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ITERATED SUMS OF FIFTH POWERS OF DIGITS

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1. Introduction. In this paper, we present and prove a variety of results about iterated sums of powers of digits and apply our results to the power five case. As described below, this extends and generalizes many results from earlier works on this topic.

Following [**3**], for e, $b \geq 2$, let $S_{e,b} : \mathbf{Z}^+ \to \mathbf{Z}^+$ map each positive integer to the sum of the eth powers of its base b digits. In other words, for $0 \leq a_i \leq b-1$,

$$S_{e,b}\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} a_i^e.$$

Let $S_{e,b}^m$ denote the m-fold iteration of $S_{e,b}$. If $S_{e,b}^m(a) = 1$ for some $m \geq 0$, then a is an *e-power b-happy number*. This is a generalization of the concept of a *happy number* which, in these terms, is a 2-power 10-happy number.

It is well known [**4**] that $S_{2,10}$ has exactly one fixed point, 1, and one cycle of length 8:

$$4 \longrightarrow 16 \longrightarrow 37 \longrightarrow 58 \longrightarrow 89 \longrightarrow 145 \longrightarrow 42 \longrightarrow 20 \longrightarrow 4.$$

Thus, for any positive integer there exists $m \geq 0$ such that $S_{2,10}^m(a) = 1$ or 4. See [**3**] for the determination of the fixed points and cycles of the functions $S_{2,b}$ and $S_{3,b}$ for $2 \leq b \leq 10$. In Section 2 of this paper, we present a method for determining the fixed points and cycles for $S_{e,b}$ in general and apply it to $S_{5,b}$ with $2 \leq b \leq 10$.

In [**1**], El-Sedy and Siksek showed that there exist arbitrarily long finite sequences of consecutive happy numbers. The present authors give more general results for $e = 2$ and $e = 3$ in [**2**]. In Section 3 of this paper, we present parallel results for $e = 5$.

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2. Cycles. Fix $e, b \geq 2$. In this section we give an algorithm for determining the fixed points and cycles of the functions $S_{e,b}$. One can easily write a computer program to find the fixed point or cycle determined by a given input. The key is finding a bound on the size of the inputs one needs to check. Such a bound is provided by the following theorem and corollary.

Theorem 1. *If $a \geq b^{e+1}$, then $S_{e,b}(a) < a$.*

Proof. Let $a \geq b^{e+1}$ and fix $r \in \mathbf{Z}$ such that $b^r \leq a < b^{r+1}$. (Note that $r > e$.) Then, in base b, a is $r+1$ digits long and so $S_{e,b}(a) \leq (r+1)(b-1)^e$. Using binomial expansion, $b^r = ((b-1)+1)^r > (b-1)^r + r(b-1)^{r-1}$. Thus, $a - S_{e,b}(a) \geq b^r - (r+1)(b-1)^e > (b-1)^r + r(b-1)^{r-1} - r(b-1)^e - (b-1)^e > 0$, since $r - 1 \geq e$. Hence, $S_{e,b}(a) < a$. $\square$

A direct application of the well-ordering principle for $\mathbf{Z}$ yields the following corollary.

Corollary 2. *For each $a \in \mathbf{Z}^+$, there is an $m \in \mathbf{Z}^+$ such that $S_{e,b}^m(a) < b^{e+1}$.*

By Corollary 2 applied to the case $e = 5$, every cycle of $S_{5,b}$ contains at least one element less than b^6. A straightforward computation then yields all fixed points and cycles of $S_{5,b}$. The results for $2 \leq b \leq 9$ are given in Table 1.

3. Consecutive 5-power b-happy numbers. We now consider sequences of consecutive 5-power b-happy numbers. We begin by recalling needed definitions and results from [**2**].

Fix $e, b \geq 2$. Let $U_{e,b}$ denote the set of all numbers in cycles (including fixed points) of $S_{e,b}$. In other words,

$$U_{e,b} = \{a \in \mathbf{Z}^+ \mid \text{for some } m \in \mathbf{Z}^+,\ S_{e,b}^m(a) = a\}.$$

We say that a finite set T is (e,b)-*good* if for each $u \in U_{e,b}$ there exist n and $m \in \mathbf{Z}^+$ such that for every $t \in T$, $S_{e,b}^m(t+n) = u$. For

TABLE 1. Fixed points and cycles of $S_{5,b}$, for small values of b.

