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# NORMALITY OF MONOMIAL IDEALS

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ABSTRACT. Given the monomial ideal $I = (x_1^{\alpha_1}, \dots, x_n^{\alpha_n}) \subset K[x_1, \dots, x_n]$, where $\alpha_i$ are positive integers and $K$ a field, let $J$ be the integral closure of $I$. It is a challenging problem to translate the question of the normality of $J$ into a question about the exponent set $\Gamma(J)$ and the Newtonian polyhedron $NP(J)$. A relaxed version of this problem is to give necessary or sufficient conditions on $\alpha_1, \dots, \alpha_n$ for the normality of $J$. We show that if $\alpha_i \in \{s, l\}$ with $s$ and $l$ arbitrary positive integers, then $J$ is normal.

**1. Introduction.** Let $I$ be an ideal in a Noetherian ring $R$. The integral closure of $I$ is the ideal $\overline{I}$ that consists of all elements of $R$ that satisfy an equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \quad a_i \in I^i.$$

The ideal $I$ is said to be integrally closed if $I = \overline{I}$. Clearly, one has that $I \subseteq \overline{I} \subseteq \sqrt{I}$. An ideal is called *normal* if all of its positive powers are integrally closed. It is known that if $R$ is a normal integral domain, then the Rees algebra $R[It] = \oplus_{n \in N} I^n t^n$ is normal if and only if $I$ is a normal ideal of $R$. This brings up the importance of normality of ideals as the Rees algebra is the algebraic counterpart of blowing up a scheme along a closed subscheme.

It is well known that the integral closure of a monomial ideal in a polynomial ring is again a monomial ideal, see [**5, 6**] for a proof. The problem of finding the integral closure for a monomial ideal $I$ reduces to finding monomials $r$, integer $i$ and monomials $m_1, m_2, \dots, m_i$ in $I$ such that $r^i + m_1 m_2 \cdots m_i = 0$, see [**5**]. Geometrically, finding the integral closure of monomial ideals $I$ in $R = K[x_0, \dots, x_n]$ is the same as finding all the integer lattice points in the convex hull $NP(I)$ (the Newton polyhedron of $I$) in $\mathbf{R}^n$ of $\Gamma(I)$ (the Newton polytope of $I$) where $\Gamma(I)$ is the set of all exponent vectors of all the monomials in $I$. This makes computing the integral closure of monomial ideals simpler.

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A power of an integrally closed monomial ideal need not be integrally closed. For example, let $J$ be the integral closure of $I = (x^4, y^5, z^7) \subset K[x, y, z]$. Then $J^2$ is not integrally closed (observe that $y^3z^3 \in J$ as $(y^3z^3)^5 = y^5y^5y^5z^7z^8 \in I^5$. Now $x^2y^4z^5 \in \overline{J^2}$ since $(x^2y^4z^5)^2 = (x^4 \cdot y^5)(y^3z^3 \cdot z^7) \in (J^2)^2$. On the other hand, we used the algebra software Singular [**3**] to show that $x^2y^4z^5 \notin J^2$). However, a nice result of Reid et al. [**4**, Proposition 3.1] states that if the first $n-1$ powers of a monomial ideal, in a polynomial ring of $n$ variables over a field, are integrally closed, then the ideal is normal. For the case $n = 2$ this follows from the celebrated theorem of Zariski [**7**] that asserts that the product of integrally closed ideals in a 2-dimensional regular ring is again integrally closed.

In general, there is no good characterization for normal monomial ideals. It is a challenging problem to translate the question of normality of a monomial ideal $I$ into a question about the exponent set $\Gamma(I)$ and the Newton polyhedron $NP(I)$. Under certain hypotheses, some necessary conditions are given. Faridi [**2**] gives necessary conditions on the degree of the generators of a normal ideal in a graded domain. Vitulli [**6**] investigated the normality for special monomial ideals in a polynomial ring over a field.

