}\right)x_n^-(a) \\
&= \int_a^b \left[-\rho(t)|(x_n^-)'(t)|^p - s(t)|x_n^-(t)|^p - f(t, x_n^+(t))x_n^-(t)\right] dt \\
&\quad - \rho(b)\Phi_p\left(\frac{B - \sigma x_n(b)}{\gamma}\right)x_n^-(b) \\
&\quad + \rho(a)\Phi_p\left(\frac{A + \beta x_n(a)}{\alpha}\right)x_n^-(a) \\
&= -\|x_n^-\|_X^p - \int_a^b f(t, x_n^+(t))x_n^-(t)\, dt \\
&\quad - \rho(b)\left|\frac{B - \sigma x_n(b)}{\gamma}\right|^{p-2}\left(\frac{Bx_n^-(b) + \sigma\left(x_n^-(b)\right)^2}{\gamma}\right)
\end{aligned}$$

$$+\rho(a)\left|\frac{A+\beta x_n(a)}{\alpha}\right|^{p-2}\left(\frac{Ax_n^-(a)-\beta(x_n^-(a))^2}{\alpha}\right)$$
$$\le -\|x_n^-\|_X^p.$$

Set $w_n^- = x_n^-/\|x_n^-\|_X$. Dividing $\|x_n^-\|_X$ on both sides of the above inequality, we have

$$\|x_n^-\|_X^{p-1} \le -\langle\varphi'(x_n), w_n^-\rangle \longrightarrow 0 \quad \text{as} \quad n\to\infty.$$

So $x_n^- \to 0$ in X. Now we shall show that (x_n^+) is bounded.

Let $H(t,x)=\int_0^x h(t,\tau)\,d\tau$ and

$$\begin{aligned} J(x_n) = \mu\frac{\gamma\rho(b)}{\sigma p}\left|\frac{B-\sigma x_n(b)}{\gamma}\right|^p &+ \mu\frac{\alpha\rho(a)}{\beta p}\left|\frac{A+\beta x_n(a)}{\alpha}\right|^p \\ &+ \rho(b)\Phi_p\left(\frac{B-\sigma x_n(b)}{\gamma}\right)x_n^+(b) \\ &- \rho(a)\Phi_p\left(\frac{A+\beta x_n(a)}{\alpha}\right)x_n^+(a). \end{aligned}$$

By (2.3) and (2.4), we have

$$\begin{aligned} \frac{\mu}{p}\|x_n\|_X^p - \|x_n^+\|_X^p = \mu\varphi(x_n) &- \langle\varphi'(x_n), x_n^+\rangle - J(x_n) \\ &+ \mu\int_a^b [F(t,x_n^+(t)) - f(t,0)x_n^-(t)]\,dt \\ &- \int_a^b f(t,x_n^+(t))x_n^+(t)\,dt. \end{aligned} \tag{2.6}$$

By (C1), (C2) and Lemma 2.3, one has

$$\begin{aligned} \mu\int_a^b [F(t,x_n^+(t)) - f(t,0)x_n^-(t)]\,dt & \\ - \int_a^b f(t,x_n^+(t))x_n^+(t)\,dt & \\ \le \mu\int_a^b H(t,x_n^+(t))\,dt & \\ \le \mu\int_a^b \left[c(t)x_n^+(t) + \frac{d(t)}{p}|x_n^+(t)|^p\right]\,dt & \\ \le \mu\|c\|_{L^1}\Delta\|x_n^+\|_X + \frac{\mu\|d\|_\infty}{p\cdot \operatorname{ess\,inf}_{[a,b]} s}\|x_n^+\|_X^p. & \end{aligned} \tag{2.7}$$

We compute

$$
\begin{aligned}
-J(x_n) &= -\rho(b)\left|\frac{B-\sigma x_n(b)}{\gamma}\right|^{p-2}\frac{Bx_n^+(b)-\sigma x_n(b)x_n^+(b)}{\gamma} \\
&\quad + \rho(a)\left|\frac{A+\beta x_n(a)}{\alpha}\right|^{p-2}\frac{Ax_n^+(a)+\beta x_n(a)x_n^+(a)}{\alpha} \\
&\quad - \frac{\mu\gamma\rho(b)}{\sigma p}\left|\frac{B-\sigma x_n(b)}{\gamma}\right|^{p} - \frac{\mu\alpha\rho(a)}{\beta p}\left|\frac{A+\beta x_n(a)}{\alpha}\right|^{p} \\
&\leq \frac{\rho(b)B(2\mu-p)}{\gamma p}\left|\frac{B-\sigma x_n(b)}{\gamma}\right|^{p-2} x_n^+(b) \\
&\quad + \frac{\rho(a)A(p-2\mu)}{\alpha p}\left|\frac{A+\beta x_n(a)}{\alpha}\right|^{p-2} x_n^+(a).
\end{aligned} \tag{2.8}
$$

