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ABSOLUTELY p-SUMMABLE SEQUENCES IN BANACH SPACES AND RANGE OF VECTOR MEASURES

M.A. SOFI

ABSTRACT. We provide characterizations of Banach spaces X such that, for a given $p \geq 1$, each absolutely p-summable sequence in X is included inside the range of an X-valued measure. Demanding the vector measure to be of bounded variation results in the class of Banach spaces having (q)-Orlicz property which corresponds to the (classical) Orlicz property for $q = 2$ (here q is conjugate to p). A similar result where the vector measure (of bounded variation) is allowed to take its values in a super space of X is also proved. In the end, examples are provided to illustrate the usefulness of the results.

1. Introduction. The recognition of sequences in a Banach space X which are contained inside the range of a vector measure is an important theme in the theory of vector measures. In this connection, quite a good deal is known regarding members of an X-valued sequence space $E(X)$ being included inside the range of a vector measure. In a series of papers [**5, 6, 7, 10**], Pineiro and his collaborators were able to achieve a complete classification of Banach spaces X for $E(X)$ consisting of all null sequences with or without the assumption of bounded variation on the vector measure μ in question. Similar results pertaining to $E(X)$ consisting of weakly p-summable sequence have been treated in [**8, 9**]. However, these results do not cover the case involving vector measures of bounded variation taking values in a superspace of X, which was accomplished by the author in [**12**] for weakly p-summable sequences in X. The methods employed in that paper also make it possible to provide an alternative proof of an earlier result of Pineiro [**9**] to the effect that Hilbert spaces are the only Banach spaces X in which null sequences, equivalently the unit ball, can be 'wrapped' inside the range

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of a vector measure of bounded variation taking its values inside a Banach space containing X as a subspace.

In the present paper, we address these questions in the context of $E(X)$ consisting of absolutely p-summable sequences for $p \geq 1$ and show that, under the assumption of bounded variation, the spaces that result in the process are precisely those having (q)-Orlicz property (q being conjugate to p) which characterize X as being finite dimensional as long as $p > 2$. This result is then used to provide another useful description of Banach spaces X in terms of (the adjoint of) l_1-valued absolutely summing maps on X such that each absolutely p-summable sequence in X is contained inside the range of an X-valued vector measure. We shall also use this occasion to provide a characterization of Banach spaces X in terms of vector measures such that for $q > 2$, l_1-valued q-summing maps on X are already absolutely summing. In the final section, examples are given to show that the extreme cases involving the range of $p \in [2, \infty)$ as guaranteed by the results of this paper are indeed attained in certain concrete situations.

2. Definitions and notation. For various concepts pertaining to Banach spaces and the theory surrounding nuclear and p-summing maps as used in this paper, we shall follow [**2**] whereas our standard reference for vector measure theory shall be [**3**]. In what follows, $X, Y, \ldots$ shall denote Banach spaces with B_X and X^* denoting the closed unit ball and the dual of X, respectively. For $p > 1, q$ shall throughout denote the conjugate of $p : 1/p + 1/q = 1$. Given a bounded linear map $T : X \to Y$, we shall say that T is

Definition 2.1. (a) *Nuclear* ($T \in N(X, Y)$) if there exist bounded sequences $\{f_n\}_{n=1}^{\infty} \subset B_{X^*}$, $\{y_n\}_{n=1}^{\infty} \subset B_Y$ and $\{\lambda_n\}_{n=1}^{\infty} \in l_1$ such that

$$T(x) = \sum_{n=1}^{\infty} \lambda_n \langle x, f_n \rangle \; y_n, \quad x \in X.$$

(b) ∞-*nuclear* ($T \in N_\infty(X, Y)$) if there are $\{f_n\}_{n=1}^{\infty} \subset X^*$, $\{y_n\}_{n=1}^{\infty} \subset Y$ with $\lim_n \; f_n = 0$, $\varepsilon_1((y_n)) < \infty$ such that

$$T(x) = \sum_{n=1}^{\infty} \langle x, f_n \rangle \; y_n, \quad \text{for all } x \in X.$$

The norm ν_∞ on $N_\infty(X,Y)$ is defined by

$$\nu_\infty(T) = \inf\left\{\sup_n \ \|f_n\| \cdot \varepsilon_1((y_n))\right\}$$

where the infimum ranges over all sequences $\{f_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ admissible in the above series. $(N_\infty(X,Y),\nu_\infty)$ then becomes a Banach space. (See [**2**, Chapter 5]).

(c) (p,q)-(*absolutely*) *summing* $(p \geq q \geq 1)$, if there exists $c > 0$ such that

$$\left(\sum_{i=1}^n \ \|Tx_i\|^p\right)^{1/p} \leq c \sup_{f\in B_{X^*}} \ \left(\sum_{i=1}^n \ |\langle x_i, f\rangle|^q\right)^{1/q}$$

for all $x_i \in X$, $1 \leq i \leq n$, $n \geq 1$.

Denoting the least such c by $\pi_{p,q}(T)$, it turns out that $\Pi_{p,q}(X,Y)$, the space of (p,q)-summing maps is a Banach space when equipped with the (p,q)-summing norm $\pi_{p,q}$. The special case $p = q$ corresponds to p-summing maps (which equals absolutely summing maps for $p = 1$) which shall be denoted by $\Pi_p = \Pi_{p,p}$. For basic properties of (p,q)-summing maps, we refer to [**2**, Chapter 10]. Here we merely recall that p-summing maps between Hilbert spaces coincide with Hilbert-Schmidt maps and that, according to Grothendieck's theorem [**2**, Chapter 1], all bounded linear maps from $L_1(\mu)$ to $L_2(\nu)$ are absolutely summing, see also [**11**, Chapter 5].