For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, let $I(\boldsymbol{\alpha})$ be the integral closure of $(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \subset K[x_1, \ldots, x_n]$. Reid et al. [**4**] showed that if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with pairwise relatively prime entries, then the ideal $I(\boldsymbol{\alpha})$ is normal if and only if the additive submonoid $\Lambda = \langle 1/\alpha_1, \ldots, 1/\alpha_n \rangle$ of $\mathbf{Q}_{\geq}$ is quasinormal, that is, whenever $x \in \Lambda$ and $x \geq p$ for some $p \in \mathbf{N}$, there exist rational numbers $y_1, \ldots, y_p$ in $\Lambda$ with $y_i \geq 1$ for all $i$ such that $x = y_1 + \cdots + y_p$. Thus, for the case where $\alpha_1, \ldots, \alpha_n$ are pairwise relatively prime, the normality condition on the $n$-dimensional monoid is reduced to the quasinormality condition on the 1-dimensional monoid. Another nice result of [**4**] is that the monomial ideal $I(\boldsymbol{\alpha})$ is normal if $\gcd(\alpha_1, \ldots, \alpha_n) > n-2$. In particular, if $n = 3$ and $\gcd(\alpha_1, \alpha_2, \alpha_3) \neq 1$, then $I(\boldsymbol{\alpha})$ is normal. Therefore, in $k[x_1, x_2, x_3]$ it remains to investigate the normality of $I(\boldsymbol{\alpha})$ whenever $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$ and the integers are not pairwise relatively prime.

An important result of Reid et al. [**4**], which we use to improve our result in this paper, is as follows. Choose $i$ and set $c = \operatorname{lcm}(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_n)$. Put $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + c, \alpha_{i+1}, \ldots, \alpha_n)$.

If $I(\boldsymbol{\alpha}')$ is normal, then $I(\boldsymbol{\alpha})$ is normal. Conversely, if $I(\boldsymbol{\alpha})$ is normal and $\alpha_i \geq c$, then $I(\boldsymbol{\alpha}')$ is normal.

The goal of this paper is to show that the integral closure of the ideal $(x_1^{\alpha_1}, \dots, x_n^{\alpha_n}) \subset K[x_1, \dots, x_n]$ is normal provided that $\alpha_i \in \{s, l\}$ with $s$ and $l$ arbitrary positive integers. The following theorem provides us with a technique that we mainly depend upon to prove integral closedness.

**Theorem 1** [**5**, Proposition 15.4.1]. *Let $I$ be a monomial ideal in the polynomial ring $R = K[x_1, \dots, x_n]$ with $K$ a field. If $I$ is primary to $(x_1, \dots, x_n)$ and $\overline{I} \cap (I : (x_1, \dots, x_n)) \subseteq I$, then $I$ is integrally closed.*

**Proposition 2** [**5**, Corollary 5.3.2]. *If $I \subseteq J$ are ideals in a ring $R$, then $J \subseteq \overline{I}$ if and only if each element in some generating set of $J$ is integral over $I$.*

**Certain normal monomial ideals.** Let $(x_1^s, \dots, x_m^s, y_1^l, \dots, y_n^l) \subset K[x_1, \dots, x_m, y_1, \dots, y_n]$ with $K$ a field, $x_i$ and $y_i$ indeterminates over $K$, and let $s$ and $l$ be positive integers such that (without loss of generality) $l \geq s$.

**Notation 3.** *For the remainder of this paper, fix positive integers $s$ and $l$ with $l \geq s$ and let $\lambda_a = \lceil a(l/s) \rceil$ where $a$ is any integer. Also, let $k$ be any positive integer.*

Let $x$ and $y$ be positive integers, and write $x = ts + r$ with $1 \leq r \leq s$. Then

$$y\lceil x/s \rceil = y\frac{x+s-r}{s} = y\frac{s-r}{s} + y\frac{x}{s} \leq y\frac{s-r}{s} + \left\lceil y\frac{x}{s} \right\rceil .$$

Therefore,

$$\left\lceil y\frac{x}{s} \right\rceil \geq y\left( \left\lceil \frac{x}{s} \right\rceil - \frac{s-r}{s} \right).$$

This inequality helps to prove the following lemma which is a key in this paper.

