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FLAT EPIMORPHISMS AND A GENERALIZED KAPLANSKY IDEAL TRANSFORM

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ABSTRACT. We generalize the notion of the Kaplansky ideal transform $\Omega(I)$ to an ideal I in an arbitrary commutative ring R by defining $\Omega(I)$ as the localization of R with respect to a certain filter of ideals. It is shown that if the total ring of quotients of R is von Neumann regular, then $\Omega(I)$ is the ring of global sections over the open set $D(I)$. Additionally for such rings, we characterize when the open set $D(I)$ is an affine scheme in terms of the flatness of $\Omega(I)$.

1. Introduction. Let R be an integral domain with quotient field K. For an ideal I, the Nagata ideal transform $N_R(I) = \cup_{n\geq 0}(R :_K I^n) = \{q \in K : I^n \subseteq (R :_K q) \text{ for some } n \geq 0\}$, where $(R :_X q) = \{r \in X : rq \in R)$, has proven to be a very useful tool in various areas of commutative ring theory. Not only in its original application by Nagata in solving Hilbert's fourteenth problem, but also in the general study of overrings, see [**1, 2, 9, 10**].

However, once one leaves the realm of Noetherian rings, the Nagata transform apparently is not as useful a tool. For nonfinitely generated ideals I of an integral domain R, a variant of this transform has been studied and proven to be of significant value. The Kaplansky (ideal) transform of R with respect to an ideal I of R is the overring

$$\Omega_R(I) := \{q \in K : I \subseteq \operatorname{Rad}(R :_R q)\}.$$

Observe that $\Omega_R(I)$ is an overring of the Nagata ideal transform of I, with the two transforms equal if I is finitely generated.

As noted by Fontana in [**4**], localizing (or Gabriel) filters of ideals and generalized rings of quotients is a natural approach to the study of ideal transforms (see Section 2 for the definitions and basic results). Carrying this notion a step further, we define a generalized Kaplansky

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ideal transform, also denoted $\Omega(I)$, for an ideal I of an arbitrary (commutative) ring in terms of the ring of quotients with respect to the localizing filter

$$\mathcal{F}_I = \{J \subseteq R : I \subseteq \mathrm{Rad}\,(J)\}.$$

When R is an integral domain, this new definition agrees with the old one, hence there is no chance of confusion. In Section 3, using [**4**] in part as a template, we generalize a number of results on the Kaplansky transform from integral domains to a more general setting. While the ideal transform is defined for an arbitrary ring, we obtain our deepest results with the additional assumption that the total ring of quotients of R, denoted $T(R)$, is von Neumann regular.

For I an ideal of R, let $Y := D(I)$, where $D(I)$ is the open subset of $X := \mathrm{Spec}\,(R)$ consisting of those primes that do not contain I. Let $\Gamma(Y, \mathcal{O}_{Y|X})$ denote the ring of global sections over the open subspace Y of X.

When R is Noetherian, Deligne's formula states that $\delta\Gamma(Y, \mathcal{O}_{Y|X}) = \varinjlim \mathrm{Hom}\,(I^n, R)$. Because of certain properties of Noetherian rings, this formula can be restated in terms of the ring of quotients of R at the localizing filter determined by I. In particular, one has that $\Gamma(Y, \mathcal{O}_{Y|X}) = \Omega(I)$. Using our more general definition of the Kaplansky transform, we show that this formula also holds in the non-Noetherian case if $T(R)$ is von Neumann regular, Theorem 3.3.

It is well known that ring homomorphisms $R \to S$ that are flat epimorphisms arise from localizing at a particularly well behaved filter of ideals, see for example [**14**]. If R is an integral domain, then the finitely generated flat overrings (which are necessarily epimorphic extensions) of R can be characterized as the Kaplansky transform $\Omega(I)$ of a finitely generated ideal I, whose associated filter $\mathcal{F}_I$ is perfect, see [**4, 13**].

For an arbitrary commutative ring we give some necessary and sufficient conditions for $\Omega(I)$ to be a finitely generated flat epimorphic R algebra. The results are made more precise under the additional assumption that $T(R)$ is von Neumann regular. This includes, for such rings, a characterization of when the open subspace $D(I)$ is an affine scheme, Theorem 3.12.

All rings are assumed to be commutative with identity. Standard notation that we will use throughout include Spec (R) to denote the set of prime ideals with the Zariski topology and Min (R) to denote the set of minimal prime ideals of R. For I an ideal of R, $V(I)$ will denote the closed subset of Spec (R) consisting of all prime ideals containing I, while $D(I) := \mathrm{Spec}\,(R) \setminus V(I)$. We call I a *regular ideal* if it contains a regular element. If S is a ring containing R, $s \in S$ and $L \subseteq S$, then $(L :_R s) := \{r \in R : sr \in L\}$. For any ring R, $T(R)$ will denote the total ring of quotients of R; in other words, R localized at the set of regular elements of R.

2. Localization. In this section we review the notion of localization (or ring of quotients) at a filter and present some of the basic results. We give special attention to those rings R such that $T(R)$ is von Neumann regular.

A collection of ideals of a commutative ring R is called a *filter* if it is closed under the operation of taking finite intersections and such that any ideal containing an element of the collection is in the collection. A filter of ideals $\mathcal{F}$ of a commutative ring R is called a *localizing*, or Gabriel, filter if it satisfies the additional property that if I is an arbitrary ideal and J is an element of $\mathcal{F}$ such that for all $a \in J$, $(I :_R a) \in \mathcal{F}$, then $I \in \mathcal{F}$.

Associated to a localizing filter $\mathcal{F}$ is a left exact functor $q_{\mathcal{F}}$ on the category of R modules, i.e., for M an R-module, $q_{\mathcal{F}}(M)$ is also an R-module, defined by

$$q_{\mathcal{F}}(M) = \bigcup_{I \in \mathcal{F}} \mathrm{Hom}\,(I, M),$$

where we identify two elements f_1, f_2 of this set if they agree on an ideal $J \in \mathcal{F}$ contained in the intersection of their domains of definition. Addition is defined in the obvious fashion. When $M = R$ we can define multiplication for $f, g \in q_{\mathcal{F}}(R)$ with domains I and J respectively as follows: The product of f and g is the composition $f \circ g$ defined on the ideal IJ. With this definition, $q_{\mathcal{F}}(R)$ is a commutative ring and the map from R to $q_{\mathcal{F}}(R)$ given by r goes to multiplication by r is a ring homomorphism. We then define a left exact, idempotent functor on the category of R-modules by $M_{\mathcal{F}} := q_{\mathcal{F}}(q_{\mathcal{F}}(M))$. For each M, there is an obvious (and canonical) R-homomorphism from

M to $M_{\mathcal{F}}$. When $M = R$ this map is a ring homomorphism. The torsion submodule, associated to $\mathcal{F}$, $\tau_{\mathcal{F}}(M)$ of a module M is defined as $\tau_{\mathcal{F}}(M) := \{m \in M : mI = 0 \text{ for some } I \in \mathcal{F}\}$. It is not difficult to see that this subset is a submodule. Moreover, one can show that

$$M_{\mathcal{F}} = \bigcup_{I \in \mathcal{F}} \mathrm{Hom}\,(I, M/\tau_{\mathcal{F}}(M)) = q_{\mathcal{F}}(M/\tau_{\mathcal{F}}(M)).$$

The module M will be called $\mathcal{F}$-torsion if $\tau_{\mathcal{F}}(M) = M$ and $\mathcal{F}$-torsion free if $\tau_{\mathcal{F}}(M) = 0$. The former occurs if and only if $M_{\mathcal{F}} = 0$, the latter if and only if the map $M \to M_{\mathcal{F}}$ is a monomorphism. When there is no ambiguity, we will omit the $\mathcal{F}$ as a subscript for τ.