We shall also say that a Banach space X verifies *Grothendieck's theorem* (or X has (GT)) if $L(X,l_2) = \Pi_1(X,l_2)$. In view of Grothendieck's theorem quoted above, L_1 has (GT). For a detailed account including further examples of (GT)-spaces, see [**11**].

Definition 2.2. For $p \geq 1$, the vector-valued sequence spaces $l_p[X]$ and $l_p\{X\}$ are defined by:

$$l_p[X] = \left\{\bar{x} = (x_n)_{n=1}^\infty \subset X : \sum_{n=1}^\infty \ |\langle x_n, x^*\rangle|^p < \infty, \quad \forall\ x^* \in X^*\right\}$$
$$l_p\{X\} = \left\{\bar{x} = (x_n)_{n=1}^\infty \subset X : \sum_{n=1}^\infty \ \|x_n\|^p < \infty\right\}$$

which turn into Banach spaces when equipped with the norms ε_p and σ_p, respectively where

$$\varepsilon_p(\bar{x}) = \sup\left\{\left(\sum_{n=1}^{\infty} |\langle x_n, x^*\rangle|^p\right)^{1/p} : x^* \in B_{X^*}\right\}, \quad \bar{x} \in l_p[X]$$

$$\sigma_p(\bar{x}) = \left(\sum_{n=1}^{\infty} \|x\|^p\right)^{1/p}, \quad \bar{x} \in l_p\{x\}.$$

Clearly, $l_p\{X\} \subset l_p[X]$ with $\varepsilon_p(\bar{x}) \leq \sigma_p(\bar{x})$ for all $\bar{x} \in l_p\{x\}$ and that equality holds precisely when X is finite-dimensional. The latter statement is the famous Dvoretzky-Rogers theorem to which we shall return in Section 3. The elements of $l_p[X]$ shall be referred to as *weakly p-summable* sequences whereas those of $l_p\{X\}$ shall be called *absolutely p-summable* sequences. An easy consequence of the uniform boundedness principle shows that $l_\infty[X] = l_\infty\{X\} = l_\infty(X)$ coincides with the space of all X-valued bounded sequences, which gets identified with $L(l_1, X)$, the space of bounded linear maps via the map:

$$l_\infty[X] \ni \bar{x} = (x_n)_{n=1}^{\infty} \longrightarrow T_{\bar{x}} \in L(l_1, X),$$

where

$$T_{\bar{x}}(\bar{\alpha}) = \sum_{n=1}^{\infty} \alpha_n x_n, \bar{\alpha} = (\alpha_n)_{n=1}^{\infty} \in l_1.$$

It is also clear that, for $X = \mathbf{K}$, the scalar field, $l_p[X] = l_p\{X\} = l_p$, the usual sequence space of all scalar sequences which are absolutely p-summable. We shall use e_i, $i \geq 1$, to denote the ith unit vector in l_p or l_p^n. An infinite sequence shall be denoted by $(x_n)_{n=1}^{\infty}$ and occasionally also by (x_n), and the symbol $\sum_n$ shall be taken to mean that n varies from 1 to ∞.

The identification: $l_\infty[X] = L(l_1, X)$ encountered above can be used to describe a useful relationship between certain variants of absolutely summing maps in $L(l_1, X)$ and sequences in X which are included inside the range of a vector measure. We shall throughout denote by $\mu : (\Omega, \Sigma) \to X$ a vector measure (v.m.) which shall be assumed to be countably additive on the σ-algebra Σ with its range being denoted by $rg(\mu)$:

$$rg(\mu) = \{\mu(A) : A \in \Sigma\}.$$

Further, μ shall be defined to be of *bounded variation* if

$$tv(\mu) = \sup_P \sum_{A \in P} \|\mu(A)\| < \infty,$$

where the supremum ranges over all (finite) partitions of Ω into pairwise disjoint members of Σ. In what follows, we shall be dealing with the following (vector-valued) sequence spaces determined by the ranges of vector measures of a special kind:

$$\begin{aligned}
R(X) &= \{\bar{x} = (x_n)_{n=1}^\infty \in l_\infty(X) \\
&\qquad \exists\ v.m.\ \mu : \Sigma \longrightarrow X, \ni:\ (x_n)_{n=1}^\infty \subset rg(\mu)\} \\
R_c(X) &= \{\bar{x} \in R(X) : rg(\mu) \text{ is compact}\} \\
R_{vbv}(X) &= \{\bar{x} \in l_\infty(X) : \exists\ X_0, \text{ a Banach space, an isometry} \\
&\qquad T : X \longrightarrow X_0, \mu : \Sigma \longrightarrow X_0 \text{ of bounded variation} \\
&\qquad \text{such that} \quad Tx_n \in rg(\mu),\ \forall\ n \geq 1\} \\
R_{bbv}(X) &= R_{vbv}(X) \quad \text{for} \quad X_0 = X^{**} \\
R_{bv}(X) &= R_{vbv}(X) \quad \text{for} \quad X_0 = X.
\end{aligned}$$

It is not difficult to see that when equipped with the 'total variation' norm $tv(\mu), R_{bv}(X)$ becomes a Banach space. The same is true of all other spaces defined above when equipped with 'natural' norms as was shown in [**6, 9**]. (See also [**12**, Theorem 3.1].) Further, we have the following useful result which will be used in the sequel.