**Lemma 4.** *If $i \in \{0, 1, \ldots, ks\}$, then $kl(ks - i - 1) + \lambda_i \geq (ks - i)(\lambda_{ks-1} - ((s - r)/s))$, where $(ks - 1)l = ts + r$ with $1 \leq r \leq s$.*

*Proof.* By Notation 3, we have

$$\begin{aligned} kl(ks - i - 1) + \lambda_i &= \left\lceil \frac{[ks(ks - i - 1) + i]l}{s} \right\rceil \\ &= \left\lceil (ks - i)\frac{(ks - 1)l}{s} \right\rceil \\ &\geq (ks - i)\left(\left\lceil \frac{(ks - 1)l}{s} \right\rceil - \frac{s - r}{s}\right) \\ &= (ks - i)\left(\lambda_{ks-1} - \frac{s - r}{s}\right). \qquad \square \end{aligned}$$

**Definition 5.** Let $F_k = \{x_{i_1} \cdots x_{i_{ks-a}} y_{j_1} \cdots y_{j_{\lambda_a}} \mid a = 0, 1, 2, \ldots, ks,$ $1 \leq i_1 \leq i_2 \leq \cdots \leq i_{ks-a} \leq m$, and $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{\lambda_a} \leq n\}$, $J_k$ the ideal generated by all the monomials in $F_k$, and $I_k = (x_1^{ks}, \ldots, x_m^{ks}, y_1^{kl}, \ldots, y_n^{kl}) \subset K[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Also, let $J = J_1$, $F = F_1$, and $I = I_1$.

**Lemma 6.** *$J_k$ is integral over the ideal $I_k$, that is, $J_k \subseteq \overline{I_k}$.*

*Proof.* By Proposition 2 it suffices to show that every element of $F_k$ is integral over $I_k$. Note that $x_{i_1}^{ksl} \cdots x_{i_{ks-a}}^{ksl} \in I_k^{l(ks-a)}$ and $y_{j_1}^{ksl} \cdots y_{j_{\lambda_a}}^{ksl} \in I_k^{s\lambda_a}$. Also note $l(ks-a)+s\lambda_a = ksl-la+s\lceil a(l/s)\rceil \geq ksl$. Therefore, $(x_{i_1} \cdots x_{i_{ks-a}} y_{j_1} \cdots y_{j_{\lambda_a}})^{ksl} \in I_k^{ksl}$. $\square$

Figure 1 is an illustration of $J_3 \subset K[x, y, z]$ with $s = 2$, $l = 7$ and $I = (x^s, y^s, z^l)$. In this case $I_3 = (x^{3s}, y^{3s}, z^{3l}) = (x^6, y^6, z^{21})$ and $F_3 = \{x^i y^j z^{\lambda_{6-(i+j)}} \mid i + j = 0, 1, 2, 3, 4, 5, 6$ and $\lambda_a = \lceil (7a)/2 \rceil\}$. The elements of $F_3$ are represented by black circles. From the figure it is clear that the set $F_3$ minimally generates $\overline{I_3}$.

Later we will prove that $J_k$ is the integral closure of $I_k$.

**Lemma 7.** $J^k = J_k$.

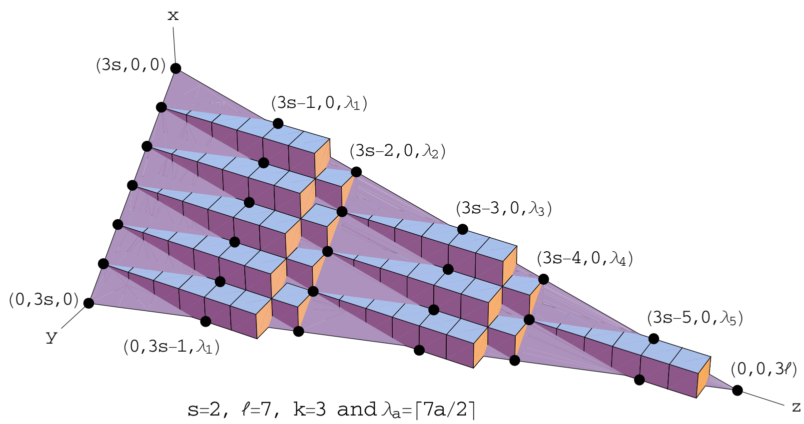


FIGURE 1.

