ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 20, Number 2, Spring 1990

ON THE MULTIPLICITY OF $\mathbf{T} \oplus \mathbf{T} \oplus \cdots \oplus \mathbf{T}$

DOMINGO A. HERRERO AND WARREN R. WOGEN

To the memory of our friends and colleagues
Constantin Apostol and Doug McMahon

1. Introduction. Let $\mathcal{L}(\mathcal{X})$ denote the algebra of all (bounded linear) operators on a complex Banach space $\mathcal{X}$. The *multiplicity* of $T \in \mathcal{L}(\mathcal{X})$ is the cardinal number defined by

$$\mu(T) = \min_{\Gamma \subset \mathcal{X}} \{\operatorname{card} \Gamma : \mathcal{X} = \bigvee \{T^k y : y \in \Gamma, k = 0, 1, 2, \dots\}\},$$

where $\bigvee \mathcal{R}$ denotes the closed linear span of the vectors in $\mathcal{R}$.

If $\mu(T)$ is finite or denumerable, then $\mathcal{X}$ is necessarily separable. Throughout this note we shall always assume that $\mathcal{X}$ is *separable* and *infinite dimensional*.

If $A \in \mathcal{L}(\mathcal{X})$ and $B \in \mathcal{L}(\mathcal{Y})$, then $A \oplus B$ denotes the direct sum of A and B acting in the usual fashion on the hilbertian direct sum $\mathcal{X} \oplus \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$. It is an easy exercise to check that $\max[\mu(A), \mu(B)] \leq \mu(A \oplus B) \leq \mu(A) + \mu(B)$.

Let $T \in \mathcal{L}(\mathcal{X})$; for each $n \geq 1$, let $T^{(n)}$ denote the direct sum of n copies of T acting in the usual fashion of the direct sum $\mathcal{X}^{(n)}$ of n copies of $\mathcal{X}$. It readily follows from the previous observations that

$$\begin{aligned} \max[\mu(T^{(m)}), \mu(T^{(n)})] = \mu(T^{(\max[m,n])}) &\leq \mu(T^{(m+n)}) \\ &\leq \mu(T^{(m)}) + \mu(T^{(n)}), m, n \geq 1. \end{aligned}$$

For which sequences $\{\mu_n\}_{n=1}^{\infty}$ of natural numbers satisfying the conditions $\mu_{\max[m,n]} \leq \mu_{m+n} \leq \mu_m + \mu_n, m, n \geq 1$, does there exist a Banach space operator T such that $\mu(T^{(n)}) = \mu_n$ for all $n = 1, 2, \dots$?

This research was partially supported by Grants of the National Science Foundation.

By combining some well-known examples and some new ones, it is possible to show that the following sequences are attainable in this way:

(A_k) $\{nk\}_{n=1}^{\infty}$ for each $k \geq 1$,
(B_k) $\{nk+1\}_{n=1}^{\infty}$ for each $k \geq 1$,
(C_k) $\{nk+2\}_{n=1}^{\infty}$ for each $k \geq 1$,
(D_k) $\{k+1, 2k, 3k, 4k, 5k, 6k, \dots\}$ for each $k \geq 1$,
(E) $\mu_n \equiv 1$,
(F) $\mu_n \equiv 2$, and
(G) $\mu_n \equiv \infty$.

Is there any other? Is $\{\mu(T^{(n)})\}_{n=1}^{\infty}$ always a convex sequence, either constant or satisfying $\mu(T^{(n)}) \geq n$ for all $n = 1, 2, \dots$?

The sequence (A_k) is attained by $A_k = S^{(k)}$, where S denotes the unilateral shift in ℓ^2 (defined by $Se_j = e_{j+1}$ for all $j = 1, 2, \dots$, with respect to some orthonormal basis $\{e_j\}_{j=1}^{\infty}, k = 1, 2, \dots$): clearly, $\mu([S^{(k)}]^{(n)}) = \mu(S^{(nk)}) \leq kn$. On the other hand, nul $S^{*^{(nk)}} :=$ dim ker $S^{*^{(nk)}} = nk$, and therefore the multiplicity cannot be smaller than nk (see, e.g., [**11**, Proposition 1(i)]). Similarly, the direct sum $S^{(\infty)}$ of denumerably many copies of S satisfies $\mu([S^{(\infty)}]^{(n)}) = \infty$ for all $n = 1, 2, \dots$, so that (G) is also attainable.

The sequence (E) is attained by a large number of examples, including the adjoints of all the unilateral weighted shifts in ℓ^2 [**9**] (see also Proposition 3.1 below). In particular, $\mu(S^{*^{(n)}}) = 1$ for all $n = 1, 2, \dots$.

This article grew out of a question of C. Apostol: Is there any Hilbert space operator T such that $\mu(T^{(2)}) = \mu(T^{*^{(2)}}) = 1$? (Clearly, neither T, nor T^*, can have an eigenvector.)

This question is affirmatively answered in §3: there exists a compact bilateral weighted shift E such that $\mu(E^{(\infty)}) = \mu(E^{*^{(\infty)}}) = 1$. Moreover, if $F = E \oplus E^*$, then an unpublished observation of J.A. Deddens [**7**] indicates that $\mu(F) = \mu(F^*) = \mu(F^{(\infty)}) = \mu(F^{*^{(\infty)}}) = 2$ (so that the sequence (F) is also attainable; see §4).