Base	Fixed Points and Cycles
2	1
3	1, 33, 34, 65
4	1, 32, 33, 245 → 488→ 308 → 245, 3 → 243 → 729 → 309 → 246 → 519 → 276 → 3
5	1, 308, 96 → 1268 → 518 → 1510 → 96, 34 → 1026 → 246 → 2050 → 276 → 34, 309 → 1089 → 1543 → 551 → 1057 → 309
6	1, 308, 2292, 2293, 2324, 3432, 3433, 6197, 128 → 518→ 128, 339 → 519 → 339, 2 → 32 → 3157 →1332 → 2, 2355 → 4425 → 3675 → 3400 → 2355, 2081 → 4636 → 2566 → 7276 → 5416 → 2081, 1004 → 4424 → 3464 → 1089 → 3369 → 1004, 520 → 1120 → 4150 → 1270 → 7275 → 4635 → 1785 → 520
7	1, 65, 1542, 3190, 3222, 3612, 3613, 4183, 9286, 4636 → 9076 → 4636, 4424 → 10934 → 9044 → 4424, 4425 → 10935 → 9045→ 4425, 551 → 4151 → 6251 → 2081 → 8051 → 551, 582 → 8802 → 3558 → 1332 → 8052 → 582, 1089 → 1269 → 9075 → 3855 → 5175 → 1089, 1543 → 1753 → 6493 → 10099 → 4393 → 8299 → 1543
8	1, 1056, 1057, 2323, 16819, 17864, 17865, 24583, 25639, 32 → 1024 → 32, 33 → 1025 → 33, 24615 → 25607 → 24615
9	1, 243, 244, 8052, 8295, 9857, 65538, 65539, 33 → 8019 → 33, 276 →8262 → 276, 1511 → 35925 → 12981 → 49851 → 21199 → 9075 → 1511, 34 → 17050 → 5448 → 18106 → 42446 → 8864 → 33256 → 9376 → 41634 → 8020 → 34, 3222 → 18074 → 24648 → 32634 →74336 → 43734 → 76680 → 15586 → 17568 → 40576 → 44694 → 41422 → 25882 → 51866 → 82376 → 99572 → 14060 → 3222
10	1, 4150, 4151, 54748, 92727, 93084, 194979, 58618 → 76438 →58618, 89883 →157596 → 89883, and seven longer cycles

$d \in \mathbf{Z}^+$, define a *d-consecutive sequence* to be an arithmetic sequence with constant difference d.

Note that if, for given values of e and b, every finite set T is (e,b)-good, then there exist arbitrarily long finite sequences of consecutive e-power b-happy numbers. (Take $T = \{1, 2, \dots, n\}$ and $u = 1$.) Similarly if, for given values of e and b, every finite set T that is contained in a single congruence class modulo d is (e,b)-good, then there exist arbitrarily long finite sequences of d-consecutive e-power b-happy numbers. (Take $T = \{d, 2d, \dots, nd\}$ and $u = 1$.)

We will need the following basic results proved in [**2**]. Let $I : \mathbf{Z}^+ \to \mathbf{Z}^+$ be defined by $I(t) = t + 1$.

Lemma 3. *If $T = \{t\} \subseteq \mathbf{Z}^+$, then T is (e,b)-good.*

Lemma 4. *Let $F : \mathbf{Z}^+ \to \mathbf{Z}^+$ be the composition of a finite sequence of the functions $S_{e,b}$ and I. If $F(T)$ is (e,b)-good, then T is (e,b)-good.*

Restricting now to the case $e = 5$, we will first prove that for any base b, every 5-power b-happy number is congruent to 1 modulo $d = \gcd(30, b-1)$ and that every $(5,b)$-good set is contained in a single congruence class modulo d. We will then show that there are infinitely many bases b for which there exist arbitrarily long finite sequences of d-consecutive 5-power b-happy numbers and that, in particular, this holds for $2 \le b \le 10$.