Substituting (2.7) and (2.8) into (2.6), in view of Lemma 2.1 (ii), one has

$$
\begin{aligned}
\left(\frac{\mu}{p}-1\right)\|x_n^+\|_X^p &\leq \mu\varphi(x_n) - \langle \varphi'(x_n), x_n^+\rangle \\
&\quad + \frac{\rho(b)B(2\mu-p)}{\gamma p}\left|\frac{B-\sigma x_n(b)}{\gamma}\right|^{p-2} x_n^+(b) \\
&\quad + \frac{\rho(a)A(p-2\mu)}{\alpha p}\left|\frac{A+\beta x_n(a)}{\alpha}\right|^{p-2} x_n^+(a) \\
&\quad + \mu\|c\|_{L^1}\Delta\|x_n^+\|_X + \frac{\mu\|d\|_\infty}{p\cdot \operatorname{ess\,inf}_{[a,b]} s}\|x_n^+\|_X^p.
\end{aligned} \tag{2.9}
$$

Suppose that (x_n^+) is unbounded. Passing to a subsequence, we may assume, if necessary, that $\|x_n^+\|_X \to \infty$ as $n\to\infty$. Dividing both sides of (2.9) by $\|x_n^+\|_X^p$, denoting $w_n^+ = x_n^+/\|x_n^+\|_X$, we have

$$
\begin{aligned}
\frac{\mu}{p}-1 &\leq \frac{\mu\varphi(x_n)}{\|x_n^+\|_X^p} - \frac{\langle \varphi'(x_n), w_n^+\rangle}{\|x_n^+\|_X^{p-1}} \\
&\quad + \frac{\rho(b)B(2\mu-p)}{\gamma p\|x_n^+\|_X^p}\left|\frac{B-\sigma x_n(b)}{\gamma}\right|^{p-2} x_n^+(b) \\
&\quad + \frac{\rho(a)A(p-2\mu)}{\alpha p\|x_n^+\|_X^p}\left|\frac{A+\beta x_n(a)}{\alpha}\right|^{p-2} x_n^+(a) \\
&\quad + \frac{\mu\|c\|_{L^1}\Delta\|x_n^+\|_X}{\|x_n^+\|_X^p} + \frac{\mu\|d\|_\infty}{p\cdot \operatorname{ess\,inf}_{[a,b]} s}.
\end{aligned} \tag{2.10}
$$

Since $\varphi(x_n)$ is bounded and $\varphi'(x_n) \to 0$, $x_n^- \to 0$ in X, letting $n \to \infty$, we have

$$\frac{\mu}{p} - 1 \leq \frac{\mu \|d\|_\infty}{p \cdot \operatorname{ess\,inf}_{[a,b]} s},$$

which contradicts (C3). Therefore, (x_n) is bounded in X.

From the reflexivity of X, we may extract a weakly convergent subsequence that, for simplicity, we call (x_n), $x_n \rightharpoonup x$. In what follows we will show that (x_n) strongly converges to x. By (2.4) we have

$$\begin{aligned}
&(2.11) \quad \langle \varphi'(x_n) - \varphi'(x), x_n - x \rangle \\
&= \int_a^b \rho(t)[\Phi_p(x_n'(t)) - \Phi_p(x'(t))] \times (x_n'(t) - x'(t)) \\
&\qquad\qquad + s(t)[\Phi_p(x_n(t)) - \Phi_p(x(t))] \times (x_n(t) - x(t))\, dt \\
&\quad - \int_a^b [f(t, x_n^+(t)) - f(t, x^+(t))](x_n(t) - x(t))\, dt \\
&\quad - \rho(b)\left[\Phi_p\left(\frac{B - \sigma x_n(b)}{\gamma}\right) - \Phi_p\left(\frac{B - \sigma x(b)}{\gamma}\right)\right] \times (x_n(b) - x(b)) \\
&\quad + \rho(a)\left[\Phi_p\left(\frac{A + \beta x_n(a)}{\alpha}\right) - \Phi_p\left(\frac{A + \beta x(a)}{\alpha}\right)\right] \times (x_n(a) - x(a)).
\end{aligned}$$