In §5 it is shown that (1) if the sequences $\{\mu_n\}$ and $\{\mu'_n\}$ are attainable, then so is $\{\max[\mu_n, \mu'_n]\}$, and (2) a general result that implies, in particular, that $\mu(S^{(k)} \oplus R) = \mu(S^{(k)}) + \mu(R) = k + \mu(R)$ for each operator R whose spectrum $\sigma(R)$ is a subset of the open unit disk. Combining these two results and the previous examples it is easily seen that (B_k), (C_k) and (D_k) are attainable.

In [**13**], the first author completely characterized those sequences $\{\mu_n\}_{n=1}^{\infty}$ such that the multiplicity of the n-th *power* of T is equal to μ_n for all $n = 1, 2, \ldots$, for some Banach space operator T: given a sequence satisfying certain (very simple) necessary conditions, a T satisfying $\mu(T^n) = \mu_n$, for all $n = 1, 2, \ldots$, is constructed by taking infinite direct sum of suitable operators *acting on finite dimensional spaces.*

But such an operator can only satisfy $\mu(T^{(n)}) \equiv \infty$, or $\mu(T^{(n)}) = nk$ for all $n = 1, 2, \ldots$ (for some $k \geq 1$). Thus, the problem we analyze here *is intrinsically infinite dimensional.* Furthermore, the "infinite power" of an operator T does not make any sense, in general; but it makes perfect sense to consider $T^{(\infty)}$, the direct sum of denumerably many copies of $T \in \mathcal{L}(\mathcal{X})$ acting on the hilbertian direct sum of denumerably many copies of $\mathcal{X}$. It will be shown in §2 that

$$\mu(T^{(\infty)}) = \sup_n \mu(T^{(n)}) = \lim_{n\to\infty} \mu(T^{(n)}).$$

That is, either $\{\mu(T^{(n)})\}$ is an unbounded sequence and $\mu(T^{(\infty)}) = \infty$, or $\{\mu(T^{(n)})\}$ is bounded, and $\mu(T^{(\infty)}) = \max_n \mu(T^{(n)})$.

The authors wish to thank Professor Gustavo Corach for calling the attention to "stable ranks" of Banach algebras (an important tool in §5).

2. The multiplicity of $\mathbf{T}^{(\infty)}$.

Theorem 2.1. $\mu(T^{(\infty)}) = \sup_n \mu(T^{(n)})$.

Proof. Clearly, it is enough to show that if $\mu(T^{(n)}) \leq m < \infty$ for all $n = 1, 2, \ldots,$ then $\mu(T^{(\infty)}) \leq m$.

Assume $m = 1$, that is, $T^{(n)}$ is *cyclic* for all $n = 1, 2, \ldots,$ and let $\mathcal{C}(T^{(n)}) = \{(y_1, y_2, \ldots, y_n) \in \mathcal{X}^{(n)} : \mathcal{X}^{(n)} = \bigvee\{T^k y_j : j = 1, 2, \ldots, n\}_{k=0}^{\infty}\}$ be the set of cyclic vectors of $T^{(n)}$.

Since $\mu(T^{(n)}) < n$ for all $n > 1$, and $\mu(T^{(n)}) \geq \sup\{\text{nul}[(\lambda - T)^{*^{(n)}} : \lambda \in \mathcal{C}]\} = n \sup\{\text{nul}[(\lambda - T)^* : \lambda \in \mathcal{C}]\}$ [**11**, Proposition 1(i)], it readily follows that T^* cannot have eigenvectors. Therefore, by using [**12**, Propositions 1(vii) and 1_n(vii) and Theorem 1] and [**15**, Theorem 1] (see also [**1** Chapter 11]), we infer that $\mathcal{C}(T^{(n)})$ is a G_δ-dense subset of $\mathcal{X}^{(n)}$. Thus, if

$$\mathcal{C}(T^{(n)})' = \{(y_j)_{j=1}^{\infty} \in \mathcal{X}^{(\infty)} : (y_1, y_2, \ldots, y_n) \in \mathcal{C}(T^{(n)})\},$$

then

$$\mathcal{C} = \cap_{n=1}^{\infty} \mathcal{C}(T^{(n)})'$$

is a G_δ-dense subset of $\mathcal{X}^{(\infty)}$.

Let $\{\lambda_j\}_{j=1}^{\infty}$ be a bounded sequence of non-zero complex numbers, and let $(y_j)_{j=1}^{\infty} \in \mathcal{X}$. By construction, $y^{[n]} := (y_1, y_2, \ldots, y_n) \in \mathcal{C}(T^{(n)}), n = 1, 2, \ldots.$

Let $\mathcal{A}(T)$ denote the weak closure of the polynomials in T and $1_{\mathcal{X}}$, and let $M_n[\mathcal{A}(T)]$ be the algebra of all $n \times n$ operator matrices with entries in $\mathcal{A}(T)$. Since $A_n = \oplus_{j=1}^{n} \lambda_j 1_{\mathcal{X}} \in \mathcal{L}(\mathcal{X}^{(n)})$ is invertible and both A_n and A_n^{-1} belong to $M_n[\mathcal{A}(T)]$, it follows from [**12**, Proposition 1_n(vi)] that

$$A_n y^{[n]} = (\lambda_1 y_1, \lambda_2 y_2, \ldots, \lambda_n y_n) \in \mathcal{C}(T^{(n)}),$$

$n = 1, 2, \ldots,$ whence it follows that $(\lambda_j y_j)_{j=1}^{\infty} \in C$.