Theorem 2.3 [**7, 9**]. *Let X be a Banach space and $\bar{x} = (x_n)_{n=1}^\infty \subset X$ a bounded sequence. Then*

a) *$\bar{x} \in R_{bv}(X)$ if and only if $T_{\bar{x}}$ is strictly integral.*

b) *$\bar{x} \in R_{bbv}(X)$ if and only if $T_{\bar{x}}$ is integral.*

c) *$\bar{x} \in R_{vbv}(X)$ if and only if $T_{\bar{x}}$ is absolutely summing.*

d) *$\bar{x} \in R_c(X)$ if and only if $T_{\bar{x}}$ is ∞-nuclear.*

3. Main results. We start with the following theorem, giving necessary and sufficient conditions guaranteeing the containment of members of $l_p\{X\}$ inside the range of an X-valued measure.

Theorem 3.1. *For $1 < p < \infty$, the following statements are equivalent for a Banach space X:*

(i) $l_p\{X\} \subset R_c(X)$,

(ii) $l_p\{X\} \subset R(X)$,

(iii) *there exists a $c > 0$ such that for all $(x_i^*)_{i=1}^n \subset X^*$, $n \geq 1$,*

$$\left(\sum_{i=1}^n \|x_i^*\|^q\right)^{1/q} \leq c\ \pi_1\left(\sum_{i=1}^n x_i^* \otimes e_i : X \longrightarrow l_1^n\right).$$

(iv) *there exists a $c > 0$ such that for all $(x_i)_{i=1}^n \subset X$, $(x_i^*)_{i=1}^n \subset X^*$ and $n \geq 1$,*

$$\sum_{i=1}^n |\langle x_i, x_i^*\rangle| \leq c\ \pi_1\left(\sum_{i=1}^n x_i^* \otimes e_i : X \longrightarrow l_1^n\right)\ \sigma_p((x_i)_{i=1}^n).$$

Proof. We shall make use of the following lemma [**10**, Proposition 2] concerning continuous linear functionals on $R(X)$:

Lemma. *Given $T \in \Pi_1(X, l_1)$ such that $T(x) = (\langle x, x_n^*\rangle)_{n=1}^\infty, x \in X$, the map: $\psi_T(\bar{x}) = \sum_{n=1}^\infty \langle x_n, x_n^*\rangle, \bar{x} = (x_n) \in R(X)$ defines a continuous linear functional on $R(X)$ such that*

$$\|\psi_T\| \leq \pi_1(T).$$

We begin by noting that (i) $\Rightarrow$ (ii) is trivial whereas (iii) $\Rightarrow$ (iv) follows from Holder's inequality. Thus, it suffices to show that (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (iii). Let $(x_i^*)_{i=1}^n \subset X^*$ be chosen arbitrarily. Then, by the above lemma, $S = \sum_{i=1}^n x_i^* \otimes e_i \in \Pi_1(X, l_1^n)$ gives rise to $\psi_S \in (R(X))^*$ where

$$\psi_S(\bar{x}) = \sum_{i=1}^n \langle x_i, x_i^*\rangle,$$

such that

$$|\psi_S(\bar{x})| \leq \pi_1(S)\ \|\bar{x}\|_{R(X)}, \quad \text{for all } \bar{x} \in R(X). \tag{1}$$

Further, by slightly modifying the proof of Theorem 3.1 ((i) $\Rightarrow$ (ii)) of [**12**], it follows that the inclusion map: $l_p\{X\} \subset R(X)$ is continuous which when combined with (1) yields $c > 0$ such that

$$|\psi_S(\bar{x})| \leq c\ \pi_1(S)\ \sigma_p(\bar{x}), \quad \text{for all } \bar{x} \in l_p\{X\}.$$

This shows that $\psi_S \in (l_p\{X\})^* = l_q\{X^*\}$ such that

$$\|\psi_S\| = \sigma_q((x_i^*)_{i=1}^n).$$

Combining these estimates gives:

$$\left(\sum_{i=1}^n \|x_i^*\|^q\right)^{1/q} = \sigma_q((x_i^*)_{i=1}^n) = \|\psi_S\| \leq c\,\pi_1\left(\sum_{i=1}^n x_i^*\otimes e_i : X \longrightarrow l_1^n\right),$$

which gives (iii).

(iv) $\Rightarrow$ (i). Let $\varphi(X)$ denote the 'eventually' zero sequences in X, consisting of sequences which are eventually zero after some term onwards. To show that (i) holds, define a map $\psi : (\varphi(X), \sigma_p) \to N_\infty(l_1, X)$ by $\psi(\bar{x}) = T_{\bar{x}}$. Here σ_p is the norm on $\varphi(X)$ induced by $l_p\{X\}$.