*Proof.* We show $J_kJ = J_{k+1}$. Let $x_{i_1}\cdots x_{i_{s-a}}y_{j_1}\cdots y_{j_{\lambda_a}} \in F$ and $x_{i_1}\cdots x_{i_{ks-b}}y_{j_1}\cdots y_{j_{\lambda_b}} \in F_k$. Multiplying these two monomials, we get $x_{h_1}\cdots x_{h_{(k+1)s-(b+a)}}y_{t_1}\cdots y_{t_{\lambda_a+\lambda_b}}$ (with $1 \le h_1 \le h_2 \le \ldots \le m$ and $1 \le t_1 \le t_2 \le \ldots \le n$). This is a multiple of $x_{h_1}\cdots x_{h_{(k+1)s-(b+a)}}y_{t_1}\cdots y_{t_{\lambda_{a+b}}} \in J_{k+1}$ as $\lambda_{a+b} \le \lambda_a+\lambda_b$. To show the other inclusion, let $x_{i_1}\cdots x_{i_{(k+1)s-a}}y_{j_1}\cdots y_{j_{\lambda_a}} \in F_{k+1}$. If $a \ge ks$, write $a = ks+r$ with $0 \le r \le s$; then $\lambda_a = \lambda_{ks+r} = \lceil (ks+r)(l/s)\rceil = kl+\lambda_r$. Thus, this monomial equals $x_{i_1}\cdots x_{i_{s-r}}y_{j_1}\cdots y_{j_{\lambda_r+kl}}$. But $y_{j_1}\cdots y_{j_{kl}} \in F_k$ and $x_{i_1}\cdots x_{i_{s-r}}y_{j_{kl+1}}\cdots y_{j_{kl+\lambda_r}} \in F$ as $0 \le s-r \le s$. If $a < ks$, then $x_{i_1}\cdots x_{i_{(k+1)s-a}}y_{j_1}\cdots y_{j_{\lambda_a}} = x_{t_1}\cdots x_{t_s}x_{h_1}\cdots x_{h_{ks-a}}y_{j_1}\cdots y_{j_{\lambda_a}} \in JJ_k$ as $x_{t_1}\cdots x_{t_s} \in J$ and $x_{h_1}\cdots x_{h_{ks-a}}y_{j_1}\cdots y_{j_{\lambda_a}} \in J_k$. $\square$

The main goal of this paper is to prove the following theorem

**Theorem 8.** *The integral closure of the ideal* $(x_1^{\alpha_1},\ldots,x_n^{\alpha_n}) \subset K[x_1,\ldots,x_n]$ *is normal, where* $\alpha_i \in \{s,l\}$ *with* $s$ *and* $l$ *arbitrary positive integers. Or, equivalently, the ideal* $J$ *is normal.*

By Lemma 6 and since $I_k \subseteq J_k$, we have

$$I_k \subseteq J_k \subseteq \overline{I_k} \subseteq \overline{J_k}.$$

We will use Theorem 1 to show that $J_k$ is integrally closed; hence, $J_k$ is the integral closure of $I_k$. Therefore, we need the following.

*Remark* 9. Let $R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$. For $1 \leq i \leq m$, it is easy to see that $(J_k : (x_i))/J_k$ is generated by $\{z_{i_1} \cdots z_{i_{ks-a-1}} w_{j_1} \cdots w_{j_{\lambda_a}} \mid a = 0, \ldots, ks-1;\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{ks-a-1} \leq m$ and $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{\lambda_a} \leq n\}$ where $z_i$ and $w_i$ are the images of $x_i$ and $y_i$, respectively, in $R/J_k$. Also, for $1 \leq j \leq n$, note that $(J_k : (y_j))/J_k$ is generated by $\{z_{i_1} \cdots z_{i_{ks-b}} w_{j_1} \cdots w_{j_{\lambda_b}-1} \mid b = 1, \ldots, ks;\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{ks-b} \leq m$ and $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{\lambda_b} \leq n\}$. As the intersection of two monomial ideals is generated by the set of the least common multiples of generators of the two ideals, it follows that $(J_k : (x_1, \ldots, x_m, y_1, \ldots, y_n))/J_k$ is generated by $\{z_{i_1} \cdots z_{i_{ks-e}} w_{j_1} \cdots w_{j_{\lambda_e}-1} \mid e = 1, \ldots, ks;\ 1 \leq i_1 \leq i_2 \leq \cdots \leq i_{ks-e} \leq m$ and $1 \leq j_1 \leq j_2 \leq \cdots \leq j_{\lambda_e} \leq n\}$.