Claim. If $\epsilon_n \downarrow 0$, $n \to \infty$, fast enough, then $(\epsilon_j y_j)_{j=1}^{\infty}$ is a cyclic vector for $T^{(\infty)}$.

Set $\epsilon_1 = 1$, and let $\{f_i\}_{i=1}^{\infty}$ be a denumerable dense subset of $\mathcal{X}$. Clearly, $\cup_{n=1}^{\infty}\{(f_{i_1}, f_{i_2}, \ldots, f_{i_n}, 0, 0, 0, \ldots)\}$ is a denumerable dense subset of $\mathcal{X}^{(\infty)}$.

Since y_1 is cyclic for T, there exists a polynomial $p^{(1)}$ (.;1) such that

$$\|f_1 - p^{(1)}(T; 1)y_1\|^2 < 1.$$

It follows that

$$\begin{aligned}
&\|(f_1,0,0,0,\dots) - p^{(1)}(T^{(\infty)};1)(\epsilon_1 y_1, \delta_2 y_2, \delta_3 y_3, \dots)\|^2 \\
&\quad = \|f_1 - p^{(1)}(T;1)y_1\|^2 + \sum_{j=2}^{\infty} \|p^{(1)}(T;1)(\delta_j y_j)\|^2 \\
&\quad \le \|f_1 - p^{(1)}(T;1)y_1\|^2 + \|p^{(1)}(T;1)\|^2 \sum_{j=2}^{\infty} \delta_j^2 \|y_j\|^2 < 1,
\end{aligned}$$

provided $0 < \delta_j \le \epsilon_j^{(1)}, j = 2,3,\dots$ (for suitably chosen constants $\epsilon_j^{(1)}, 0 < \epsilon_j^{(1)} \le 1$).

Suppose we have already chosen $\epsilon_1 = 1, \epsilon_j^{(1)}, j \ge 2, \epsilon_j^{(2)}, j \ge 3, \dots, \epsilon_j^{(k-1)}, j \ge k$, and $\epsilon_j = \min[\epsilon_j^{(i)} : i = 1,2,\dots,j-1], j = 2,3,\dots,k$. Since $(\epsilon_j y_j)_{j=1}^k \epsilon\, \mathcal{C}\,(T^{(k)})$, there exist polynomials $p^{(k)}$ $(.; i_1.i_2,\dots,i_k)$ such that

$$\|(f_{i_1}, f_{i_2}, \dots, f_{i_k}) - p^{(k)}(T^{(k)}; i_1, i_2, \dots i_k)(\epsilon_1 y_1, \epsilon_2 y_2, \dots, \epsilon_k y_k)\|^2 < \frac{1}{k}$$

for each k-tuple $(i_1, i_2, \dots, i_k)$ with $1 \le i_h \le k$.

Clearly,

$$\begin{aligned}
&\|(f_{i_1}, f_{i_2}, \dots, f_{i_k}, 0,0,0,\dots) - p^{(k)}(T^{(\infty)}; i_1, i_2, \dots, i_k) \\
&\qquad (\epsilon_1 y_1, \epsilon_2 y_2, \dots, \epsilon_k y_k, \delta_{k+1} y_{k+1}, \delta_{k+2} y_{k+2}, \dots)\|^2 \\
&\quad = \|(f_{i_1}, f_{i_2}, \dots, f_{i_k}) - p^{(k)}(T^{(k)}; i_1, i_2, \dots, i_k)(\epsilon_1 y_1, \epsilon_2 y_2, \dots, \epsilon_k y_k)\|^2 \\
&\quad + \max\{\|p^{(k)}(T; r_1, r_2, \dots r_k)\|^2 \sum_{j=k+1}^{\infty} \delta_j^2 \|y_j\|^2 : \\
&\qquad (r_1, r_2, \dots, r_k) \in \{1,2,\dots,k\}^{(k)}\} < \frac{1}{k},
\end{aligned}$$

provided $0 < \delta_j \le \epsilon_j^{(k)}, j = k+1, k+2, \dots$ (for suitably chosen constants $\epsilon_j^{(k)}, 0 < \epsilon_j^{(k)} \le \epsilon_k$).

Define $\epsilon_{k+1} = \min\,[\epsilon_{k+1}^{(1)}, \epsilon_{k+1}^{(2)}, \dots, \epsilon_{k+1}^{(k)}]$. It is immediate that $(\epsilon_1 y_1, \epsilon_2 y_2, \dots)$ is a cyclic vector for $T^{(\infty)}$; that is, $\mu(T^{(\infty)}) = 1$.