One other important result that will be needed from the theory on rings of quotients is that if $\alpha : M \to N$ is a monomorphism between R-modules, then the induced map $\alpha_{\mathcal{F}} : M_{\mathcal{F}} \to N_{\mathcal{F}}$ is an isomorphism if and only if $N/\alpha(M)$ is $\mathcal{F}$-torsion, see for example, [**7**, Proposition 6.2]. For more on the basic definitions and results on Gabriel filters and localizations, see [**7, 14**].

There is another way to define the module of quotients $M_{\mathcal{F}}$ with respect to a localizing filter $\mathcal{F}$. While this method is not as transparent, it is often more useful. First suppose that M is $\mathcal{F}$-torsion free. Let $E(M)$ denote the injective envelope of M, and let $\pi : E(M) \to E(M)/M$ denote the canonical surjection. Then $M_{\mathcal{F}}$ is canonically isomorphic to $\pi^{-1}(\tau(E(M)/M))$. For an arbitrary module M, $M_{\mathcal{F}}$ is isomorphic to the previous construction applied to $M/\tau(M)$. From this construction it is clear that if $\alpha : M \to M_{\mathcal{F}}$ is the canonical map, then $M_{\mathcal{F}}/\alpha(M)$ is $\mathcal{F}$-torsion.

The most common example of a localizing filter is given by a multiplicative set $S \subset R$. The associated filter $\mathcal{F}$ is the set of ideals $\{J \subseteq R :$ such that $J \cap S \neq \varnothing\}$. For $P \in \mathrm{Spec}\,(R)$, the adjectives P-torsion and P-torsion free will mean with respect to the filters determined by the set theoretic complement of P.

Our first lemma collects some elementary facts about localization of a commutative ring that will be used throughout the paper.

Lemma 2.1. *Let $\mathcal{F}$ be a localizing filter on the ring R. Then the following hold*:

(1) *If $a \in R$ is regular, then so is the image of a in $R_{\mathcal{F}}$.*

(2) *If R is a reduced ring, then so is $R_{\mathcal{F}}$.*

Proof. (1) Clearly $a \notin \tau_{\mathcal{F}}(R) = \tau(R)$, since $\tau(R)$ consists of zero divisors. Let $b \in R$ be such that $ab \in \tau(R)$. Then there exists $J \in \mathcal{F}$ such that $abJ = 0$. Since a is a regular element, $bJ = 0$. Thus $b \in \tau(R)$ and so its image in $R_{\mathcal{F}}$ is zero.

(2) First we will show that $R/\tau(R)$ is reduced. Let $a \in R$ be such that $a^n \in \tau(R)$ for some $n \geq 1$. Then, for some $J \in \mathcal{F}$, $a^n J = 0$. Since R is reduced, $aJ = 0$. Hence, $a \in \tau(R)$ and we have the claim.

Let $x \in R_{\mathcal{F}}$ be a nilpotent element. Thus, $x^n = 0$ for some $n > 0$. We also know that there exists an ideal $J \subset R$ with $J \in \mathcal{F}$, such that $xJ \subseteq R/\tau(R) \subseteq R_{\mathcal{F}}$. Clearly xJ is a nilpotent ideal of $R/\tau(R)$, which, by our initial claim, forces $xJ = 0$. Since $R_{\mathcal{F}}$ is torsion free over $\mathcal{F}$ we have that $x = 0$. □

Our next result is part of the folklore of general localization theory, which we present for the sake of completeness. It reduces localization at a filter $\mathcal{F}$ to the case where the ring R is $\mathcal{F}$-torsion free. First we need a definition. Let R be any ring, and let $\mathcal{F}$ be any localizing filter on R. Let $R' = R/\tau(R)$ and denote by π the canonical surjection $\pi : R \to R'$. We can use $\mathcal{F}$ to define a filter $\mathcal{F}'$ on R' via $\mathcal{F}' = \{J \subseteq R' : \pi^{-1}(J) \in \mathcal{F}\}$. One checks that in fact $\mathcal{F}'$ is a localizing filter on R'.

Proposition 2.2. *Let $\mathcal{F}$ be a localizing filter on R and $\mathcal{F}'$ the filter on $R' := R/\tau_{\mathcal{F}}(R)$ induced by $\mathcal{F}$. Then $R_{\mathcal{F}} \simeq R'_{\mathcal{F}'}$.*

Proof. First observe that R' is torsion free with respect to $\mathcal{F}'$. Thus $R'_{\mathcal{F}'} = \cup_{J \in \mathcal{F}'} \mathrm{Hom}\,(J, R')$. Then we can define a map

$$f : R'_{\mathcal{F}'} \longrightarrow R_{\mathcal{F}}$$

by sending $g : J \to R'$ to $g \circ \pi : \pi^{-1}(J) \to R'$. It is not difficult to check that f is a well defined ring homomorphism.

To see that f is injective, suppose that $h : J \to R'$, for $J \in \mathcal{F}'$, is in the kernel of f. In other words, there exists $L \subseteq \pi^{-1}(J)$ with $L \in \mathcal{F}$ such that $h \circ \pi$ restricted to L is the zero map. Thus, $h \circ \pi$ induces a map from $\pi^{-1}(J)/L \to R'$. But the domain of this map is $\mathcal{F}$-torsion (since R/L is $\mathcal{F}$-torsion), while the range is $\mathcal{F}$-torsion free. Hence, this induced map is the zero map, and so the original map h would also have to be the zero map (since π is a surjection). Thus, f is injective.

To check that f is surjective, let $g : L \to R'$ where $L \in \mathcal{F}$. Since R' is $\mathcal{F}$-torsion free, one has that $L \cap \tau(R) \subseteq \ker g$. Thus, g induces a map from $L/(L \cap \tau(R))$ to R'. Since $L/(L \cap \tau(R)) \simeq (L+\tau(R))/\tau(R)$, we have a map from $(L + \tau(R))/\tau(R)$ to R'. Moreover, $(L + \tau(R))/\tau(R) \in \mathcal{F}'$ and so we have an element of $R'_{\mathcal{F}'}$ that is mapped to g by f. Thus the map between the two rings of quotients is an isomorphism. □

We want to examine reduced rings R whose total ring of quotients $T(R)$ is a von Neumann regular ring. An important fact to recall about such rings is that every prime ideal of R that is not a minimal prime contains a regular element.

Lemma 2.3. *Suppose that $T(R)$ is von Neumann regular. Let $\mathcal{F}$ be a localizing filter on R and set $\tau = \tau_{\mathcal{F}}$. Then $\tau(R) = \cap_{P \in X} P$ where $X = \{P \in \mathrm{Min}\,(R) : \tau(R) \subseteq P\}$.*

Proof. By Lemma 2.1 (2), $\tau(R)$ is a radical ideal. Thus, $\tau(R)$ is the intersection of elements of $\mathrm{Spec}\,(R)$ that are minimal over the torsion ideal. On the other hand, if $Q \in \mathrm{Spec}\,(R)$ is minimal over $\tau(R)$, then its image in $R/\tau(R)$ consists of zero divisors of $R/\tau(R)$, see [**10**, Corollary 2.2]. However, as $T(R)$ is von Neumann regular, we can apply Lemma 2.1 (1). Thus, the image in $R/\tau(R)$ of every $Q \in \mathrm{Spec}\,(R) \setminus \mathrm{Min}\,(R)$ is a regular ideal of $R/\tau(R)$. Hence, all the prime ideals of R that are minimal over $\tau(R)$ are elements of $\mathrm{Min}\,(R)$. □

Proposition 2.4. *Suppose that $T(R)$ is von Neumann regular and $\mathcal{F}$ is a localizing filter on R. Let X be as in Lemma 2.3, and let $Y \subseteq \mathrm{Spec}\,(T(R))$ be the primes of $T(R)$ that lay over X, i.e., there is a bijection between the sets Y and X given by $Q \mapsto Q \cap R$. Let $L := \cap_{Q \in Y} Q$. Then $L = \tau(T(R))$ and $T(R)/L$ is the total ring of* *quotients of $R/\tau(R)$. In particular, $R/\tau(R)$ has a total ring of quotients that is von Neumann regular.*

Proof. First we show that $L = \tau(T(R))$. By Lemma 2.3, $\tau(R) = \cap_{P\in X} P$ and, since X consists of minimal primes of R, this last set equals $L \cap R$. Thus, $\tau(T(R)) = \tau(R)T(R) \subseteq L$. Conversely, if $x \in L$, then $xb \in R$ for some regular element $b \in R$. Furthermore, since $x \in L$, $xb \in P$ for each $P \in X$. Thus, $xb \in \cap_{P\in X} P = \tau(R)$.