Lemma 5. *Fix $b \ge 2$. Let $d = \gcd(30, b-1)$. For each $x, m \in \mathbf{Z}^+$,*

$$S_{5,b}^m(x) \equiv x \pmod{d}.$$

Proof. First note that it follows from Fermat's little theorem that for $p \in \{2, 3, 5\}$ and for all $a \in \mathbf{Z}^+$, $a^5 \equiv a \pmod{p}$. Letting $x = \sum_{i=0}^n a_i b^i$, note that for any prime p dividing d,

$$x = \sum_{i=0}^n a_i b^i \equiv \sum_{i=0}^n \equiv a_i \equiv \sum_{i=0}^n a_i^5 \equiv S_{5,b}(x) \pmod{p}.$$

Since d is a product of distinct primes, we get that $S_{5,b}(x) \equiv x \pmod{d}$. A simple induction on m completes the proof. $\square$

Corollary 6. *Every 5-power b-happy number is congruent to 1 modulo d, where $d = \gcd(30, b-1)$.*

Corollary 7. *Every $(5, b)$-good set is contained in a single congruence class modulo $d = \gcd(30, b-1)$.*

The following lemma is key to the remaining results in this section. We prove it by generalizing the methods used to prove [**2**, Lemma 9].

Lemma 8. *Fix $b \geq 2$ and let $d = \gcd(30, b-1)$. If for each k, $0 \leq k < b-1$, there exists some $c \in \mathbf{Z}$ such that $5c(c^3+2c^2+2c+1) \equiv kd \pmod{b-1}$, then each finite set of positive integers contained in a single congruence class modulo d is $(5, b)$-good.*

Proof. Since every set is $(5, 2)$-good [**3**, Theorem 4], assume that $b > 2$. By Lemma 3, every one-element set is $(5, b)$-good and so by induction we may assume that T has $N > 1$ elements and that each set of fewer than N elements contained in a single congruence class modulo d is $(5, b)$-good.

Let $t_1 > t_2 \in T$. We consider three cases. In each we show that there exists a function F of the type described in Lemma 4 such that $F(t_1) = F(t_2)$.

First, if t_1 and t_2 have the same nonzero digits in base b, then $S_{5,b}(t_1) = S_{5,b}(t_2)$. So we let $F = S_{5,b}$. Next, if t_1 and t_2 do not have the same nonzero digits in base b, but $t_1 \equiv t_2 \pmod{b-1}$, then let $v = (t_1 - t_2)/(b-1) \in \mathbf{Z}^+$. Fix $r \in \mathbf{Z}^+$ so that $b^r > bv + t_2$ and let $m = b^r + v - t_2$. Then $I^m(t_1)$ and $I^m(t_2)$ have the same nonzero digits in base b. Therefore, we let $F = S_{5,b}I^m$.

Finally, if neither of the above holds, let $u = t_1 - t_2$. By the assumption on T, $u \equiv 0 \pmod{d}$. Using Lemma 5, $S_{5,b}(u-1) \equiv u - 1 \equiv -1 \pmod{d}$. Thus $-1 - S_{5,b}(u-1)$ is a multiple of d. By assumption then, there exists a c, $0 \leq c < b-1$, such that

$$(1) \qquad 5c(c^3 + 2c^2 + 2c + 1) \equiv -1 - S_{5,b}(u-1) \pmod{b-1}.$$

Fix $r' \in \mathbf{Z}^+$ such that $b^{r'} > t_1 + 1$, and let $m' = (c+1)b^{r'} - t_2 - 1 > 0$. Then

$$S_{5,b}(t_1 + m') = (c+1)^5 + S_{5,b}(u-1) \equiv c^5 \pmod{b-1}$$

and

$$S_{5,b}(t_2 + m') = c^5 + r'(b-1)^5 \equiv c^5 \pmod{b-1}.$$

Hence, $S_{5,b}(t_1 + m') \equiv S_{5,b}(t_2 + m') \pmod{b-1}$. Using the above argument, we can let $F = S_{5,b} I^m S_{5,b} I^{m'}$, for some appropriately chosen $m \geq 0$. Hence, in each case, there exists a map F as in Lemma 4 such that $F(t_1) = F(t_2)$.