The case $1 < m < \infty$ can be handled in exactly the same way, with $\mathcal{X}$ replaced by $\mathcal{X}^{(m)}$; the set $\mathcal{C}_m(T^{(n)})$ of all multicyclic m-tuples of $T^{(n)}$ is a G_δ-dense subset of $(\mathcal{X}^{(n)})^{(m)}$ for all $n = 1, 2, \ldots$. The details are left to the reader. □

3. A Hilbert space operator T such that both $\mathbf{T}^{(\infty)}$ and $\mathbf{T}^{*(\infty)}$ are cyclic. In a certain sense, "most" Hilbert space operators satisfying $\mathcal{H} = \bigvee\{\ker T^n\}_{n=1}^\infty$ satisfy $\mu(T^{(\infty)}) = 1$:

PROPOSITION 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ ($\mathcal{H}$ a separable Hilbert space) and assume that*

$$T = \left(\begin{array}{cccccccccccc} 0 & 0 & \cdots & 0 & T_{1,r_1} & \cdots & \cdots & \cdots & & & & \\ & 0 & 0 & \cdots & \cdots & \cdots & 0 & T_{2,r_2} & \cdots & & & \\ & & 0 & & 0 & \cdots & \cdots & \cdots & \cdots & 0 & T_{3,r_3} & \cdots \\ & & & & \cdot & \cdot & & & & & & \\ & \mathrm{O} & & & & \cdot & \cdot & & & & & \\ & & & & & & \cdot & \cdot & & & & \end{array}\right)\begin{array}{c}\mathcal{H}_1\\ \mathcal{H}_2\\ \mathcal{H}_3\\ \cdot\\ \cdot\\ \cdot\end{array},$$

where $\mathcal{H} = \oplus_{j=1}^\infty \mathcal{H}_j$, $\{r_j\}_{j=1}^\infty$ is strictly increasing and T_{j,r_j} has dense range for all $j = 1, 2, \ldots$; then T is cyclic. Furthermore, $(T^k)^{(\infty)}$ is cyclic for all $k = 1, 2, \ldots$.

(The proof follows by minor modifications of the proof of Theorem 2 of [**18**], or Lemma 7 of [**17**]. Observe that $(T^k)^{(\infty)}$ always has a matrix of the same kind as T; therefore, it suffices to show that T is a cyclic operator.)

In all these examples, T has nontrivial kernel and T^* has no eigenvectors. Clearly, $T^{*(2)}$ cannot be cyclic.

Here is an example in which both $T^{(\infty)}$ and $T^{*(\infty)}$ are cyclic.

EXAMPLE. Let $\{e_n\}_{-\infty}^\infty$ be an orthonormal basis of the Hilbert space $\mathcal{H}$, and define $E \in \mathcal{L}(\mathcal{H})$ by $Ee_n = w_n e_{n+1}, n \in \mathbf{Z}$, where

$$w_n = \begin{cases} 1, & \text{if } n(0) := 0 \leq n < n(1), \\ \frac{1}{2k,} & \text{if } -n(2k) \leq n < -n(2k-2), k = 1, 2, \ldots, \\ \frac{1}{2k-1,} & \text{if } n(2k-1) \leq n < n(2k+1), k = 1, 2, \ldots, \end{cases}$$

for a certain strictly increasing sequence $\{n(k)\}_{k=0}^{\infty}$ tending to infinity "very fast" (in a sense to be specified later).

CLAIM. If $n(k) \to \infty (k \to \infty)$ fast enough, then

$$x = \sum_{k=1}^{\infty} \frac{1}{k} e_{-n(2k-1)} \in \mathcal{C}(E) \quad \text{and} \quad y = \sum_{k=1}^{\infty} \frac{1}{k} e_{n(2k)} \in \mathcal{C}(E^*).$$

For each $h > 0$, we have

$$\begin{aligned} &\left\| \frac{hE^{n(2h-1)}x}{\|E^{n(2h-1)}e_{-n(2h-1)}\|} - e_0 \right\| \\ &= \left\| \sum_{k=1}^{h-1} \left(\frac{h}{k}\right) \frac{E^{n(2h-1)}e_{-n(2k-1)}}{\|E^{n(2h-1)}e_{-n(2h-1)}\|} + \sum_{k=h+1}^{\infty} \left(\frac{h}{k}\right) \frac{E^{n(2h-1)}e_{-n(2k-1)}}{\|E^{n(2h-1)}e_{-n(2h-1)}\|} \right\| \\ &\leq C(h-1)\left(\frac{h-1}{h}\right)^{n(2h-1)} + \sum_{k=h+1}^{\infty} \left(\frac{h}{k}\right)^{n(2h-1)}, \end{aligned}$$

and

$$\begin{aligned} &\left\| \frac{hE^{*n(2h)}y}{\|E^{*n(2h)}e_{n(2h)}\|} - e_0 \right\| \\ &= \left\| \sum_{k=1}^{h-1} \left(\frac{h}{k}\right) \frac{E^{*n(2h)}e_{n(2k)}}{E^{*n(2h)}e_{n(2h)}} + \sum_{k=h+1}^{\infty} \left(\frac{h}{k}\right) \frac{E^{*n(2h)}e_{n(2k)}}{\|E^{*n(2h)}e_{n(2h)}\|} \right\| \\ &\leq C(h-1)\left(\frac{2h-3}{2h-1}\right)^{n(2h)} + \sum_{k=h+1}^{\infty} \left(\frac{2h-1}{2k-1}\right)^{n(2h)}, \end{aligned}$$

where $C(h-1)$ is a constant depending only on $h-1$ (and $n(1), n(2), \ldots, n(2h-2)$).