Claim. *ψ is continuous.*

Choose $\bar{x} = (x_1, x_2, \dots, x_n, 0, 0, 0, \dots) \in \varphi(X)$, $n \geq 1$. Then, using trace duality applied to $T_{\bar{x}}$ as a map in $N_\infty(l_1^n, X)$, (iv) yields

$$\begin{aligned}
&\nu_\infty(\psi(\bar{x}))\\
&= \nu_\infty(T_{\bar{x}})\\
&= \sup\{|\mathrm{trace}\,(T_{\bar{x}} \circ S) : S \in \Pi_1(X, l_1^n), \pi_1(S) \leq 1\}\\
&\quad = \sup\left\{\left|\sum_{i=1}^n \langle x_i, S^*e_i^*\rangle\right| : S = \sum_{i=1}^n S^*e_i^* \otimes e_i \in \Pi_1(X, l_1^n), \pi_1(S) \leq 1\right\}\\
&\leq c\ \sup\left\{\pi_1\left(\sum_{i=1}^n S^*e_i^* \otimes e_i : X \longrightarrow l_1^n\right)\sigma_p(\bar{x}) : \pi_1(S) \leq 1\right\}\\
&\leq c\ \sigma_p(\bar{x}).
\end{aligned}$$

This shows that ψ is continuous and, therefore, can be extended as a continuous linear map to $l_p\{X\}$ which contains $\varphi(X)$ as a dense subspace. As is easily seen, the extended map, denoted again by ψ, is given by:

$$\psi(\bar{x}) = T_{\bar{x}},$$

which means that $T_{\bar{x}} \in N_\infty(l_1, X)$. Invoking Theorem 2.3(d), it follows that $\bar{x} \in R_c(X)$, and the proof is completed. □

Remark 3.2. Using the same approach as in the above theorem and recalling the identification: $R_{vbv}(X) = \Pi_2(l_1, X)$ (Theorem 2.3(c)), leads to the following theorem pertaining to the containment of $l_p\{X\}$ into $R_{vbv}(X)$.

Theorem 3.3. *For a Banach space X and $p \geq 1$, the following statements are equivalent*:

(i) $l_p\{X\} \subset R_{vbv}(X)$,

(ii) *there exists a $c > 0$ such that for all $(x_i^*)_{i=1}^n \subset X^*$ and $n \geq 1$,*

$$\left(\sum_{i=1}^n \ \|x_i^*\|^q \right)^{1/q} \leq c \ \pi_2 \left(\sum_{i=1}^n x_i^* \otimes e_i : X \longrightarrow l_1^n \right)$$

(iii) *there exists a $c > 0$ such that for all $(x_i)_{i=1}^n \subset X, (x_i^*)_{i=1}^n \subset X^*$ and $n \geq 1$,*

$$\sum_{i=1}^n |\langle x_i, x_i^* \rangle| \leq c \ \pi_2 \left(\sum_{i=1}^n x_i^* \otimes e_i : X \longrightarrow l_1^n \right) \ \sigma_p((x_i)_{i=1}^n).$$

Here in the proof of the implication (i) $\Rightarrow$ (ii), we use the fact that every map in $\Pi_2(X, l_1)$ acts as a continuous linear functional on $R_{vbv}(X)$. A proof of this statement is included in [**12**, Theorem 3.1] which also includes a proof of a similar result pertaining to the inclusion $l_p[X] \subset R_{vbv}(X)$. Proceeding on similar lines and using the fact that a bounded linear map in $L(X, l_1)$ induces a continuous linear functional on $I(l_1, X)$, the space of integral maps $(= R_{bbv}(X))$, see [**4**, Theorem 6.16(a)], we can state and prove the following theorem on the containment of $l_p\{X\}$ into $R_{bbv}(X)$.

Theorem 3.4. *For $1 < p < \infty$, the following statements are equivalent for a Banach space X:*

(i) $l_p\{X\} \subset R_{bbv}(X)$.

(ii) *there exists a $c > 0$ such that for all $(x_i^*)_{i=1}^n \subset X^*$ and $n \geq 1$,*

$$\left(\sum_{i=1}^n \|x_i^*\|^q\right)^{1/q} \leq c \left\| \left(\sum_{i=1}^n x_i^* \otimes e_i : X \longrightarrow l_1^n\right) \right\|.$$

(iii) *there exists a $c > 0$ such that for all $(x_i)_{i=1}^n \subset X, (x_i^*)_{i=1}^n \subset X^*$ and $n \geq 1$,*

$$\sum_{i=1}^n |\langle x_i, x_i^* \rangle| \leq c \left\| \left(\sum_{i=1}^n x_i^* \otimes e_i : X \longrightarrow l_1^n\right) \right\| \; \sigma_p(x_i)_{i=1}^n.$$

Before we state the next corollary, let us recall that a Banach space is said to have (q)-*Orlicz property*, $1 \leq q < \infty$, if unconditionally convergent series in X are absolutely q-convergent (summable). It is a highly nontrivial theorem of Talagrand that, for $q > 2$, cotype q spaces are exactly those which have (q)-Orlicz property!

Corollary 3.5. *For $p > 1$, $l_p\{X\} \subset R_{bbv}(X)$ if and only if X^* has (q)-Orlicz property. In particular, for $p > 2$, $l_p\{X\} \subset R_{bbv}(X)$ exactly when X is finite-dimensional.*

Proof. It is enough to observe that $L(X, l_1)$ can be isometrically identified with $l_1[X^*]$ and that unconditionally convergent series in X^* correspond to (a subspace of) $l_1[X^*]$. Combined with Theorem 3.4 (i)–(ii), it follows that unconditionally convergent series in X^* are absolutely q-convergent. Finally, the finite-dimensionality part of the corollary is a consequence of the Dvoretzky-Rogers theorem for $1 \leq q < 2$, see [**2**, Theorem 10.5].