By $x_n \rightharpoonup x$ in X, we have (x_n) uniformly converges to x in $C([a,b])$. So

$$(2.12) \qquad \begin{gathered} \int_a^b [f(t, x_n^+(t)) - f(t, x^+(t))](x_n(t) - x(t))\, dt \longrightarrow 0, \\ x_n(b) \to x(b), \quad x_n(a) \to x(a) \quad \text{as} \quad n \to \infty. \end{gathered}$$

By $\varphi'(x_n) \to 0$ and $x_n \rightharpoonup x$, we have

$$(2.13) \qquad \langle \varphi'(x_n) - \varphi'(x), x_n - x \rangle \longrightarrow 0 \quad \text{as} \quad n \to \infty.$$

By (2.2) of [**18**], there exist $c_p, d_p > 0$ such that

$$\begin{aligned}
(2.14)\quad & \int_a^b \rho(t)[\Phi_p(u'(t)) - \Phi_p(v'(t))] \times (u'(t) - v'(t)) \\
&\qquad + s(t)[\Phi_p(u(t)) - \Phi_p(v(t))] \times (u(t) - v(t))\, dt \\
&\qquad \geq \begin{cases} c_p \int_a^b [\rho(t)|u'(t) - v'(t)|^p + s(t)|u(t) - v(t)|^p]\, dt, \\ \qquad \text{if} \quad p \geq 2; \\ d_p \int_a^b (\rho(t)|u'(t) - v'(t)|^2)/((|u'(t)| + |v't)|^{2-p}) \\ +(s(t)|u(t) - v(t)|^2)/((|u(t)| + |v(t)|)^{2-p})\big]\, dt, \\ \qquad \text{if} \quad 1 < p < 2. \end{cases}
\end{aligned}$$

If $p \geq 2$, then (2.11), (2.12), (2.13) and (2.14) yield that $\|x_n - x\|_X \to 0$ in X.

If $1 < p < 2$, then by Hölder's inequality, for $u, v \in X$, we obtain

$$\begin{aligned}
(2.15)\quad & \int_a^b \rho(t)|u'(t) - v'(t)|^p\, dt \\
&\leq \left(\int_a^b \frac{\rho(t)|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}}\, dt \right)^{p/2} \\
&\quad \times \left(\int_a^b \rho(t)(|u'(t)| + |v'(t)|)^p\, dt \right)^{(2-p)/2} \\
&\leq \left(\int_a^b \frac{\rho(t)|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}}\, dt \right)^{p/2} 2^{(p-1)(2-p)/2} \\
&\quad \times \left(\int_a^b \rho(t)\, [|u'(t)|^p + |v'(t)|^p]\, dt \right)^{(2-p)/2} \\
&\leq 2^{(p-1)(2-p)/2} \left(\int_a^b \frac{\rho(t)|u'(t) - v'(t)|^2}{(|u'(t)| + |v'(t)|)^{2-p}}\, dt \right)^{p/2} \\
&\quad \times (\|u\|_X + \|v\|_X)^{((2-p)p)/2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}(2.16)\qquad &\int_a^b s(t)|u(t)-v(t)|^p\,dt\\ &\qquad\le 2^{(p-1)(2-p)/2}\bigg(\int_a^b \frac{s(t)|u(t)-v(t)|^2}{(|u(t)|+|v(t)|)^{2-p}}dt\bigg)^{p/2}\\ &\qquad\quad\times(\|u\|_X+\|v\|_X)^{((2-p)p)/2}.\end{aligned}$$