One sees that, if $n(2h-1) \geq n(2h-1)'$ and $n(2h) \geq n(2h)'$ (for sufficiently large $n(2h-1)' > 2n(2h-2)$ and $n(2h)' > 2n(2h-1)$, then

$$\left\| \frac{hE^{n(2h-1)}x}{\|E^{n(2h-1)}e_{-n(2h-1)}\|} - e_0 \right\| < \frac{1}{h}$$

and

$$\left\| \frac{hE^{*^{n(2h)}}y}{\|E^{*^{n(2h)}}e_{n(2h)}\|} - e_0 \right\| < \frac{1}{h}$$

for all $h = 1, 2, \dots$.

It readily follows that $e_0 \in \mathcal{M}_+ \cap \mathcal{M}_-$, where

$$\mathcal{M}_+ = \bigvee \{E^k x\}_{k=0}^{\infty} \text{ and } \mathcal{M}_- = \bigvee \{E^{*^k} y\}_{k=0}^{\infty}.$$

A fortiori, $\mathcal{M}_+ \supset \mathcal{H}_+ = \bigvee\{e_n\}_{n=0}^{\infty}$, and $\mathcal{M}_- \supset \mathcal{H}_- = \bigvee\{e_{-n}\}_{n=0}^{\infty}$.

Let $P_+(P_-)$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_+(\mathcal{H}_-$, respectively). It is easily seen that $P_- f \in \mathcal{M}_+$ for all f in $\mathcal{M}_+$. Since, for each $h > 0$ and all k sufficiently large,

$$\begin{aligned} &\left\| k \prod_{j=0}^{n(2k-1)-h} \frac{1}{w_{-n(2k-1)+j}} P_- E^{n(2k-1)-h} x - e_h \right\|^2 \\ &\qquad = k^2 \sum_{i=k+1}^{\infty} \prod_{j=0}^{n(2k-1)-h} \left(\frac{w_{-n(2i-1)+j}}{w_{-n(2k-1)+j}} \right)^2 \to 0, k \to \infty, \end{aligned}$$

we deduce that $e_{-1}, e_{-2}, \dots \in \mathcal{M}_+$.

Hence, $\mathcal{M}_+ = \mathcal{H}$; that is, $x \in \mathcal{C}(E)$.

The same argument shows that $y \in \mathcal{C}(E^*)$.

Define $\mathbf{N}_j = \{2^{j-1}(2r-1)\}_{r=1}^{\infty}$ $(j = 1, 2, \dots)$. Ad hoc modifications of the above proof show that if

$$f_j = \sum_{k \in \mathbf{N}_j} \frac{1}{k} e_{-n(2k-1)} \quad \text{and} \quad g_j = \sum_{k \in \mathbf{N}_j} \frac{1}{k} e_{n(2k)'}$$

then

$$(f_1, f_2, \dots) \in \mathcal{C}(E^{(\infty)}) \quad \text{and} \quad (g_1, g_2, \dots) \in \mathcal{C}(E^{*^{(\infty)}}).$$

REMARKS 3.2. (i) E and E^* are compact operators without eigenvalues. By using [**12**, Proposition 1(vii) and Theorem 1] it is not difficult to check that $\mathcal{C}(E^{(\infty)}) \cap \mathcal{C}(E^{*^{(\infty)}})$ is actually a G_δ-dense subset of $\mathcal{H}^{(\infty)}$.

(ii) Minor modifications of the proof show that, given a two-sided sequence $\{w'_n\}_{-\infty}^{\infty}$ of positive numbers, we can find E as in the example whose weight sequence $\{w_n\}_{-\infty}^{\infty}$ satisfies $0 < w_n \le w'_n$ for all $n \in \mathbf{Z}$.

4. The "transpose" of a Hilbert space operator. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of the Hilbert space $\mathcal{H}$, and let $T \in \mathcal{L}(\mathcal{H})$. T admits a unique matrix representation $(t_{ij})_{i,j=1}^{\infty}$ with respect to this basis; moreover, the "transpose" matrix

$$^tT = (t_{ji})_{i,j=1}^{\infty}$$

is also the matrix of an operator acting on this space.

The "transpose operator" ^tT is not uniquely determined by T; it actually depends on T and on the basis $\{e_j\}_{j=1}^{\infty}$. Nevertheless, two transposes of a given operator are always unitarily equivalent. Indeed, if $\overline{T} = (\bar{t}_{ij})_{i,j=1}^{\infty}$ is the "conjugate operator," then $T^* = {}^t\overline{T}$. If $T = (t'_{ij})_{i,j=1}^{\infty}$ with respect to the orthonormal basis $\{f_j\}_{j=1}^{\infty}$ and $U = (u_{ij})_{i,j=1}^{\infty}$ is the unitary operator defined by $Ue_j = f_j, j = 1, 2, \ldots$, then the matrix of UTU^* with respect to $\{e_j\}_{j=1}^{\infty}$ coincides with $(t'_{ij})_{i,j=1}^{\infty}$. We have

$$^tT' = (t'_{ji})_{i,j=1}^{\infty} = {}^t(UTU^*) = \overline{(UTU^*)}^* = \overline{(UTU^*)} = \overline{U}\, {}^tT\overline{U}^*,$$

where $\overline{U}$ is the "conjugate" of U with respect to the basis $\{e_j\}_{j=1}^{\infty}$.

^tT behaves, in every sense, like the "mirror image" of T. Recall that an operator T is semi-Fredholm if ran T is closed and either nul T or nul T^* is finite dimensional. In this case, the index is defined by ind $T =$ nul $T -$ nul T^* (see, e.g., [**3**]).