Remark 3.6. An alternative and direct proof of the above corollary which is interesting in its own right may be sketched as follows.

Indeed, assume that X^* has the (q)-Orlicz property, and fix $\bar{x} = (x_n) \in l_p\{X\}$. We show that $(\alpha_n x_n) \in R_{bbv}(X)$ for all $\bar{\alpha} = (\alpha_n) \in c_0$, so that by virtue of [**5**, Theorem 1], it follows that $\bar{x} \in R_{bbv}(X)$. Now, given $T = \sum_{n=1}^{\infty} x_n^* \otimes e_n \in K(X, l_1)$, we see that $(x_n^*) \in l_1[X^*]$ so that the (q)-Orlicz property of X^* combined with Holder's inequality yields $c > 0$ such that

$$\begin{aligned}\sum_{n=1}^{\infty} \|x_n\| \, \|x_n^*\| &\leq \bigg(\sum_{n=1}^{\infty} \|x_n\|^p\bigg)^{1/p} \bigg(\sum_{n=1}^{\infty} \|x_n^*\|^q\bigg)^{1/q} \\ &\leq c \; \sigma_p(\bar{x})\varepsilon_1((x_n^*)),\end{aligned}$$

which proves that the map $\psi : K(X, l_1) \to l_1\{X^*\}$ given by: $\psi(T) = (\|x_n\| x_n^*)_{n=1}^{\infty}$ is well defined and continuous. Dualizing and denoting by I the class of integral operators, we get, by virtue of [**4**, Chapter 19],

$$\psi^* : l_\infty\{X^{**}\} \longrightarrow I(l_1, X^{**})$$

where

$$\psi^*((x_n^{**}))(T) = \langle (x_n^{**}), \psi(T)\rangle = \sum_{n=1}^{\infty} x_n^{**}(x_n^*) \; \|x_n\| = \operatorname{trace}(ST),$$

and $S = \sum_{n=1}^{\infty} e_n^* \otimes \|x_n\| \; x_n^{**} \in I(l_1, X^{**})$. This shows that $\psi^*((x_n^{**})) = S$, so that in particular, ψ^* actually maps $c_0(X)$ into $I(l_1, X)$ and that

$$\psi^*(\bar{y}) = \sum_{n=1}^{\infty} e_n^* \otimes \|x_n\| y_n, \quad \bar{y} = (y_n) \in c_0(X).$$

An application of Theorem 2.3(b) shows that $(\|x_n\| y_n) \in R_{bbv}(X)$. In particular, $(\alpha_n x_n) \in R_{bbv}(X)$ for all $\bar{\alpha} = (\alpha_n) \in c_0$, and this completes the argument.

Conversely, assume that $l_p\{X\} \subset R_{bbv}(X)$. By Theorem 2.3(b), the map $\psi : l_p\{X\} \to I(l_1, X)$ where $\psi(\bar{x}) = T_{\bar{x}}$ is well defined and also continuous. Noting that each $\bar{x}$ in $l_p\{X\}$ is a limit of its 'nth-sections' in $l_p\{X\}$ and that $N(l_1, X)$ is a closed subspace of $I(l_1, X)$, it follows that ψ actually maps $l_p\{X\}$ into $N(l_1, X)$. Taking conjugates gives: $\psi^* : L(X, l_1^{**}) \to l_q\{X^*\}$ where $\psi^*(S)(\bar{x}) = \text{trace}\,(T_{\bar{x}} \circ S)$, for all $S \in L(X, l_1^{**})$ and $\bar{x} \in l_p\{X\}$. Finally, let $\sum_{n=1}^{\infty} x_n^*$ be unconditionally convergent in X^*. Then, for $S = \sum_{n=1}^{\infty} x_n^* \otimes e_n \in L(X, l_1)$, we have $\psi^*(S)(\bar{x}) = \sum_{n=1}^{\infty} \langle x_n, x_n^{\infty} \rangle$, for all $\bar{x} = (x_n) \in l_p\{X\}$, which yields that $\psi^*(S) = (x_n^*) \in l_q\{X^*\}$ and, therefore, X^* has the (q)-Orlicz property.

Remark 3.7. Corollary 3.5 provides a refinement of the results of Pineiro [**6, 8**] pertaining to the description of Banach spaces X such that $c_0(X) \subset R_{bbv}(X)$ or $l_p[X] \subset R_{bbv}(X)$ for $p > 2$. The special case of our corollary corresponding to $p = 2$ was treated by Pineiro in [**7**]. See also [**1**, Corollary 6(c)].

Remark 3.8. It is possible to interpret the above results in terms of linear operators between X and l_1. Thus, we have

a) $l_p\{X\} \subset R(X) \Rightarrow \Pi_1(X, l_1) \subset N_q(X, l_1)$,

b) $l_p\{X\} \subset R_{vbv}(X) \Rightarrow \Pi_2(X, l_1) \subset N_q(X, l_1)$,

c) $l_p\{X\} \subset R_{bbv}(X) \Rightarrow L(X, l_1) = N_q(X, l_1)$.

Here N_q stands for q-nuclear maps, see [**2**, Chapter 5]. Back to Theorem 3.4, where the proof of the equivalence (i) $\Leftrightarrow$ (iii) can be generalized with suitable modifications to assert the following:

Proposition 3.9. *For a bounded linear operator $T : X \to Y$, it holds that T maps sequences $\bar{x} = (x_n)$ in X from $l_p\{X\}$ into $(T(x_n)) \in R_{bbv}(Y)$ if and only if $T^* : Y^* \to X^*$ is $(q,1)$-summing.*

A more general result, subsuming the above result and involving the so called "(p, q)-summing multipliers" is also true. A proof of that statement shall appear elsewhere.