Clearly, $R/\tau(R)$ is naturally a subring of $T(R)/L$. Furthermore, since $T(R)$ is von Neumann regular, so is $T(R)/L$. Therefore, to show that $T(R)/L$ is the total ring of quotients of $R/\tau(R)$, it will suffice to show that for any $0 \neq q \in T(R)/L$, there exists a regular element $t \in R/\tau(R)$, such that $0 \neq qt \in R/\tau(R)$.

Let $x \in T(R)$ be a preimage of q. Then there exists a regular element $s \in R$ such that $0 \neq xs \in R$. However, by Lemma 2.1 (1), the image of s in $R/\tau(R)$ is regular. Therefore, it will suffice to show that the image of xs in $R/\tau(R)$ is not zero, in other words, to show that $xs \notin \tau(R)$. However, notice that if Q is any prime ideal of $T(R)$, then $x \in Q$ if and only if $xs \in Q \cap R$. Thus, $x \in L$ if and only if $xs \in L \cap R = \tau(R)$. Since the image of x in $T(R)/L$ is not zero, we can conclude that $xs \notin \tau(R)$ and we have the proof. □

Corollary 2.5. *Suppose that the total ring of quotients of R is von Neumann regular. Let $\mathcal{F}$ be a localizing filter on R, $\tau = \tau_{\mathcal{F}}$, and J an ideal of R such that $\tau(R) \subseteq J$. Then J contains a regular element of R if and only if the image of J in $R/\tau(R)$ contains a regular element of $R/\tau(R)$.*

Proof. Note that $\tau(T(R)) = \tau(R)T \subseteq JT$. Let $R' = R/\tau(R)$, and let J' denote the image of J in R'. By the Proposition, $T(R') = T(R)/\tau(T(R))$. Thus, $J'T(R') = T(R')$ if and only if $JT(R) = T(R)$. Hence, J' contains a regular element of R' if and only if J contains a regular element of R. □

Let $\mathcal{F} \subseteq \mathcal{G}$ be localizing filters on the ring R. Then it follows from the definition that there is a natural ring homomorphism from $R_{\mathcal{F}}$ to $R_{\mathcal{G}}$. In particular, if $\mathcal{G}$ is the filter determined by the regular elements of R, and if $\mathcal{F}$ is a filter such that every $J \in \mathcal{F}$ is a regular ideal, then $R \subseteq R_{\mathcal{F}} \subseteq R_{\mathcal{G}} = T(R)$. Furthermore, it follows from the alternate definition of localization that in this case $R_{\mathcal{F}} = \{q \in T(R) : qJ \subseteq R$ for some $J \in \mathcal{F}\}$. We can generalize this notion. $\square$

Proposition 2.6. *Let $T(R)$ be von Neumann regular, $\mathcal{F}$ a localizing filter on R and $\tau = \tau_{\mathcal{F}}$. If, for each $J \in \mathcal{F}$, $J + \tau(R)$ is a regular ideal of R, then $R_{\mathcal{F}} \subseteq T(R/\tau(R))$. Furthermore, in this case $R_{\mathcal{F}} = R_{(\mathcal{F})}/\tau(R_{(\mathcal{F})})$ where $R_{(\mathcal{F})} = \{q \in T(R) : qJ \subseteq R$ for some $J \in \mathcal{F}\}$.*

Proof. Again let $R' = R/\tau(R)$. First observe that by Proposition 2.4 the von Neumann regular ring $T(R)/\tau(T(R))$ equals $T(R')$. Now assume that for each $J \in \mathcal{F}$, $J + \tau(R)$ is a regular ideal of R. Let $\mathcal{F}'$ be the filter on R' induced by $\mathcal{F}$. Then by Lemma 2.1 (1) every element of $\mathcal{F}'$ is a regular ideal of R'. Thus $\mathcal{F}'$ is a subfilter of the filter determined by the regular elements of R'. Hence $R_{\mathcal{F}}$ is a subring of the total ring of quotients of R'. Furthermore, we know that

$$R_{\mathcal{F}} = (R')_{\mathcal{F}'} = \{q \in T(R)/\tau(T(R)) : qL \subseteq R' \text{ for some } L \in \mathcal{F}'\}.$$

Let $\pi : T(R) \to T(R')$ be the canonical surjection. Since for any $L \in \mathcal{F}'$, $\pi^{-1}(L) \in \mathcal{F}$, it follows that for $q \in R_{\mathcal{F}}$, $\pi^{-1}(q) \subset R_{(\mathcal{F})}$. Thus $R_{\mathcal{F}} \subseteq R_{(\mathcal{F})}/\tau(R_{(\mathcal{F})})$.

For the reverse containment, let $x \in R_{(\mathcal{F})}$. Then $xJ \subseteq R$ for some $J \in \mathcal{F}$. Furthermore, the image of $x \cdot \tau(R)$ in R' is $\mathcal{F}$-torsion, yet $T(R')$ is $\mathcal{F}$-torsion free. Thus, it follows that $\pi(x \cdot \tau(R)) = 0$. Therefore, $\pi(x) \cdot (J + \tau(R))/\tau(R) \subseteq R'$. Since $(J + \tau(R))/\tau(R) \in \mathcal{F}'$, we have $\pi(x) \in R_{\mathcal{F}'} = R_{\mathcal{F}}$ and thus $R_{(\mathcal{F})}/\tau(R_{(\mathcal{F})}) = R_{\mathcal{F}}$ as claimed. $\square$

We will obtain a partial converse of this last result in the next section.

3. The generalized Kaplansky transform. Let R be an integral domain, K its quotient field and $I \subset R$ an ideal. The (Nagata) ideal transform $N_R(I) = \{q \in K : qI^n \subseteq R$ for some $n \geq 1\}$ of an ideal I of R has proven very useful in the study of Noetherian domains. Also note that $N_R(I) = \lim_{\to} \operatorname{Hom}(I^n, R)$, which of course is reminiscent of the definition of localization at a filter. However, for the non-Noetherian case, there appear to be many advantages in using the ideal transform introduced by Kaplansky, see for example, [**4, 5**]

$$\Omega_R(I) = \Omega(I) = \{q \in K : I^n \subseteq \text{ Rad}\,(R :_R qR)\}.$$

It is not difficult to check that $N(I) \subseteq \Omega(I)$ with equality if I is finitely generated.

In this section we introduce a generalized version of the Kaplansky transform that is applicable to commutative rings that are not domains. Let R be an arbitrary commutative ring, and let I be an ideal of R. Define a filter of ideals on R via $\mathcal{F}_I := \{J \subset R : I \subseteq \text{ Rad}\,(J)\} = \{J \subset R :$ for all $y \in I$ there exists $n \geq 1$ such that $y^n \in J\}$. One checks that $\mathcal{F}_I$ is in fact a localizing filter on R. Therefore for an arbitrary ring R and ideal I we can define the *generalized Kaplansky transform* of the ideal I as the ring $R_{\mathcal{F}_I}$, which we also denote by $\Omega(I)$. This definition is motivated by the observation that, for an arbitrary ideal I of a domain R, the usual Kaplansky transform is equal to $R_{\mathcal{F}_I}$, see [**4**, Lemma 4.3], so there is no ambiguity. For the sake of brevity, we denote the torsion submodule of a module M with respect to this filter by $\tau_I(M)$.