The following result resumes the most important properties of the transpose operators. The proofs are left to the reader.

PROPOSITION 4.1. (i) $\sigma({}^tT) = \sigma(T)$

(ii) $\mathrm{nul}(\lambda - {}^tT)^k = \mathrm{nul}[(\lambda - T)^*]^k$ *and* $\mathrm{nul}\,[(\lambda - {}^tT)^*]^k = \mathrm{nul}(\lambda - T)^k$ *for all* $\lambda \in \mathcal{C}$ *and all* $k = 1, 2, \ldots$.

(iii) $\inf\{\|(\lambda - {}^tT)x\| : \ \|x\| = 1, x \perp \ker(\lambda - {}^tT)\} = \inf\{\|(\lambda - T)x\| : \|x\| = 1, x \perp \ker(\lambda - T)\}$.

In particular, $\operatorname{ran}(\lambda - {}^tT)$ *is closed if and only if* $\operatorname{ran}(\lambda - T)$ *is closed.*

(iv) $\lambda - {}^tT$ *is semi-Fredholm if and only if* $\lambda - T$ *is semi-Fredholm; in this case,* $\operatorname{ind}(\lambda - {}^tT) = -\operatorname{ind}(\lambda - T)$.

(v) $f({}^tT) = {}^tf(T)$ *and* $\|f({}^tT)\| = \|f(T)\|$ *for each function* f *analytic on some neighborhood of* $\sigma(T)$.

In [**7**], J.A. Deddens proved that if $T = (t_{ij})_{i,j=1}^{\infty}$ and t_{ij} is *real* for all (i,j), then $T \oplus T^*$ cannot be cyclic. It is obvious that in this case $T^* = {}^tT$. Thus, the following proposition is a mild improvement of Deddens's result; the proof follows by the same argument. (We include it here for completeness.)

Proposition 4.2. *Let* $T = (t_{ij})_{i,j=1}^{\infty}$ *be the matrix of the Hilbert space operator* T *with respect to the orthonormal basis* $\{e_j\}_{j=1}^{\infty}$*, and let* ${}^tT = (t_{ji})_{i,j=1}^{\infty}$*; then* $T \oplus {}^tT$ *is not cyclic.*

Proof. Observe that $(T^k e_i, e_j) = (e_i, {T^*}^k e_j) = \overline{({T^*}^k e_j, e_i)} = (({}^tT)^k e_j, e_i)$ for all $i,j = 1,2,\ldots$. For any $f = \sum_{j=1}^{\infty} a_j e_j$ and $g = \sum_{j=1}^{\infty} b_j e_j$ in $\mathcal{H}$ $(f,g$ non-zero vectors), define

$$\overline{f} = \sum_{j=1}^{\infty} \overline{a}_j e_j \quad \text{and} \quad \overline{g} = \sum_{j=1}^{\infty} \overline{b}_j e_j;$$

then

$$\begin{aligned}((T \oplus {}^tT)^k (f,g), (\overline{g}, -\overline{f})) &= (T^k f, \overline{g}) - (({}^tT)^k g, \overline{f}) \\ &= \Big(\sum_{i=1}^{\infty} a_i T^k e_i, \sum_{j=1}^{\infty} \overline{b}_j e_j\Big) \\ &\quad - \Big(\sum_{j=1}^{\infty} b_j ({}^tT)^k e_j, \sum_{i=1}^{\infty} \overline{a}_i e_i\Big) \\ &= \sum_{i,j=1}^{\infty} [a_i b_j (T^k e_i, e_j) - b_j a_i (({}^tT)^k e_j, e_i)] = 0\end{aligned}$$

for all $k = 0, 1, 2, \ldots$.

Hence, $(\overline{f}, -\overline{g})$ is a non-zero vector orthogonal to

$$\bigvee\{(T \oplus^t T)^k (f,g)\}_{k=0}^{\infty}.$$

Therefore no vector is cyclic for $T \oplus^t T$. $\square$

Clearly, the results of Propositions 4.1 and 4.2 remain true if ${}^tT$ is defined as the transpose of T with respect to an orthonormal basis ordered as $\mathbf{Z}$. Hence, we have

Corollary 4.3. *Let E be the bilateral weighted shift defined in §3, and let $F = E \oplus E^*$; then*

$$\mu(F^{(n)}) = \mu({F^*}^{(n)}) = \mu(F^{(\infty)}) = \mu({F^*}^{(\infty)}) = 2$$

for all $n = 1, 2, \ldots$.

Proof. Clearly, $E^* = {}^tE$. By Proposition 4.2, $F = E \oplus E^*$ cannot be cyclic. Since both E and E^* are cyclic, we deduce that $\mu(F) = 2$. Since $F^* = E^* \oplus E$ is unitarily equivalent to F, we also have $\mu(F^*) = 2$.

The same argument shows that all the operators $F^{(n)}, {F^*}^{(n)}, n = 1, 2, \ldots, F^{(\infty)}$ and ${F^*}^{(\infty)}$ have multiplicity 2. $\square$

Remarks 4.4. (i) Suppose $\mu(T) = \mu({}^tT) \geq 2$. Does it follow that $\mu(T \oplus^t T) \geq 3$? The answer is NO; that is, Proposition 4.2 cannot be improved in this direction: take $T = F$! Then ${}^tT = {}^tF$ is unitarily equivalent to F and, therefore, $T \oplus^t T$ is unitarily equivalent to $F^{(2)}$, but $\mu(F^{(2)}) = 2$.