We can now use Proposition 3.9 to give another useful characterization of Banach spaces X such that absolutely p-summable sequences in X are included inside the range of an X-valued measure.

Theorem 3.10. *For a Banach space X and $p > 1$, the following statements are equivalent*:

(i) $l_p\{X\} \subset R(X)$.

(ii) $\Pi_1(X,Y) \subset \Pi^d_{q,1}(X,Y)$, *for all Banach spaces* Y.

(iii) $\Pi_1(X,l_1) \subset \Pi^d_{q,1}(X,l_1)$.

Here $\Pi^d_{q,1}$ *stands for those operators whose adjoint is* $(q,1)$*-summing.*

Proof. (i) $\Rightarrow$ (ii): Let $T \in \Pi_1(X,Y)$ and $\bar{x} = (x_n) \in l_p\{X\}$ be arbitrarily chosen. In view of Proposition 3.9, it suffices to show that $(T(x_n)) \in R_{bbv}(X)$. By Theorem 3.1 ((i) $\Leftrightarrow$ (ii)), it follows that $\bar{x} \in R_c(X)$ and, therefore, by [**10**, Proposition 1.4] applied to $\bar{x}$, there exists an unconditionally convergent series $\sum_n y_n$ in X such that $x_n \in \sum_{m=1}^\infty [-y_m, y_n] = \{x \in X : x = \sum_{m=1}^\infty \alpha_m y_m$, for some $\bar{\alpha} = (\alpha_n) \in B_{l_\infty}\}$. By the definition of T, we have $\sum_{m=1}^\infty \|Ty_m\| < \infty$, so that by virtue of [**6**, Proposition 2.1], $(Tx_n) \in R_{bv}(X)$.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Here again we invoke Theorem 3.1 to prove our assertion by showing that (iii) of Theorem 3.1 holds. To this end, fix $n \geq 1$ and $(x_i^*)_{i=1}^n \subset X^*$. Then for $S = \sum_{i=1}^n x_i^* \otimes e_i \in \Pi_1(X, l_1^n)$, we have $S \in \Pi^d_{q,1}(X, l_1^n)$. Now (iii) yields that there exists $c > 0$ such that

$$(2) \qquad \pi_{q,1}(T^*) = \pi^d_{q,1}(T) \leq c\, \pi_1(T), \quad \text{for all } T \in \Pi_1(X, l_1^n),\ n \geq 1.$$

By the given hypothesis, $S^* \in \Pi_{q,1}(l_\infty^n, X^*)$ which translates into the estimate

$$(3) \qquad \left(\sum_{i=1}^m \|S^*(\bar{\alpha}_i)\|^q\right)^{1/q} \leq \pi_{q,1}(S^*) \sup\left\{\sum_{i=1}^m |\langle \bar{\alpha}_i, \bar{\beta}\rangle| : \bar{\beta} \in B_{l_1^n}\right\}$$

for all $(\bar{\alpha}_i)^m \subset l_\infty^n$ and $m \geq 1$.

Combining (2) and (3) and noting that $S^*(e_i) = x_i^*$, $1 \leq i \leq n$, we get

$$\left(\sum_{i=1}^n \|x_i^*\|^q\right)^{1/q} \leq c\, \pi_1\left(\sum_{i=1}^n x_i^* \otimes e_i : X \longrightarrow l_1^n\right)$$

which was required to be proved.

A similar result involving the containment of $l_p\{X\}$ inside $R_{vbv}(X)$ can be stated as follows:

Theorem 3.11. *For a Banach space X and $p > 1$, the following statements are equivalent*:

(i) $l_p\{X\} \subset R_{vbv}(X)$,

(ii) $\Pi_2(X, Y) \subset \Pi^d_{q,1}(X, Y)$, *for all Banach spaces Y*,

(iii) $\Pi_2(X, l_1) \subset \Pi^d_{q,1}(X, l_1)$.

The proof of the above statement follows exactly on the lines of Theorem 3.10, except that in the case of implication (i) $\Rightarrow$ (ii), we use the easily checked fact that a 2-summing map pushes sequences in X from $R_{vbv}(X)$ into $R_{bv}(Y)$. This is a consequence of Theorem 2.3(c) combined with the well-known fact [**2**, Theorem 5.31] that a composite of 2-summing maps is always nuclear.

Proceeding on similar lines, we can state and prove the analogous statement regarding $R_{bv}(X)$.

Theorem 3.12. *For a Banach space X and $1 < p < \infty$, the following statements are equivalent*:

(i) $l_p\{X\} \subset R_{bv}(X)$,

(ii) $l_p\{X\} \subset R_{bbv}(X)$,

(iii) $L(X, Y) = \Pi^d_{q,1}(X, Y)$, *for all Banach spaces Y*

(iv) $L(X, l_1) = \Pi^d_{q,1}(X, l_1)$.

The above argument can be slightly modified to give proofs of analogous statements involving the spaces of weakly p-summable sequences, with the ideal $\Pi_{q,1}$ now being replaced by $\Pi_{1,q,1}$, the ideal of $(1, q, 1)$-summing maps.