So (2.14), (2.15) and (2.16) yield

$$\begin{aligned}(2.17)\qquad &\int_a^b \rho(t)[\Phi_p(x_n'(t))-\Phi_p(x'(t))](x_n'(t)-x'(t))\\ &\qquad + s(t)[\Phi_p(x_n(t))-\Phi_p(x(t))](x_n(t)-x(t))\,dt\\ &\ge d_p\int_a^b\bigg[\rho(t)\frac{|x_n'(t)-x'(t)|^2}{(|x_n'(t)|+|x'(t)|)^{2-p}}+s(t)\frac{|x_n(t)-x(t)|^2}{(|x_n(t)|+|x(t)|)^{2-p}}\bigg]\,dt\\ &\ge \frac{d_p}{2^{((p-1)(2-p))/p}(\|x_n\|_X+\|x\|_X)^{2-p}}\\ &\quad\times\bigg\{\bigg(\int_a^b\rho(t)|x_n'(t)-x'(t)|^p\,dt\bigg)^{2/p}\\ &\qquad+\bigg(\int_a^b s(t)|x_n(t)-x(t)|^p\,dt\bigg)^{2/p}\bigg\}\\ &\ge\frac{d_p}{2^{((p-1)(2-p))/p}\max\{2^{(2/p)-1},1\}}\times\frac{\|x_n-x\|_X^2}{(\|x_n\|_X+\|x\|_X)^{2-p}}.\end{aligned}$$

Then (2.11)–(2.13) and (2.17) yield that $\|x_n-x\|_X\to 0$ in X as $1<p<2$, that is, (x_n) strongly converges to x in X. $\square$

3. Main results.

Theorem 3.1. *Suppose that* (C1)–(C3) *hold. Furthermore, we assume* (C4) *that there exists* $R_0>0$ *such that*

$$\begin{aligned}\bigg[\frac{1}{p}-\frac{\|d\|_\infty}{p\cdot\operatorname{ess\,inf}_{[a,b]}s}\bigg]R_0^p>{}&\frac{\|l\|_\infty(b-a)\Delta^\mu R_0^\mu}{\mu}+\|c\|_{L^1}\Delta R_0\\ &+\frac{\rho(b)|B|^p}{\sigma p\gamma^{p-1}}+\frac{\rho(a)|A|^p}{\beta p\alpha^{p-1}}.\end{aligned}$$

Then problem (1.3) *has at least two positive classical solutions* x_0, x^* *with* $\|x_0\|_X < R_0$.

Proof. We complete the proof by three steps:

Step 1. By Lemma 2.6 the functional φ satisfies the Palais-Smale condition.

Step 2. We shall show that there exists an $R > 0$ such that the functional φ has a local minimum $x_0 \in B_R := \{x \in X : \|x\|_X < R\}$.

Let $R > 0$ which will be determined later. First we claim that $\overline{B}_R$ is bounded and weak sequentially closed. In fact, let $(u_n) \subseteq \overline{B}_R$ and $(u_n) \rightharpoonup u$ as $n \to \infty$, by the Mazur theorem [**14**], there exists a sequence of convex combinations

$$v_n = \sum_{j=1}^{n} \alpha_{n_j} u_j, \quad \sum_{j=1}^{n} \alpha_{n_j} = 1, \quad \alpha_{n_j} \geq 0, \quad j \in N$$

such that $v_n \to u$ in X. Since $\overline{B}_R$ is a closed convex set, $(v_n) \subset \overline{B}_R$ and $u \in \overline{B}_R$. Now we claim that φ has a minimum $x_0 \in \overline{B}_R$. We will show that φ is weak sequentially lower semi-continuous on $\overline{B}_R$. For this, let

$$\varphi^1(x) = \frac{1}{p} \int_a^b [\rho(t)|x'(t)|^p + s(t)|x(t)|^p] \, dt$$

and

$$\begin{aligned} \varphi^2(x) = & -\int_a^b \left[F(t, x^+(t)) - (f(t,0), x^-(t))\right] dt \\ & + \frac{\gamma\rho(b)}{\sigma p} \left|\frac{B - \sigma x(b)}{\gamma}\right|^p + \frac{\alpha\rho(a)}{\beta p} \left|\frac{A + \beta x(a)}{\alpha}\right|^p, \end{aligned}$$

then $\varphi(x) = \varphi^1(x) + \varphi^2(x)$. By $x_n \rightharpoonup x$ on X we have (x_n) uniformly converges to x in $C([a,b])$. So φ^2 is weak sequentially continuous. Clearly φ^1 is continuous which, together with the convexity of φ^1, we have that φ^1 is weak sequentially lower semi-continuous. Therefore, φ is weak sequentially lower semi-continuous on $\overline{B}_R$. Besides, X is a reflexive Banach space, $\overline{B}_R$ is a bounded and weak sequentially closed set, so our claim follows from Lemma 2.4. Without loss of generality, we assume that $\varphi(x_0) = \min_{x\in\overline{B}_R} \varphi(x)$. Now we will show that

$$\varphi(x_0) < \inf_{x\in\partial B_R} \varphi(x). \tag{3.1}$$

If this is true, the result of Step 2 holds.