(ii) As J.A. Deddens observed in [**7**], it follows from Proposition 4.2 that $S \oplus S^*$ is not cyclic. (S = the unilateral shift. This was also observed by N.K. Nikol'skiĭ, V.V. Peller and V.I. Vasjunin; see [**9**, p. 283].) By using this result, we can now answer the question in the last line of [**11**, p. 98]: Let $T_{ab} = S \oplus (a + bS^*), b \neq 0$; then T_{ab} is cyclic if and only if $|a| + |b| > 1$. Indeed, according to this reference, it only remains to consider the case $|a| + |b| = 1$. If $a = 0$

and $|b| = 1$, we are done because bS^* is unitarily equivalent to $|b|S^*$. Assume that $|b| < 1$, and let $\phi(\lambda) = \overline{a} + \overline{b}\lambda$. If $C_\phi \in \mathcal{L}(H^2)$ is defined by $(C_\phi f)(\lambda) = (f \circ \phi)(\lambda) = f(\overline{a} + \overline{b}\lambda), f \in H^2$, then C_ϕ and C_ϕ^* are injective operators with dense range and $S^* C_\phi^* = C_\phi^*(a + bS^*)$, so that

$$(S \oplus S^*)(1 \oplus C_\phi^*) = (1 \oplus C_\phi^*)[S \oplus (a + BS^*)].$$

Since $1 \oplus C_\phi^*$ has dense range, we see that

$$2 = \mu(S \oplus S^*) \leq \mu[S \oplus (a + BS^*)] \leq \mu(S) + \mu(a + bS^*) = 2,$$

that is, $\mu[S \oplus (a + bS^*)] = 2$ for $|a| + |b| \leq 1, b \neq 0$ (see [**11**, Proposition 1(vi)]).

5. New examples from the old ones.

PROPOSITION 5.1. *Let $T_j (T_j \in \mathcal{L}(\mathcal{X}_j))$ be a finite or denumerable family of operators, and let*

$$T = \oplus_j [2^{-j} 1_{\mathcal{X}_j} + 4^{-j}(1 + \|T_j\|)^{-1} T_j] \in \mathcal{L}(\oplus_j \mathcal{X}_j);$$

then

$$\mu(T^{(n)}) = \sup_j \mu(T_j^{(n)}) \ for\ all\ n = 1, 2, \ldots.$$

PROOF. Let $A_j = 2^{-j} 1_{\mathcal{X}_j} + 4^{-j}(1 + \|T_j\|)^{-1} T_j$; then $T = \oplus_j A_j$ and the spectrum of the direct summand A_k is a clopen subset of $\sigma(T)$ included in the band $\{\lambda \in \mathcal{C} : 2^{-k} - 4^{-k} < \text{Re}\ \lambda < 2^{-k} + 4^{-k}\}$. Since this band does not intersect $\sigma(\oplus_{j \neq k} A_j)$, it follows from Runge's theorem (see, e.g., [**8**]) that there exists a sequence $\{p_{k,h}\}_{h=1}^\infty$ of polynomials such that

$$p_{k,h}(\lambda) \to \begin{cases} 1, & \text{uniformly on a neighborhood of } \sigma(A_k) \\ 0, & \text{uniformly on a neighborhood of } \sigma(T) \backslash \sigma(A_k), \end{cases}$$

$k = 1, 2, \ldots$.

Therefore, $P_k =$ the projection of $\oplus_j \mathcal{X}_j$ onto $\mathcal{X}_k$ along $\oplus_{j \neq k} \mathcal{X}_j$ is a norm limit of polynomials in T. It follows that, if $f = (f_j) \in \oplus_j \mathcal{X}_j$, then $f_k = P_k f \in \bigvee \{T^h f\}_{h=0}^\infty$.

It is easily seen that f is cyclic for T if and only if f_k is cyclic for T_k, for each $k = 1, 2, \ldots$. More generally, $(f^{(1)}, f^{(2)}, \ldots, f^{(m)})$ is a multicyclic m-tuple for T if and only if $(P_k f^{(1)}, P_k f^{(2)}, \ldots, P_k f^{(m)})$ is a multicyclic m-tuple for T_k for each $k = 1, 2, \ldots$.

Thus, $\mu(T) = \sup_j \mu(T_j)$.

The same argument shows that $\mu(T^{(n)}) = \sup_j \mu(T_j^{(n)})$ for all $n = 1, 2, \ldots$. $\square$

Corollary 5.2. *The sequence* $(D_1) = \{2, 2, 3, 4, 5, 6, \ldots\}$ *is attainable.*

Proof. Apply the above result to $T_1 = S =$ shift and $T_2 = F$, defined as in Corollary 4.3. $\square$

The sequence $(B_k) = \{nk + 1\}_{n=1}^{\infty}$ can be attained as follows: let B be the bilateral weighted shift defined by

$$Be_j = \begin{cases} 2e_{j+1}, & \text{if } j \geq 0, \\ e_j, & \text{if } j < 0. \end{cases}$$

It is not difficult to check that $\sigma(B) = \{\lambda \in \mathcal{C} : 1 \leq |\lambda| \leq 2\}$ and $\lambda - B$ is a semi-Fredholm operator of index -1 for all λ in the interior of $\sigma(B)$; moreover,

$$\mathcal{H} = \bigvee \{B^k e_0\}_{-\infty}^{\infty} = [\mathcal{A}^a(B)e_0]^{-},$$

where $\mathcal{A}^a(B)$ denotes the weak closure of the rational functions with poles outside $\sigma(B)$; that is, B is *rationally cyclic*. However, B is not cyclic because $\{\lambda \in \mathcal{C} : \operatorname{ind}(\lambda - B) = -1\}$ is connected, but not simply connected (see [**10, 11, 14**], or [**1**, Chapter 11]).