**Lemma 10.** *The ideal $J_k$ is integrally closed.*

*Proof.* By Theorem 1 we need to show that none of the preimages, in $K[x_1, \ldots, x_m, y_1, \ldots, y_n]$, of the monomial generators of $(J_k : (x_1, \ldots, x_m, y_1, \ldots, y_n))/J_k$ is in $\overline{J_k}$. Assume not, that is, assume $\sigma = x_{i_1} \cdots x_{i_{ks-e}} y_{j_1} \cdots y_{j_{\lambda_e}-1} \in \overline{J_k}$ for some $e \in \{1, \ldots, ks\}$. This implies $\sigma^d \in J_k^d$ for some positive integer $d$; thus, $\sigma^d = x_{i_1}^d x_{i_2}^d \cdots x_{i_{ks-e}}^d y_{j_1}^d \cdots y_{j_{\lambda_e}-1}^d$ equals the following product of products of the generators of $J_k$

$$
\begin{aligned}
\beta & \prod_{1 \leq j_1 \leq \cdots \leq j_{kl} \leq n} (y_{j_1} y_{j_2} \cdots y_{j_{kl}})^{c_{j_1, \ldots, j_{kl}}} \\
& \prod_{\substack{1 \leq i_1 \leq m \\ 1 \leq j_1 \leq \cdots \leq j_{\lambda_{ks-1}} \leq n}} (x_{i_1} y_{j_1} y_{j_2} \cdots y_{j_{\lambda_{ks-1}}})^{l_{i_1, j_1, \ldots, j_{\lambda_{ks-1}}}} \\
& \prod_{\substack{1 \leq i_1 \leq i_2 \leq m \\ 1 \leq j_1 \leq \cdots \leq j_{\lambda_{ks-2}} \leq n}} (x_{i_1} x_{i_2} y_{j_1} y_{j_2} \cdots y_{j_{\lambda_{ks-2}}})^{l_{i_1, i_2, j_1, \ldots, j_{\lambda_{ks-2}}}} \\
& \qquad \vdots
\end{aligned}
$$

$$\prod_{\substack{1\le 1i_1\le i_2\le\cdots\le i_{ks-2}\le m\\ 1\le j_1\le\cdots\le j_{\lambda_2}\le n}} (x_{i_1}\cdots x_{i_{ks-2}}y_{j_1}\cdots y_{j_{\lambda_2}})^{l_{i_1,i_2,\dots,i_{ks-2},j_1,\dots,j_{\lambda_2}}}$$
$$\prod_{\substack{1\le i_1\le i_2\le\cdots\le i_{ks-1}\le m\\ 1\le j_1\le\cdots\le j_{\lambda_1}\le n}} (x_{i_1}\cdots x_{i_{ks-1}}y_{j_1}\cdots y_{j_{\lambda_1}})^{l_{i_1,i_2,\dots,i_{ks-1},j_1,\dots,j_{\lambda_1}}}$$
$$\prod_{1\le i_1\le i_2\le\cdots\le i_{ks}\le m} (x_{i_1}x_{i_2}\cdots x_{i_{ks}})^{l_{i_1,i_2,\dots,i_{ks}}}$$

where $\beta$ is some monomial, $c_{j_1,\dots,j_{kl}}$ and $l_{i_1,\dots,i_t,j_1,\dots,j_{\lambda_{ks-t}}}$ (with $1\le t\le ks$) are nonnegative integers. For $1\le t\le ks$, let $L_t=\sum_{\substack{1\le i_1\le i_2\le\cdots\le i_t\le m\\ 1\le j_1\le\cdots\le j_{\lambda_{ks-t}}\le n}} l_{i_1,\dots,i_t,j_1,\dots,j_{\lambda_{ks-t}}}$, and let $C=\sum_{1\le j_1\le\cdots\le j_{kl}\le n}\times c_{j_1,\dots,j_{kl}}$. By summing powers, we have

$$L_{ks}+L_{ks-1}+\cdots+L_3+L_2+L_1+C=d. \tag{1}$$

Also, by the total-degree count of the monomial $x_{i_1}\cdots x_{i_{ks-e}}$, we have the following equality

$$(ks)L_{ks}+(ks-1)L_{ks-1}+\cdots+3L_3+2L_2+L_1+\varepsilon=(ks-e)d \tag{2}$$

where $\varepsilon$ is the total-degree of the monomial $x_{i_1}\cdots x_{i_{ks-e}}$ in $\beta$. By the total-degree count of the monomial $y_1\cdots y_n$, we must have the following inequality

$$\begin{aligned}\lambda_1L_{ks-1}+\lambda_2L_{ks-2}+\cdots+\lambda_{ks-3}L_3+\lambda_{ks-2}L_2+\lambda_{ks-1}L_1+Ckl\\ \le(\lambda_e-1)d.\end{aligned} \tag{3}$$

We finish the proof by showing that (1), (2) and (3) cannot hold simultaneously.