We will generalize some of the known results on $\Omega(I)$ to arbitrary (commutative) rings. For our first theorem in this direction, let $Y := D(I)$ and $X := \text{Spec}\,(R)$ (recall that $D(I)$ is the open subset of $\text{Spec}\,(R)$ consisting of those elements that do not contain I). Let $\Gamma(Y, \mathcal{O}_{Y|X})$ denote the ring of global sections over the open subspace Y of X. For an arbitrary Noetherian ring R, Deligne's formula states that $\Gamma(Y, \mathcal{O}_{Y|X}) = \lim_{\rightarrow} \text{Hom}\,(I^n, R)$. Furthermore, it has been shown [**3**, Lemma 2.5] that if R is Noetherian, then

$$\varinjlim \text{Hom}\,(I^n, R) = \varinjlim \text{Hom}\,(I^n, R/\tau_I(R)).$$

Thus in our notation, Deligne's formula translates to the statement that $\Gamma(Y, \mathcal{O}_{Y|X}) = \Omega(I)$, for any ideal I of Noetherian ring R. Additionally this formula has been shown to hold for finitely generated ideals of an arbitrary (commutative) ring [**3**, Proposition 2.12]. Generalizing in a different direction, we show that the equation $\Gamma(Y, \mathcal{O}_{Y|X}) = \Omega(I)$ holds if I is any ideal of a ring R whose total ring of quotients is von Neumann regular, Theorem 3.3.

We also show that certain finitely generated (as algebras), flat epimorphic ring homomorphisms from R are of the form $R \to \Omega(I)$. Finally, in a generalization of [**4**, Theorem 4.4], we obtain more precise conditions for when this occurs under the assumption that the total ring of quotients of R is von Neumann regular, see Theorem 3.12. These conditions relate the flatness of $\Omega(I)$ to when $D(I)$ is an affine scheme.

We proceed to generalize Deligne's formula. If $I = aR$ is a principal ideal, then the ring of global sections over $D(I)$ is just R_a, i.e., R localized at the multiplicative set $\{a, a^2, a^3, \dots\}$. First we need a simple lemma regarding R_a.

Lemma 3.1. *Let $I = aR$ be a principal ideal. Then $\Omega(I) = R_a$.*

Proof. The filter $\mathcal{F}_I$ clearly consists of all ideals J that contain a power of a. This is precisely the filter determined by the multiplicative set $\{1, a, a^2, a^3 \dots\}$. Since R_a is localization at this filter, we have the desired equality. □

Next we give a helpful description of $\tau_I(R)$ when R is reduced.

Lemma 3.2. *Let I be an ideal of R, and suppose that R is a reduced ring. Let $Z = \mathrm{Min}\,(R) \cap D(I)$. Then $\tau_I(R) = \cap_{P\in Z} P = (0 :_R I)$.*

Proof. Let $L = \cap_{P\in Z} P$. Then IL is contained in every element of $\mathrm{Min}\,(R)$. Since R is reduced, we have $IL = 0$. Thus, $L \subseteq \tau_I(R)$. Conversely, let $a \in \tau_I(R)$. Then $aJ = 0$ for some $J \in \mathcal{F}_I$. Thus, by definition, $I \subset \mathrm{Rad}\,(J)$. Clearly $D(I) \subset D(J)$, and hence $Z = \mathrm{Min}\,(R) \cap D(I) \subseteq \mathrm{Min}\,(R) \cap D(J)$. Since $aJ = 0$, it follows that $a \in Q$ for all $Q \in \mathrm{Min}\,(R) \cap D(J)$. Thus, $a \in Q$ for all $Q \in Z$. Therefore, the sets L and $\tau_I(R)$ are equal. As for the second equality, as noted earlier, $IL = 0$. Hence, $L \subseteq (0 :_R I)$. On the other hand, since $I \in \mathcal{F}_I$, it is clear that $(0 :_R I) \subset \tau_I(R) = L$. □

Theorem 3.3. *Suppose that $T(R)$ is von Neumann regular. Then, for any ideal $I \subset R$, one has*

$$\Gamma(Y, \mathcal{O}_{Y|X}) = \Omega(I),$$

where $X = \mathrm{Spec}\,(R)$ *and* $Y = D(I)$.

Proof. Let $S = \{s_i\}_{i\in\Gamma}$ be a generating set for I. For $s_i \in S$, we let R_i denote R localized at the element s_i. For $(k, m) \in \Gamma \times \Gamma$, we let R_{km} denote R localized at the element $s_k s_m$. Then, for each $k, m \in \Gamma$ there is a canonical map $R_k \to R_{km}$. Thus, for each $i \in \Gamma$, there are two maps from R_i into $\prod_{(k,m)\in\Gamma\times\Gamma} R_{km}$. This induces two maps g and h from $\prod_{t\in\Gamma} R_t$ to $\prod_{(k,m)\in\Gamma\times\Gamma} R_{km}$.

One defines the difference map $f : \prod_{t\in\Gamma} R_t \to \prod_{(k,m)\in\Gamma\times K} R_{km}$ via $f := g - h$. Since the sets $\{D(s_k)\}_{k\in\Gamma}$ form an open cover of Y, it follows from Lemma 3.1 and the definition of a sheaf, see for example, [**15**, Remark 1.8], that there is a canonical embedding α such that the following is an exact sequence:

$$0 \longrightarrow \Gamma(Y, \mathcal{O}_{Y|X}) \xrightarrow{\alpha} \prod_{k\in\Gamma} R_k \xrightarrow{f} \prod_{(k,m)\in\Gamma\times\Gamma} R_{km}.$$

Since localization is idempotent and since $\mathcal{F}_I \subseteq \mathcal{F}_J$ whenever $J \subseteq I$, we know that $(R_k)_{\mathcal{F}_I} = R_k$. Furthermore, since localization is left exact, it commutes with products. Thus, when we localize the above sequence at the filter $\mathcal{F}_I$, it remains exact and by Lemma 3.1 the two terms on the right are unchanged. Hence, the remaining term is unchanged. Thus $(\Gamma(Y, \mathcal{O}_{Y|X}))_{\mathcal{F}_I} = \Gamma(Y, \mathcal{O}_{Y|X})$. We will conclude our proof by showing that $\Omega(I) = (\Gamma(Y, \mathcal{O}_{Y|X}))_{\mathcal{F}_I}$.

It is clear from the last argument that $\Gamma(Y, \mathcal{O}_{Y|X})$ is $\mathcal{F}_I$-torsion free. Let $\alpha : R \to \Gamma(Y, \mathcal{O}_{Y|X})$ be the canonical map with R' the image of α. Then for each $P \in D(I)$ we know that $R'_P = \Gamma(Y, \mathcal{O}_{Y|X})_P = R_P$. Hence, $\Gamma(Y, \mathcal{O}_{Y|X})/R'$ is P-torsion. In particular, for $P \in D(I)$ and any $q \in \Gamma(Y, \mathcal{O}_{Y|X})$ we have $(R' :_R q) \not\subset P$. Therefore $I \subset \mathrm{Rad}\,(R' :_R q)$ for any $q \in \Gamma(Y, \mathcal{O}_{Y|X})$. Hence, $\Gamma(Y, \mathcal{O}_{Y|X})/R'$ is $\mathcal{F}_I$-torsion. Thus, if we can show that $\ker\alpha = \tau_I(R)$, then using the alternate definition of localization, we would have $\Omega(I) = \Gamma(Y, \mathcal{O}_{Y|X})_{\mathcal{F}_I}$.