We conclude this section by including another useful characterization of Banach spaces X for which $\Pi_q(X, l_1) = \Pi_1(X, l_1), q \geq 2$. For $q = 2$ and recalling that 2-summing maps coincide with 2-integral maps, we recover Pineiro's theorem to the effect that the indicated equality holds exactly when sequences in X included inside the ranges of X-valued measures are already contained inside the ranges of vector measures of bounded variation taking values in a space larger than X.

Theorem 3.13. *For a Banach space X and $p \geq 2$, the following statements are equivalent*:

(i) *Every sequence $\bar{x} = (x_n)$ in X contained inside the range of an X-valued measure μ induces a p-integral operator $T_{\bar{x}} \in L(l_1, X)$.*

(ii) *Same as* (i) *with the range of μ being relatively compact.*

(iii) $\Pi_q(X, l_1) = \Pi_1(X, l_1)$.

Proof. We shall briefly sketch the proof of (iii) $\Rightarrow$ (i) as the proof of (ii) $\Rightarrow$ (iii) follows by reversing the steps involved in the proof of (iii) $\Rightarrow$ (i). Likewise, (i) $\Rightarrow$ (ii) follows on the lines of Theorem 3.1 ((i) $\Rightarrow$ (iii)).

Assume that (iii) holds, and let $\bar{x} = (x_n) \in R(X)$. Then Proposition 2 of [**9**] applies to assert that $\sum_n |\langle x_n, x_n^* \rangle| < \infty$, for all $S = \sum_n x_n^* \otimes e_n \in \Pi_1(X, l_1)$. Combined with (iii), this leads to the existence of a map $\psi : \Pi_q(X, l_1) \to l_1$, where $\psi(S) = (\langle x_n, x_n^* \rangle)$ for $S = \sum_n x_n^* \otimes e_n \in \Pi_q(X, l_1)$. Dualizing, we get

$$\begin{aligned} \psi^* : l_\infty &\longrightarrow I_p(l_1, X^{**}), \quad \text{where} \\ (\psi^*(\bar{\alpha}))(S) &= \sum_n e_n^*(\bar{\alpha}) \langle x_n, x_n^* \rangle \\ &= \text{trace}\left(\left(\sum_n e_n^* \otimes \alpha_n x_n \right) \circ \left(\sum_n x_n^* \otimes e_n \right) \right) \\ &= \left\langle \sum_n e_n^* \otimes \alpha_n x_n, S \right\rangle. \end{aligned}$$

Equivalently, $\psi^*(\bar{\alpha}) = \sum_n e_n^* \otimes \alpha_n x_n \in I_p(l_1, X)$ for all $\bar{\alpha} \in l_\infty$. In particular, $T_{\bar{x}} = \sum_n e_n^* \otimes x_n \in I_p(l_1, X)$ and (i) is established.

4. Examples. In this final section, we apply the results of Section 3 to examine the extreme cases involving the range of $p \geq 1$ that can occur in certain concrete situations. To this end, we introduce the following sets of real scalars associated with a Banach space X. (See [**8, 9**].)

$$\begin{aligned} r_a(X) &= \{p \in [1, 2] : l_p\{X\} \subset R_{bbv}(X)\} \\ s_a(X) &= \{p \in (2, \infty) : l_p\{X\} \subset R(X)\} \\ v_a(X) &= \{p \in (2, \infty) : l_p\{X\} \subset R_{vbv}(X)\}. \end{aligned}$$

The reason why the range of p in each of the above sets has been restricted as indicated follows from known results in the theory of vector measures where, for instance, it is well known that $l_2[X] \subset R(X)$. This explains the choice of range of p in $s_a(X)$ whereas the case of $v_a(X)$ follows from [**12**, Corollary 3.2] where it is shown that, in fact, $l_2[X] \subset R_{vbv}(X)$. Finally, the fact that for $p > 2$, the inclusion $l_p\{X\} \subset R_{bbv}(X)$ forces X to be finite-dimensional (Corollary 3.5) explains why we restrict $p \leq 2$ in the definition of $r_a(X)$. Also, it follows that $v_a(X) \subset s_a(X)$, see [**7**].

Example 4.1. Let X be a Banach space such that X^* has cotype 2. By Corollary 3.5, it follows easily that $r_a(X) = [1, 2]$. In particular, $r_a(l_p) = [1, 2]$ for $p \geq 2$. The same also holds for L_p-spaces for $p \geq 2$. This is in sharp contrast with the corresponding situation for weakly s-summable sequences in l_p where it is known [**8**, Section 3] that for each $s \in (1, 2]$, there exist weakly s-summable sequences in l_p, $p \geq 1$, which are not contained inside the range of an l_p-valued measure of bounded variation! The same is true for L_p spaces, $1 \leq p < \infty$. Further, there exist Banach spaces X with $r_a(X) = [1, 2]$ but X^* lacks the cotype 2 property. An example to this effect was discovered in his seminal work by Talagrand [**13**]. It is also easy to see that $r_a(l_p) = [1, p]$ for $1 \leq p < 2$. This is a consequence of Corollary 3.5 combined with the fact that $l_p^* = l_q$ has cotype r for $r \geq q$ but no cotype s for $s < q$. Further, we also have $r_a(l_\infty) = \phi$. Obviously, these statements are also valid for infinite dimensional L_p-spaces, $1 \leq p \leq \infty$.