This implies that $F(T)$ has fewer elements than does T. Further, the elements of T are in a single congruence class modulo d and this property is preserved by both I and $S_{5,b}$, implying that the same holds for $F(T)$. Therefore, by the induction hypothesis, $F(T)$ is $(5, b)$-good. Hence, by Lemma 4, T is $(5, b)$-good. □

The following theorem provides two infinite families of values of b that satisfy the hypothesis of Lemma 8 and hence for which there exist arbitrarily long finite d-consecutive sequences of b-happy numbers. It should be noted, however, that the applicability of the lemma is quite limited. For example, it can be shown that any value of b for which $b-1$ is divisible by a prime greater than five fails the hypothesis.

Theorem 9. *For any $r \in \mathbf{Z}^+$, let $b = 2^r + 1$ or $5 \cdot 2^r + 1$. Then, for $d = \gcd(30, b-1)$, there exist arbitrarily long finite d-consecutive sequences of b-happy numbers.*

Proof. Fix $r \in \mathbf{Z}^+$. As noted at the beginning of the section, it suffices to show that every finite set of positive integers contained in a single congruence class modulo d is $(5, b)$-good.

For $b = 5 \cdot 2^r + 1$, $d = 10$. By Lemma 8, it suffices to show that for every k, $0 \leq k < 5 \cdot 2^r$, there exists some $c \in \mathbf{Z}$ such that $5c(c^3 + 2c^2 + 2c + 1) \equiv 10k \pmod{5 \cdot 2^r}$ or equivalently, $c(c^3 + 2c^2 + 2c + 1) \equiv 2k \pmod{2^r}$. Letting $F(x) = x(x^3 + 2x^2 + 2x + 1) - 2k$ and applying Hensel's lemma, with $x_0 = 1$, yields the desired result.

For $b = 2^r + 1$, $d = 2$, and so it suffices to show that, for every k, $0 \le k < 2^r$, there exists some $c \in \mathbf{Z}$ such that $5c(c^3 + 2c^2 + 2c + 1) \equiv 2k \pmod{2^r}$. Again the result is immediate using Hensel's lemma. □

Finally, we show that, for $2 \le b \le 10$, there exist arbitrarily long finite sequences of d-consecutive 5-power b-happy numbers. Again we begin with a lemma.

Lemma 10. *Every finite set of positive integers is* $(5, 8)$*-good.*

Proof. Let T be a finite set of positive integers. By the definition of $U_{5,8}$, there exists some $m \in \mathbf{Z}^+$ such that $S_{5,8}^m(T) \subseteq U_{5,8}$. By Lemma 4, it therefore suffices to prove that $U_{5,8}$ is $(5, 8)$-good.

From Table 1, $U_{5,8} = \{1, 32, 33, 1024, 1025, 1056, 1057, 2323, 16819, 17864, 17865, 24583, 24615, 25607, 25639\}$. Direct computation shows that

$$S_{5,8}^{17} I^{10} S_{5,8}^{24} I^{10} S_{5,8}^{25} I^4 S_{5,8}^{31} I^2 (U_{5,8}) = \{32\}.$$

Therefore, by Lemmas 3 and 4, $U_{5,8}$ is $(5, 8)$-good. □

Corollary 7, Lemma 10, and a direct verification that the hypothesis of Lemma 8 holds for $b = 2, 3, 4, 5, 6, 7, 9, 10$ easily prove the following theorem and corollary.

Theorem 11. *Let* $2 \le b \le 10$ *and let* T *be a finite set of positive integers. Then* T *is* $(5, b)$*-good if and only if* T *is contained in a single congruence class modulo* $d = \gcd(30, b - 1)$.

Corollary 12. *For each* $2 \le b \le 10$*, there exist arbitrarily long finite sequences of* d*-consecutive* 5*-power* b*-happy numbers, where* $d = \gcd(30, b - 1)$.

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