Since $\Gamma(Y, \mathcal{O}_{Y|X})$ is $\mathcal{F}_I$-torsion free, $\tau_I(R) \subseteq \ker\alpha$. We also know from Lemma 3.2 that $\tau_I(R)$ is the intersection of the elements in the set $\mathrm{Min}\,(R) \cap D(I)$. Thus, if $\ker\alpha$ is strictly bigger than $\tau_I(R)$, it follows that $\ker\alpha \not\subseteq P$ for some $P \in D(I)$. Hence, for this P, $R'_P = 0$. But this would be a contradiction with the fact that the maps $R \to \Gamma(Y, \mathcal{O}_{Y|X}) \to R_P$ compose to give the canonical map from R to R_P. Thus, $\ker\alpha = \tau_I(R)$, and our proof is complete. □

A ring homomorphism $\alpha : R \to S$ is called an epimorphism, if it is an epimorphism in the category of *Rings*. This is equivalent to the statement that the multiplication map $S \otimes_R S \to S$ is an isomorphism. There is a well-known correspondence between flat epimorphism and the ring of quotients with respect to a certain kind of localizing filter. This relation will be crucial to what follows. If $\alpha : R \to S$ is a flat epimorphism, then the collection of ideals $\mathcal{F} = \{I \subseteq R : \alpha(I)S = S\}$ is a localizing filter and there is a canonical isomorphism $S \simeq R_{\mathcal{F}}$. Conversely, if the family $\mathcal{F}$ of ideals J such that $\sigma(J)T = T$ is a localizing filter such that there is a ring isomorphism $\sigma : T \to R_{\mathcal{F}}$ where $\sigma\alpha : R \to R_{\mathcal{F}}$ is the canonical map, then $\alpha : R \to T$ is a flat epimorphism, see [**14**, Theorem 2.1]. We call such an $\mathcal{F}$ a *perfect filter* (for more information on such filters, see [**14**]). If $\mathcal{F}$ is a perfect filter, then for any ideal J of R, one has that $J_{\mathcal{F}} = \alpha(J)R_{\mathcal{F}}$, where $\alpha : R \to R_{\mathcal{F}}$ is the canonical map. Moreover, localization at a perfect filter is an exact functor.

In general it is possible to have two distinct filters $\mathcal{F}$ and $\mathcal{G}$ such that $R_{\mathcal{F}} = R_{\mathcal{G}}$. However, if $\mathcal{F}$ is a perfect filter, then $\mathcal{F} \subseteq \mathcal{G}$. This is a consequence of the next result (which undoubtedly is known to some).

Lemma 3.4. *Let R be any ring and $\mathcal{F}$ a localizing filter on R. Let $\alpha : R \to R_{\mathcal{F}}$ be the canonical morphism. If J is an ideal of R such that $\alpha(J)R_{\mathcal{F}} = R_{\mathcal{F}}$, then $J \in \mathcal{F}$.*

Proof. We prove the contrapositive. Suppose that $J \notin \mathcal{F}$. Since the annihilator of the image of 1 in R/J is J, it follows that R/J is not $\mathcal{F}$-torsion. Thus, the canonical morphism $R/J \to (R/J)_{\mathcal{F}}$ is not the zero map. Since localization is left exact, we get the following commutative diagram where the horizontal sequences are exact and the vertical maps are canonical:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J & \longrightarrow & R & \longrightarrow & R/J & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_{\mathcal{F}} & \longrightarrow & R_{\mathcal{F}} & \longrightarrow & (R/J)_{\mathcal{F}} & &
\end{array}$$

Since the diagram is commutative, it follows that $\alpha(J)R_{\mathcal{F}} \subseteq J_{\mathcal{F}}$. Furthermore, since the map $R \to R/J$ is a surjection, we know that the composed map $R \to (R/P)_{\mathcal{F}}$ is not the zero morphism. Hence, the image of $R_{\mathcal{F}}$ in $(R/P)_{\mathcal{F}}$ is not zero. Therefore, $\alpha(J)R_{\mathcal{F}} \subseteq J_{\mathcal{F}} \neq R_{\mathcal{F}}$. □

We are now able to present a partial converse to Proposition 2.6 as promised.

Corollary 3.5. *Let $\mathcal{F}$ be a perfect filter on R such that $R_{\mathcal{F}} \subset T(R)$. Then $\mathcal{F} \subseteq \mathcal{G}$, where $\mathcal{G}$ is the filter determined by the regular elements of R. In particular, each $J \in \mathcal{F}$ is a regular ideal of R.*

Proof. It is not difficult to check that the intersection of two localizing filters is again a localizing filter. Thus, $\mathcal{H} := \mathcal{F} \cap \mathcal{G}$ is a localizing filter on R. We claim that $R_{\mathcal{F}} = R_{\mathcal{H}}$. Since $\mathcal{H} \subseteq \mathcal{G}$, it follows that $R_{\mathcal{H}}$ is a subring of $T(R)$; to be precise, $R_{\mathcal{H}} = \{q \in T(R) : (R :_R q) \in \mathcal{H}\}$. If $q \in R_{\mathcal{F}} \subset T(R)$, then clearly $(R :_R q) \in \mathcal{F} \cap \mathcal{G}$. Thus, $R_{\mathcal{F}} \subset R_{\mathcal{H}}$.

Conversely, using the alternate definition of localization, we see that $T(R)/R_{\mathcal{F}}$ is $\mathcal{F}$-torsion free and hence $\mathcal{H}$-torsion free. Thus, $R_{\mathcal{H}} \subseteq R_{\mathcal{F}}$, and so we have equality of the two rings. However, since $\mathcal{F}$ is perfect, it follows from Lemma 3.4 that this equality of the ring of quotients implies that $\mathcal{F} \subseteq \mathcal{H}$. Hence, the two filters are equal, which proves the result. □

Recall that if R is a domain, then any flat overring S, i.e., S contains R and is contained in the quotient field of R, is an epimorphic extension (this will also follow from Lemma 3.8). Thus, the following two results are a generalization of what is essentially [**11**, Theorem 5] from the case when R is an integral domain to arbitrary commutative rings. We note that our proofs are relatively simple adaptations of the proof found in [**4**, Proposition 2.9].

Proposition 3.6. *Let $I \subset R$ be an ideal, and let $\alpha : R \to \Omega(I)$ be the canonical map. If $\alpha(I)\Omega(I) = \Omega(I)$, then $\mathcal{F}_I$ is a perfect filter (so $\mathcal{F}_I = \{J \subset R : J\Omega(I) = \Omega(I)\}$) and $\Omega(I)$ is a finitely generated, as an algebra, flat epimorphic extension of R. Furthermore, there is a finitely generated ideal $I' \subseteq I$, such that* $\mathrm{Rad}\,(I) = \mathrm{Rad}\,(I')$, *so $\mathcal{F}_{I'} = \mathcal{F}_I$.*

Proof. First we will show that $\mathcal{F}_I = \{J \subseteq R : \alpha(J)\Omega(I) = \Omega(I)\}$ and hence it is a perfect filter. Let $J \in \mathcal{F}$, so $I \subseteq \mathrm{Rad}\,(J)$. Thus, if $P \in V(J)$, then $I \subseteq P$. Therefore, $\alpha(P)\Omega(I) = \Omega(I)$. It is not difficult to see that this implies that $\alpha(J)\Omega(I) = \Omega(I)$ (if not let Q be a prime of $\Omega(I)$ that contains $\alpha(J)\Omega(I)$. Now pull Q back to R and obtain a contradiction.)