Example 4.2. According to a theorem of Pineiro and Rodriguez Piazza [**10**, Theorem 4.4], given a Banach space X such that X^* is a subspace of an L^1-space, it holds that $c_0(X) \subset R(X)$. In particular, for these spaces, $s_a(X) = (2, \infty)$. However, there are situations when the other extreme situation can occur, viz., $s_a(X) = \phi$. This happens, for instance, in the case of an infinite-dimensional Banach space X of cotype 2 such that X^* has (GT). Indeed, the cotype 2 property of X yields $\Pi_2(X, l_1) = \Pi_1(X, l_1)$ whereas (GT)-property of X^* gives: $\Pi_2(X, l_1) = L(X, l_1)$. Combining these two equalities gives $c > 0$ such that

$$\pi_1(T) \leq c\,\|T\|, \quad \text{for all } T \in L(X, l_1).$$

Assume, on the contrary, that there exists $2 < p < \infty$ such that

$p \in s_a(X)$. By Theorem 3.1, we can choose $c' > 0$ such that, for $n \geq 1$, we have

$$\begin{aligned}\left(\sum_{i=1}^n \|x_i^*\|^q\right)^{1/q} &\leq c'\pi_1\left(\sum_{i=1}^n x_i^* \otimes e_i : X \longrightarrow l_1^n\right)\\ &\leq cc' \left\|\left(\sum_{i=1}^n x_i^* \otimes e_i : X \longrightarrow \ell_1^n\right)\right\|\\ &= cc' \ \sup\left\{\sum_{i=1}^n |\langle x, x_i^*\rangle| : x \in B_X\right\},\end{aligned}$$

which shows that X^* has (q)-Orlicz property where $q < 2$. An application of Dvoretzky-Rogers theorem (refer to the proof of Corollary 3.5) yields that $\dim X < \infty$!

The above conclusion provides a strengthening of Pineiro's observation [**9**] that for infinite dimensional Banach spaces of cotype 2 with X^* having (GT) and for $p > 2$, there exist *weakly* p-summable sequences in X which are not contained inside the range of an X-valued measure.

Example 4.3. It was shown in [**7**], see also [**12**], that for a Hilbert space X, all X-valued null sequences are included inside the range of a vector measure of bounded variation taking its values in a superspace of X. This yields, in particular, that $v_a(X) = (2, \infty)$ whenever X is a Hilbert space. On the other hand, Theorem 3.3 yields that $v_a(X) = \phi$ whenever X is an infinite-dimensional space with X^* having (GT). Indeed, assuming the contrary yields the existence of $2 < p < \infty$ and $c_1 > 0$ such that, for each $n \geq 1$,

$$\left(\sum_{i=1}^n \|x_i^*\|^q\right)^{1/q} \leq c_1\pi_2\left(\sum_{i=1}^n x_i^* \otimes e_i : X \longrightarrow l_1^n\right).$$

Also, the (GT)-property of X^* gives $c_2 > 0$ such that

$$\pi_2(T) \leq c_2\|T\|, \quad \text{for all } T \in L(X, l_2).$$

In particular, given $n \geq 1$ and $(x_i^*)_{i=1}^n \subset X^*$, we have

$$\pi_2\left(\sum_{i=1}^n x_i^* \otimes e_i : X \longrightarrow l_1^n\right) \leq c_2 \left\|\left(\sum_{i=1}^n x_i^* \otimes e_i : X \longrightarrow l_1^n\right)\right\|. \tag{5}$$

Combining (4) with (5) gives

$$\left(\sum_{i=1}^{n} \|x_i^*\|^q\right)^{1/q} \leq c \ \sup\left\{\sum_{i=1}^{n} |\langle x_i^*, x\rangle| : x \in B_X\right\}, \quad \text{for all } n \geq 1.$$

In other words, X^* has (q)-Orlicz property for $q < 2$ which forces X to be finite dimensional by Dvoretzky-Rogers theorem referred to above.

The above discussion when applied to the disc algebra $A(D)$ yields that, for *each* $p > 2$, there exists an absolutely p-summable sequence in $A(D)$ which is not contained inside the range of a vector measure of bounded variation, regardless of the superspace X (containing $A(D)$) in which the vector measure is allowed to take its values. The same is also true for Pisier's space or any $C(K)$-space. However, something more can be said about Pisier's space P. In fact for $X = P$, Example 4.2 yields for each $p > 2$, the existence of an absolutely p-summable sequence in X which is not contained inside the range of an X-valued measure, with or without bounded variation! On the other hand, Example 4.1 tells us that each absolutely p-summable sequence, in $X = A(D)$ or P, can be 'wrapped' inside the range of an X^{**}-valued measure of bounded variation as long as $1 \leq p \leq 2$. The last statement is reminiscent of a well-known theorem of Diestel and Anantharaman to the effect that, given a Banach space X and $1 \leq p \leq 2$, every weakly p-summable sequence in X can be enclosed inside the range of an X-valued measure, not necessarily having bounded variation.

We conclude with the following open problems belonging to this circle of ideas which are motivated by the above discussion.

Problem 1. *Let X be a Banach space such that $v_a(X) = (2, \infty)$. Does it follow that X is a Hilbert space?*

Problem 2. *Do there exist Banach spaces X such that $s_a(X) = (2, \infty)$ but X^* is not a subspace of L^1? (A special case of Problem* 2 *also appears in* [**9**].)

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Department of Mathematics, University of Kashmir, Srinagar - 190 006, India
Email address: aminsofi@rediffmail.com