In fact, for any $x \in \partial B_R$, we have by (2.3), (C1) and Lemma 2.3,

$$\begin{aligned}
\varphi(x) &\geq \frac{R^p}{p} - \int_a^b F(t, x^+(t))\,dt \\
&\geq \frac{R^p}{p} - \int_a^b \left[\frac{l(t)|x^+(t)|^\mu}{\mu} + c(t)x^+(t) + \frac{d(t)|x^+(t)|^p}{p}\right] dt \\
&\geq \frac{R^p}{p} - \frac{\|l\|_\infty(b-a)}{\mu}\|x\|_\infty^\mu - \|c\|_{L^1}\|x\|_\infty \\
&\quad - \frac{\|d\|_\infty}{p\cdot \operatorname{ess\,inf}_{[a,b]} s}\|x\|_X^p \\
&\geq \frac{R^p}{p} - \frac{\|l\|_\infty(b-a)}{\mu}\Delta^\mu\|x\|_X^\mu - \|c\|_{L^1}\Delta\|x\|_X \\
&\quad - \frac{\|d\|_\infty}{p\cdot \operatorname{ess\,inf}_{[a,b]} s}\|x\|_X^p \\
&= \frac{R^p}{p} - \frac{\|l\|_\infty(b-a)}{\mu}\Delta^\mu R^\mu - \|c\|_{L^1}\Delta R \\
&\quad - \frac{\|d\|_\infty}{p\cdot \operatorname{ess\,inf}_{[a,b]} s}R^p.
\end{aligned}$$

So

$$\inf_{x\in\partial B_R} \varphi(x) \geq \frac{R^p}{p} - \frac{\|l\|_\infty(b-a)}{\mu}\Delta^\mu R^\mu - \|c\|_{L^1}\Delta R - \frac{\|d\|_\infty}{p\cdot \operatorname{ess\,inf}_{[a,b]} s}R^p.$$

Noticing $\varphi(x_0) \leq \varphi(0) = (\rho(b)|B|^p)/(\sigma p\gamma^{p-1}) + (\rho(a)|A|^p)/(\beta p\alpha^{p-1})$, by (C4) there exists an $R_0 > 0$ such that $\varphi(x) > \varphi(0) \geq \varphi(x_0)$ for any $x \in \partial B_{R_0}$. So (3.1) holds and $x_0 \in B_{R_0}$.

Step 3. We shall show that there exists an x_1 with $\|x_1\| > R_0$ such that $\varphi(x_1) < \inf_{x\in\partial B_{R_0}} \varphi(x)$.

Let $\widetilde{e}(t) = 1 \in X, \overline{\lambda} > 0$. Then

$$\begin{aligned}\varphi(\overline{\lambda}\widetilde{e}) &= \frac{\overline{\lambda}^p}{p}\int_a^b s(t)\,dt - \int_a^b F(t,\overline{\lambda})\,dt \\ &\quad + \frac{\gamma\rho(b)}{\sigma p}\left|\frac{B-\sigma\overline{\lambda}}{\gamma}\right|^p + \frac{\alpha\rho(a)}{\beta p}\left|\frac{A+\beta\overline{\lambda}}{\alpha}\right|^p \\ &= \frac{\overline{\lambda}^p}{p}\int_a^b s(t)\,dt - \int_a^b \left[\frac{l(t)\overline{\lambda}^\mu}{\mu} + H(t,\overline{\lambda})\right] dt \\ &\quad + \frac{\gamma\rho(b)}{\sigma p}\left|\frac{B-\sigma\overline{\lambda}}{\gamma}\right|^p + \frac{\alpha\rho(a)}{\beta p}\left|\frac{A+\beta\overline{\lambda}}{\alpha}\right|^p \\ &\leq \frac{\overline{\lambda}^p}{p}\int_a^b s(t)\,dt - \int_a^b \frac{l(t)\overline{\lambda}^\mu}{\mu}\,dt \\ &\quad + \frac{\gamma\rho(b)}{\sigma p}\left|\frac{B-\sigma\overline{\lambda}}{\gamma}\right|^p + \frac{\alpha\rho(a)}{\beta p}\left|\frac{A+\beta\overline{\lambda}}{\alpha}\right|^p.\end{aligned} \tag{3.2}$$