Conversely, if J is an ideal of R such that $\alpha(J)\Omega(I) = \Omega(I)$, then by Lemma 3.4, $J \in \mathcal{F}_I$. Hence, $\mathcal{F}_I$ is as claimed and so it is a perfect filter. Thus, $R \to \Omega(I)$ is a flat epimorphism.

Next we show the existence of I' with the stated properties. Since $\alpha(I)\Omega(I) = \Omega(I)$, there exists $x_1, x_2, \dots, x_n \in I$ and $s_1, s_2, \dots, s_n \in \Omega(I)$ such that

$$\sum_{i=1}^{n} \alpha(x_i)s_i = 1.$$

Let $I' = (x_1, \dots, x_n)$. Clearly $\alpha(I')\Omega(I) = \Omega(I)$. Thus $I' \in \mathcal{F}_I$. Hence $I \subseteq \mathrm{Rad}\,(I')$. Since $I' \subseteq I$, we also have $\mathrm{Rad}\,(I) = \mathrm{Rad}\,(I')$.

Finally we show that $\Omega(I)$ is finitely generated over R. Clearly we may assume that I is finitely generated. Let $R' = \alpha(R) \subseteq \Omega(I)$. We claim that $\Omega(I) = R'[s_1, \dots s_n]$. Let $t \in \Omega(I)$, so $(R' :_R t) \in \mathcal{F}_I$. Since I is finitely generated, there exists $M > 0$ such that $I^M \subseteq (R' :_R t)$. Since $\alpha(I)R'[s_1, \dots, s_n] = R'[s_1, \dots, s_n]$, it follows that $\alpha(I^M)R'[s_1, \dots, s_n] = R'[s_1, \dots, s_n]$. Hence,

$$1 = \sum_{k=1}^{r} \alpha(x_k)z_k \text{ with } x_k \in I^M$$

and

$$z_k \in R'[s_1, \dots, s_n], \quad r \geq 1.$$

Since $x_k \in I^M$ and $\alpha(I^M)t \subseteq R'$, we have

$$t = \sum_{k=1}^{r}(\alpha(x_k)t)z_k \in R'[s_1, \ldots, s_n],$$

and the proof is complete. □

We also have a partial converse to the above, which is really just the proof (i) ⇒ (ii) of [**4**, Proposition 2.9].

Proposition 3.7. *Let $R \hookrightarrow S$ be an embedding of rings that makes S into a finitely generated, as an algebra, flat epimorphic extension. Then $S = \Omega(I)$ for some finitely generated ideal I of R such that $\mathcal{F}_I$ is a perfect filter.*

Proof. Since S is a flat epimorphic extension of R, the family of ideals $\mathcal{F} = \{J \subseteq R : JS = S\}$ is a perfect filter on R. We will show that there is an ideal I of R such that $\mathcal{F} = \mathcal{F}_I$.

For each $t \in S$, the ideal $(R :_R t)$ is an element of $\mathcal{F}$, since by the alternate definition of localization the module S/R is $\mathcal{F}$-torsion. Let $S = R[s_1, \ldots, s_n]$, and set

$$I = (R :_R s_1) \bigcap (R :_R s_2) \bigcap \cdots \bigcap (R :_R s_n).$$

Thus, $I \in \mathcal{F}$, since $\mathcal{F}$ is a filter, so $IS = S$.

If $J \in \mathcal{F}$, then $JS = S$. Thus, there exists $j_1, \ldots, j_r \in J$ and $t_1, \ldots, t_r \in S$ such that $\sum_{k=1}^{r} j_k t_k = 1$. Since $t_k \in R[s_1, \ldots, s_n]$, there exists $N \geq 0$ such that $t_k I^N \subseteq R$ for each k, $1 \leq k \leq r$. Therefore,

$$I^N = I^N \cdot 1 = I^N \sum_{k=1}^{r} j_k t_k = \sum_{k=1}^{r} j_k t_k I^N \subseteq J,$$

whence $J \in \mathcal{F}_I$, from which we deduce that $\mathcal{F} \subseteq \mathcal{F}_I$.

In the other direction suppose that $J \in \mathcal{F}_I$. Thus, $I \subseteq \mathrm{Rad}\,(J)$. Since $IS = S$ we can, as in the proof of Proposition 3.6 (the first paragraph), conclude that $JS = S$, i.e., $J \in \mathcal{F}$. Hence, $\mathcal{F} = \mathcal{F}_I$.

Since $\mathcal{F}_I$ is a perfect filter, we can use Proposition 3.6 to find a finitely generated ideal $I' \subseteq I$ so that $\mathrm{Rad}\,(I) = \mathrm{Rad}\,(I')$. Thus, we are done. □

Observe that if the ideal I of R contains a regular element, then every element of $\mathcal{F}_I$ contains a regular element. This follows for if $J \in \mathcal{F}_I$, then $\mathrm{Rad}\,(J)$ contains a regular element a. Hence, $a^m \in J$ for some $m > 0$. In particular, $\mathcal{F}_I \subseteq \mathcal{G}$, where $\mathcal{G}$ is the filter determined by the set of regular elements of R. Thus, $R_{\mathcal{F}_I} = \{t \in T(R) : tJ \subseteq R$ for some $J \in \mathcal{F}_I\}$.

When we make the additional assumption that the total ring of quotients of R is von Neumann regular, we are able to obtain further results. We first record some known results on flat epimorphisms that will be needed in what follows.

Lemma 3.8. *Let R be a (commutative) ring. Then the following statements hold.*

(1) *Let $R \subset S \subset T$ be ring inclusions such that T is a flat epimorphism over R. Then S flat over R implies that S is also an epimorphic extension of R;*

(2) *if the total ring of quotients $T(R)$ is von Neumann regular, then any ring that contains R as a flat epimorphic extension is contained in $T(R)$;*

(3) *a ring homomorphism $\alpha : R \to S$ is a flat epimorphism if and only if for each $P \in \mathrm{Spec}\,(R)$, either $\alpha(P)S = S$ or $\alpha \otimes_R S_P : R_P \to S_P$ is an isomorphism.*

Proof. For (1), see [**14**, Proposition 2.4] and for (2), see [**6**, Theorem 4.3.7]. Finally, (3) is from [**13**, Proposition 2.4]. □

Corollary 3.9. *Assume $T(R)$ is von Neumann regular, and let $R \subseteq S \subset T(R)$. Then the following are equivalent:*

(1) *S is flat and finitely generated over R.*

(2) *There exists a finitely generated ideal I containing a regular element such that $S = \Omega(I)$ and $I\Omega(I) = \Omega(I)$.*

Proof. (1) $\Rightarrow$ (2). By Lemma 3.8 (1), S is also an epimorphic extension of R. Therefore, by Proposition 3.7 the ideal I exists with all the properties stated, with the possible exception that it need contain a regular element. However, since the map from R to $\Omega(I) = S$ is a monomorphism, $(I :_R 0) = 0$. Because I is finitely generated, by [**10**, Theorem 4.5] it must contain a regular element.

(2) $\Rightarrow$ (1). This follows immediately from Proposition 3.6. □

Let $\mathcal{F}$ be a localizing filter on R, and let $P \in \operatorname{Spec}(R)$. Since the annihilator of every element of R/P is P, it follows that R/P is either $\mathcal{F}$-torsion or $\mathcal{F}$-torsion free, depending on whether $P \in \mathcal{F}$ or not. In a similar argument, for $Q \in \operatorname{Spec}(R_{\mathcal{F}})$, $R_{\mathcal{F}}/Q$ is either $\mathcal{F}$-torsion or $\mathcal{F}$-torsion free, depending on whether or not $\alpha^{-1}(Q)$ is an element of $\mathcal{F}$ (here α denotes the canonical map from R to $R_{\mathcal{F}}$). In particular, there is an assignment from the set $Z := \{Q \in \operatorname{Spec}(R_{\mathcal{F}}) : R_{\mathcal{F}}/Q$ is $\mathcal{F}$-torsion free$\}$ to the set $Y := \{P \in \operatorname{Spec}(R) : R/P$ is $\mathcal{F}$-torsion free$\}$ via $Q \mapsto \alpha^{-1}(Q)$. In fact we can say more about this function.