From (1) and (2),

$$C=(ks-1)L_{ks}+(ks-2)L_{ks-1}+\cdots+2L_3+L_2+\varepsilon-(ks-e-1)d. \tag{4}$$

Recall $(ks-1)l=ts+r$ with $1\le r\le s$ and $\lambda_{ks-1}<\lambda_{ks}=kl$. Now consider the lefthand side of (3),

$$
\begin{aligned}
&\lambda_1 L_{ks-1} + \lambda_2 L_{ks-2} + \cdots + \lambda_{ks-3} L_3 + \lambda_{ks-2} L_2 + \lambda_{ks-1} L_1 + Ckl \\
&= \left[\sum_{i=0}^{ks-1} [kl(ks-1-i) + \lambda_i] L_{ks-i}\right] + \varepsilon kl - kl(ks-e-1)d \text{ (by (4))} \\
&\geq \left[\sum_{i=0}^{ks-1} (ks-i)(\lambda_{ks-1} - \frac{s-r}{s}) L_{ks-i}\right] + \varepsilon kl - kl(ks-e-1)d \\
&\hfill \text{(by Lemma 4)} \\
&\geq \left(\lambda_{ks-1} - \frac{s-r}{s}\right)(ks-e)d - kl(ks-e-1)d \text{ (by (2))} \\
&= \frac{(ks-1)l}{s}(ks-e)d - kl(ks-e-1)d \\
&= \left(\frac{e}{s} l\right) d \\
&> (\lambda_e - 1)d.
\end{aligned}
$$

This is a contradiction to (3) as required. □

*Proof of Theorem* 8. The proof follows by the above lemma and Lemma 7. □

We have already proved that if $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ with the entries of $\boldsymbol{\alpha}$ consisting of two positive integers, then $I(\boldsymbol{\alpha})$, the integral closure of $(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \subset K[x_1, \ldots, x_n]$, is normal. Noting that the ideal $I(x^4, y^5, z^7) \subset K[x, y, z]$ is not normal, the following question arises: when is $I(\boldsymbol{\alpha})$ normal provided that $\boldsymbol{\alpha}$ consists of three distinct positive integers? In the proposition below we give a partial answer for this question.

**Theorem 11** [**4**, Theorem 5.1]. *Let* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$, $c =$ lcm $(\alpha_1, \ldots, \alpha_{n-1})$. *Let* $I(\boldsymbol{\alpha})$ *be the integral closure of* $(x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}) \subset K[x_1, \ldots, x_n]$ *and* $I(\boldsymbol{\alpha}')$ *the integral closure of* $(x_1^{\alpha_1}, \ldots, x_{n-1}^{\alpha_{n-1}}, x_n^{\alpha_n + c}) \subset K[x_1, \ldots, x_n]$. *If* $I(\boldsymbol{\alpha}')$ *is normal, then* $I(\boldsymbol{\alpha})$ *is normal. Conversely, if* $I(\boldsymbol{\alpha})$ *is normal and* $\alpha_n \geq c$, *then* $I(\boldsymbol{\alpha}')$ *is normal.*

**Proposition 12.** *If* $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ *with* $\alpha_i \in \{s, l\}$ *for* $i = 1, \ldots, n-1$ *such that* $s$ *divides* $l$ *and* $l$ *divides* $\alpha_n$, *then* $I(\boldsymbol{\alpha})$ *is normal.*

*Proof.* We proceed by induction on the integer $q = \alpha_n/l$. By Theorem 8 the ideal $I(\boldsymbol{\alpha})$ is normal whenever $q = 1$. Note $l = \operatorname{lcm}\{s, l\}$ as $s$ divides $l$. Assume $I(\boldsymbol{\alpha})$ is normal for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n-1}, ql)$ with $\alpha_i \in \{s, l\}$ for $i = 1, \ldots, n-1$. Then, by Theorem 11, $I(\boldsymbol{\alpha}')$ is normal where $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_{n-1}, ql + l)$. $\square$

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