Since $\mu > p$, we have $\lim_{\overline{\lambda}\to+\infty}\varphi(\overline{\lambda}\widetilde{e}) = -\infty$. So there exists a sufficiently large $\lambda_0 > 0$ with $\|\overline{\lambda}_0\widetilde{e}\| > R_0$ such that $\varphi(\overline{\lambda}_0\widetilde{e}) < \inf_{x\in\partial B_{R_0}}\varphi(x)$. Therefore, let $x_1 = \overline{\lambda}_0\widetilde{e}$ and $\varphi(x_1) < \inf_{x\in\partial B_{R_0}}\varphi(x)$.

Lemma 2.5 now gives the critical value

$$c = \inf_{h\in\Gamma}\max_{t\in[0,1]}\varphi(h(t)),$$

where

$$\Gamma = \{h \mid h : [0,1] \longrightarrow E \quad \text{is continuous and} \quad h(0) = x_0, h(1) = x_1\},$$

that is, there exists an $x^* \in X$ such that $\varphi'(x^*) = 0$. Therefore, x_0 and x^* are two critical points of φ, $\|x_0\|_X < R_0$, and hence they are classical solutions of (2.1). Lemma 2.2 means x_0 and x^* are positive classical solutions of problem (1.3). □

Corollary 3.2. *Suppose that* (C1) *holds. Moreover, we assume*

(C2′) *there exist* $0 \leq s < p$, $c \in L^1([a,b],[0,+\infty))$ *and* $d \in C([a,b],[0,+\infty))$ *such that*

$$h(t,x) \leq c(t) + d(t)\Phi_s(x);$$

(C4′) *there exists an* $R_0 > 0$ *such that*

$$\frac{1}{p}R_0^p > \frac{\|l\|_\infty (b-a)\Delta^\mu R_0^\mu}{\mu} + \|c\|_{L^1}\Delta R_0 + \frac{\|d\|_\infty}{p \cdot \operatorname{ess\,inf}_{[a,b]} s} R_0^s + \frac{\rho(b)|B|^p}{\sigma p \gamma^{p-1}} + \frac{\rho(a)|A|^p}{\beta p \alpha^{p-1}}.$$

Then problem (1.3) *has at least two positive classical solutions* x_0 *and* x^* *with* $\|x_0\|_X < R_0$.

Corollary 3.3. *Suppose that* (C1) *and* (C2′) *hold. Moreover, we assume*

(C5) *there exists an* $R_0 > 0$ *such that*

$$\frac{1}{p}R_0^p > \frac{\|l\|_\infty (b-a)\Delta^\mu R_0^\mu}{\mu} + \|c\|_{L^1}\Delta R_0 + \frac{\|d\|_\infty}{p \cdot \operatorname{ess\,inf}_{[a,b]} s} R_0^s.$$

Then problem (1.3) *with* $A = B = 0$ *has at least two positive solutions* x_0 *and* x^*.

Example 3.1. Consider the following second-order boundary value problem

(3.3)
$$\begin{cases} -(\Phi_3(x'))' + \Phi_3(x) = (1+t)/(48)\Phi_5(x) + \lambda(t^3/8) + \lambda(t/5)\Phi_3(x) \\ \qquad t \in [0,1] \\ 4x'(0) - x(0) = -1 \\ \qquad 8x'(1) + x(1) = 2. \end{cases}$$

Corresponding to (1.3), $p = 3$, $\alpha = 4$, $\beta = 1$, $\gamma = 8$, $\sigma = 1$, $l(t) = (1+t)/48$, $c(t) = \lambda(t^3/8)$, $d(t) = \lambda(t/5)$,

$$f(t,x) = \frac{1+t}{48}\Phi_5(x) + \lambda\left[\frac{t^3}{8} + \frac{t}{5}\Phi_3(x)\right].$$

Clearly, $f(t,0) = \lambda(t^3/8) \not\equiv 0$. $f(t,x)$ satisfies the conditions (C1)(C2). By computing, let $R_0 = 1$; (C3) and (C4) are satisfied. By Theorem 3.1, BVP (3.3) has at least two positive solutions for $\lambda \in (0,1)$. □

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