Lemma 3.10. *Let $\mathcal{F}$ be a localizing filter on R, and let $\alpha : R \to R_{\mathcal{F}}$ be the canonical map. Let $Z := \{Q \in \operatorname{Spec}(R_{\mathcal{F}}) : R_{\mathcal{F}}/Q$ is $\mathcal{F}$-torsion free$\}$ and $Y := \{P \in \operatorname{Spec}(R) : R/P$ is $\mathcal{F}$-torsion free$\}$. Then the assignment $Q \mapsto \alpha^{-1}(Q)$ is a bijection from Z to Y.*

Proof. As we have already seen, α^{-1} defines a map from Z to Y. We will show that the map $P \mapsto P_{\mathcal{F}}$ is the inverse assignment.

First we must show that for $P \in Y$, $P_{\mathcal{F}}$ is a prime ideal of $R_{\mathcal{F}}$. Since R/P is $\mathcal{F}$-torsion free, $\tau(R) \subseteq P$. Thus, by appealing to Proposition 2.2 we may assume that R is $\mathcal{F}$-torsion free, and hence $R \subseteq R_{\mathcal{F}}$. Now suppose that $a, b \in R_{\mathcal{F}}$ such that $ab \in P_{\mathcal{F}}$. Since both $R_{\mathcal{F}}/R$ and $P_{\mathcal{F}}/P$ are $\mathcal{F}$-torsion, by the definition of a filter, there exist $J \in \mathcal{F}$ such that both aJ and bJ are contained in R, while $abJ^2 \subset P$. Since P is prime, either $aJ \subseteq P \subseteq P_{\mathcal{F}}$ or $bJ \subseteq P \subset P_{\mathcal{F}}$. But $R_{\mathcal{F}}/P_{\mathcal{F}} \subseteq (R/P)_{\mathcal{F}}$ is $\mathcal{F}$-torsion free. Hence $a \in P_{\mathcal{F}}$ or $b \in P_{\mathcal{F}}$. Which proves the ideal is prime.

For each $Q \in Z$, $Q/(Q \cap R)$ is $\mathcal{F}$-torsion (since it is contained in $R_{\mathcal{F}}/R$). Thus, $(Q \cap R)_{\mathcal{F}} = Q_{\mathcal{F}}$. On the other hand, consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q & \longrightarrow & R_{\mathcal{F}} & \longrightarrow & R_{\mathcal{F}}/Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_{\mathcal{F}} & \longrightarrow & R_{\mathcal{F}} & \longrightarrow & (R_{\mathcal{F}}/Q)_{\mathcal{F}} & &
\end{array}$$

where the horizontal rows are exact and the vertical maps are the canonical injective maps (the middle one of course is bijective). It follows from a standard diagram chase that the leftmost vertical map must also be onto. In particular, $(Q \cap R)_{\mathcal{F}} = Q_{\mathcal{F}} = Q$. Thus, the assignments are one sided inverses of each other.

Finally, we must show that for $P \in Y$, $P_{\mathcal{F}} \cap R = P$. It is clear that $P \subseteq P_{\mathcal{F}}$ and $P_{\mathcal{F}}/P$ is $\mathcal{F}$-torsion. Furthermore ,$(P_{\mathcal{F}} \cap R)/P \subseteq R/P$. Since by hypothesis the latter module is $\mathcal{F}$-torsion free, it follows that $P_{\mathcal{F}} \cap R = P$. □

Next we present one more lemma before moving onto our final theorem of the section.

Lemma 3.11. *Suppose that $T(R)$ is von Neumann regular. Let I be an ideal of R such that $I + \tau_I(R)$ is a regular ideal. Then for any $J \in \mathcal{F}_I$, $J + \tau_I(R)$ is a regular ideal. In particular, $\Omega(I) \subset T(R/\tau_I(R))$.*

Proof. Let $J \in \mathcal{F}_I$, then $I \subseteq \text{Rad}\,(J)$. Thus, $I + \tau_I(R) \subseteq \text{Rad}\,(J + \tau_I(R))$ and hence $\text{Rad}\,(J + \tau_I(R))$ contains a regular element of R. As before we see that this implies that $J + \tau_I(R)$ is a regular ideal of R. Therefore, by Lemma 2.1 (1) every ideal of $\mathcal{F}'$, the filter on $R/\tau_I(R)$ induced by $\mathcal{F}_I$, contains a regular element of $R/\tau_I(R)$. Thus, by Proposition 2.2 we have the concluding remark of the Lemma. □

We are almost ready to move onto generalizing Theorem 4.4 of [**4**]. In that theorem, where R is assumed to be an integral domain, one condition used is the assumption that, for certain prime ideals P, $\Omega(I) \not\subset R_P$. In our next result we work with localizations that are not embeddings. Therefore, we need an alternate formulation of this condition. To that end we claim that when R is a domain, the condition that $\Omega(I) \not\subset R_P$ is equivalent to the condition that $\Omega(I)/R$ is not P-torsion. To see why, let K denote the quotient field of R, then

$$\begin{aligned}\Omega(I) \not\subset R_P &\Longleftrightarrow \exists\, q \in K, \text{ such that } q \in \Omega(I) \text{ and } q \notin R_P\\ &\Longleftrightarrow I \subseteq \operatorname{Rad}(R :_R q) \text{ and } (R :_R q) \subseteq P\\ &\Longleftrightarrow q \in \Omega(I) \text{ and the image of } q \text{ in } \Omega(I)/R\\ &\qquad\qquad\qquad \text{is not } P\text{-torsion.}\end{aligned}$$

Our last theorem is an adaption of Theorem 4.4 of [**4**] from the case where R is an integral domain, to where $T(R)$ is von Neumann regular.

Theorem 3.12. *Assume that $T(R)$ is von Neumann regular, and let I be an ideal of R. Set $X := \operatorname{Spec}(R)$, $Y := D(I)$, $W := \operatorname{Spec}(\Omega(I))$, and let $\alpha : R \to \Omega(I)$ denote the canonical map. Then the following statements are equivalent*:

(1) *Y is an affine open subspace of X*;

(2) *the canonical morphism*

$$(W, \mathcal{O}_W) \longrightarrow (Y, \mathcal{O}_Y), \quad Q \longmapsto \alpha^{-1}(Q),$$

is a scheme-isomorphism;

(3) $\alpha(I)\Omega(I) = \Omega(I)$;

(4) *for each $P \in V(I)$, $\alpha(P)\Omega(I) = \Omega(I)$*;

(5) *$\mathcal{F}_I$ is a perfect filter* (*so in particular $\alpha : R \to \Omega(I)$ is a flat epimorphism*);

(6) *$I + (0 :_R I)$ contains a regular element of R, $\Omega(I)$ is flat over R, and for each $P \in V(I)$, $\Omega(I)/\alpha(R)$ is not P-torsion.*

If the above equivalent conditions hold, then $\Omega(I) = \overline{\Omega}(I)/(0 :_{\overline{\Omega}(I)} I)$, where $\overline{\Omega}(I) = \{q \in T(R) : qJ \subseteq R$ for some $J \in \mathcal{F}_I\}$.

Proof. Let $Y' := D_W(I\Omega(I))$.

(1) $\Leftrightarrow$ (2). By Theorem 3.3 Y is an affine subspace of X if and only if the canonical map $Y \to \operatorname{Spec}(\Gamma(Y, \mathcal{O}_{X|Y})) = W$ [**8**, I.2.3.2] defines a scheme isomorphism.

$(2) \Leftrightarrow (3)$. This follows directly from Lemma 3.10, since $I\Omega(I) = \Omega(I)$ if and only if $Y' = W$.

$(3) \Leftrightarrow (4)$. Clearly (3) implies (4). The reverse direction is the same argument as used in the proof of Proposition 3.6 (first paragraph).

$(4) \Rightarrow (5)$. This is immediate from Proposition 3.6.

$(5) \Rightarrow (6)$. The map $\alpha : R \to \Omega(I)$ is a flat epimorphism. Now suppose that for some $P \in \operatorname{Spec}(R)$, $I \subseteq P$, yet $\Omega(I)/\alpha(R)$ is P-torsion. Then we have the following commutative diagram:

$$\begin{array}{ccc} R & \longrightarrow & \Omega(I) \\ \downarrow & & \downarrow \\ R_P & \longrightarrow & \Omega(I)_P. \end{array}$$

Since $\Omega(I)/\alpha(R)$ is P-torsion, the bottom horizontal map is onto. Hence, the image of PR_P in $\Omega(I)_P$ is a prime ideal. Let Q denote this ideal pulled back to $\Omega(I)$. Thus, we have $P\Omega(I) \subset Q$. But this is a contradiction to the assumption that $P\Omega(I) = \Omega(I)$ for all $P \in \operatorname{Spec}(R)$ with $I \subseteq P$.

Finally we have to show that $I + (0 :_R I)$ contains a regular element. First note that $(0 :_R I) = \tau(R)$ by Lemma 3.2. Let $\mathcal{F}'$ be the filter induced by $\mathcal{F}_I$ on the ring $R' = R/\tau(R)$. It is easy to see that $\mathcal{F}'$ is a perfect filter on R'. Furthermore, by Lemma 3.8 (2) $R_{\mathcal{F}} = R'_{\mathcal{F}'} \subset T(R')$. Hence, each element of $\mathcal{F}'$ contains a regular element of R', Corollary 3.5. Therefore, we can apply Corollary 2.5 to conclude that every element of $\mathcal{F}$ that contains $\tau(R)$ contains a regular element of R.

$(6) \Rightarrow (4)$. Note that $\Omega(I)$ is necessarily flat over R', since the canonical map $R \to R'$ is an epimorphism. Furthermore, by Lemma 3.11, $R_{\mathcal{F}} \subseteq T(R')$. Since $\Omega(I)$ is flat over R', by Lemma 3.8 (1), $\Omega(I)$ is a flat epimorphic extension of R'. Thus, $R \to \Omega(I)$ is a flat epimorphism and whence $\Omega(I)$ is the localization of R at a perfect filter.

Let $P \in \operatorname{Spec}(R)$; then $\alpha(P)\Omega(I)$ is the localization of P at a perfect filter. In particular, either $\alpha(P)\Omega(I) = \Omega(I)$ or $\alpha(P)\Omega(I)$ is a prime ideal of $\Omega(I)$ and $\alpha^{-1}(P) = P$. Now suppose that $P \in V(I)$ and yet $\alpha(P)\Omega(I) \neq \Omega(I)$. Then, by Lemma 3.8 (3), the map $\alpha \otimes R_P : R_P \to \Omega(I)_P$ is an isomorphism. This implies that $\Omega(I)/\alpha(R)$ is P-torsion, a contradiction with our assumption. Hence, $\alpha(P)\Omega(I) = \Omega(I)$ for all $P \in V(I)$.

Finally, to see the concluding statement, note that by Proposition 3.11, every ideal in $\mathcal{F}'$ contains a regular element of R'. Thus, by Proposition 2.6, $\Omega(I) = \overline{\Omega}(I)/\tau_I(\overline{\Omega}(I))$. However, $\tau_I(\overline{\Omega}(I)) = \tau_I(T(R)) \bigcap \overline{\Omega}(I))$. Since $T(R)$ is a localization of R at a multiplicative set, it follows that $\tau_I(T(R)) = \tau_I(R)T(R) = (0 :_R I)T(R) = (0 :_{T(R)} IT(R))$. Therefore, $\tau_I(\overline{\Omega}(I)) = (0 :_{T(R)} IT(R)) \cap \overline{\Omega}(I) = (0 :_{\overline{\Omega}(I)} I\overline{\Omega}(I)) = (0 :_{\overline{\Omega}(I)} I)$. □

We can translate the above equivalent conditions to one that is internal to the total ring of quotients of R.

Corollary 3.13. *Suppose that $T(R)$ is von Neumann regular and I is an ideal of R. Let $\alpha : R \to \Omega(I)$ be the canonical map, and let $\overline{\Omega}(I)$ denote the subring of $T(R)$ as given in the last result. Then the following conditions are equivalent*:

(1) $\alpha(I)\Omega(I) = \Omega(I)$;

(2) $I + (0 :_R I)$ *is a regular ideal and* $I\overline{\Omega}(I) + (0 :_{\overline{\Omega}(I)} I) = \overline{\Omega}(I)$.

If I is finitely generated, the above is equivalent to: $I\overline{\Omega}(I) + (0 :_{\overline{\Omega}(I)} I) = \overline{\Omega}(I)$.

Proof. (1) $\Rightarrow$ (2). By the Theorem, the kernel of the projection $\overline{\Omega}(I) \to \Omega(I)$ is $(0 :_{\overline{\Omega}(I)})$ and $I + (0 :_R I)$ contains a regular element. Thus, we are done.

(2) $\Rightarrow$ (1). Since $I + (0 :_R I)$ contains a regular element, by Lemma 3.11 every ideal of the form $J + (0 :_R I)$, $J \in \mathcal{F}_I$, contains a regular element. Thus, by Proposition 2.6, $\Omega(I) = \overline{\Omega}(I)/\tau_I(\overline{\Omega}(I))$. From the Theorem we know that $\tau_I(\overline{\Omega}(I)) = (0 :_{\overline{\Omega}(I)} I)$. Hence it is clear that the second assumption in (2) implies that $\alpha(I)\Omega(I) = \Omega(I)$.

To show the concluding statement it will suffice to prove that $I + (0 :_R I)$ is a regular ideal whenever I is finitely generated. As already observed, if I is finitely generated, either I is a regular ideal or there exists $0 \neq s \in R$ such that $sI = 0$. In the former case we are done. Assume the latter; then $IT(R)$ is a proper, finitely generated ideal of the von Neumann regular ring $T(R)$. Thus, it is generated by an idempotent e. Furthermore, there exists a regular element $b \in R$ such that $(1-e)b \in R$. Clearly $(1-e)b \in (0:_R I)$ and $(I,(1-e)b)T(R) = T(R)$. Since $(I,(1-e))$ is finitely generated, it contains a regular element t. In other words, $t \in (I,(1-e)) \subseteq I + (0:_R I)$, so we are done. $\square$

If R is a domain and I an ideal of R, then whenever $\Omega(I)$ is flat, the map $R \to \Omega(I)$ is also an epimorphism. We conclude with an example of a von Neumann regular ring R that contains an ideal I such that $\Omega(I)$ is flat over R yet $R \to \Omega(I)$ is not an epimorphism.

Example 3.14. Let F be any field, and let T be a product of copies of F indexed by the natural numbers. Let R be the subring of T generated by the full socle I of T, i.e., $I = \sum F$, along with all elements of the form $a \cdot 1$ where $a \in F$ (thus, R consists of all sequences that are eventually constant). Then R is von Neumann regular. Additionally, T is an essential extension of R and $T \cdot I \subseteq R$ (more precisely, I is an ideal of T); thus, $\Omega(I) = T$. Clearly T is flat over R, since R is von Neumann regular. On the other hand, since R is von Neumann regular, it has no proper monic flat epimorphic extensions.

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