Furthermore, by combining the results in these references with [**11**, Propositions 1(i) and 2], we infer that $\mathcal{A}^a(B^{(n)})$ has multiplicity n and

$$\mu(B^{(n)}) = n + 1 \quad \text{for all } n = 1, 2, \ldots.$$

Thus, $B_k = B^{(k)}$ satisfies

$$\mu(B_k^{(n)}) = \mu(B^{(nk)}) = nk + 1 \quad \text{for all } n = 1, 2, \ldots.$$

Unfortunately, the above results cannot be modified to obtain the sequences $\{nk+2\}_{n=1}^{\infty}$. This can be done by using the following quantitative version of [**11**, Proposition 4].

THEOREM 5.3. *Suppose* $T \in \mathcal{L}(\mathcal{X}), R \in \mathcal{L}(\mathcal{Y}), \mathcal{X} = \bigvee\{T^k y_j : j = 1,2,\dots,m\}_{k=0}^{\infty}$, *and there exist a Jordan curve* $\gamma \subset \{\lambda \in \mathcal{C} : \mathrm{nul}(\lambda - T)^* = m\}$ *and a function* $\phi : \gamma \to \mathcal{X}^{*(m)}, \phi = (\phi_1, \phi_2, \dots, \phi_m)$, *such that*

(i) $\ker(\lambda - T)^* = \bigvee\{\phi_i(\lambda)\}_{i=1}^m$ *for each* $\lambda \in \gamma$;

(ii) $\|\phi(\lambda)\| \leq C$ *and* $|\det(\phi_i(\lambda)y_j)_{i,j=1}^m| \geq \delta > 0$ (*for some positive constants* C, δ) *for all* $\lambda \in \gamma$; *and*

(iii) $\sigma(R)$ *is included in* interior(γ) (= *the bounded component of* $\mathcal{C} \setminus \gamma$).

Then

$$\mu(T \oplus R) = \mu(T) + \mu(R) = m + \mu(R).$$

We shall need an auxiliary result:

LEMMA 5.4. *Let* γ *and* R *be as in Theorem 5.3, and let*

$$\mathcal{A}(\gamma) = \{f : f \textit{ is continuous on } \hat{\gamma}, \textit{analytic on } \mathrm{interior}(\gamma)\}$$

(*sup norm*, $\hat{\gamma} = \gamma \cup$ interior(γ)). *If* M_λ = "multiplication by λ" *on* $\mathcal{A}(\gamma)$, *then* $\mu(M_\lambda \oplus R) = \mu(R) + 1$.

PROOF. It is obvious that $\mu(M_\lambda \oplus R) \leq \mu(M_\lambda) + \mu(R) = \mu(R) + 1$.

Assume that $\mu(M_\gamma \oplus R) = m < \infty$, and let $(f_1, y_1), (f_2, y_2), \dots, (f_m, y_m) \in \mathcal{A}(\gamma) \oplus \mathcal{Y}$ be a multicyclic m-tuple for $M_\lambda \oplus R$, that is,

$$A(\gamma) \oplus \mathcal{Y} = \bigvee\{(M_\lambda \oplus R)^k (f_j, y_j) : j = 1,2,\dots,m\}_{k=0}^{\infty}.$$

Then $\mathcal{Y} = \bigvee\{R^k y_j : j = 1,2,\dots,m\}_{k=0}^{\infty}$ and $\mathcal{A}(\gamma) = \bigvee\{(M_\lambda)^k f_j : j = 1,2,\dots,m\}_{k=0}^{\infty}$, and therefore the ideal generated by $f_1, f_2, \dots, f_m$ coincides with $\mathcal{A}(\gamma)$. It follows from [**5**, Theorem 1.2 or Theorem 3.11] or [**16**] that there exist functions $h_1, h_2, \ldots, h_m \in \mathcal{A}(\gamma)$ such that h_1 is *invertible* and

$$h_1 f_1 + h_2 f_2 + \cdots + h_m f_m = e_0,$$

where $e_0(\lambda) \equiv 1$ on $\hat{\gamma}$.

Observe that $h_j(M_\lambda \oplus R) = h_j(M_\lambda) \oplus h_j(R)$ (where $h_j(M_\lambda) =$ "multiplication by h_j," and $h_j(R)$ defined via functional calculus, $j = 1, 2, \ldots, m$) is a well-defined norm-limit of polynomials in $M_\lambda \oplus R$; moreover, since h_1 is invertible in $\mathcal{A}(\gamma)$, so is the operator $h_1(M_\lambda \oplus R)$. Hence the $m \times m$ operator matrix

$$L = \begin{pmatrix} h_1(M_\lambda \oplus R) & h_2(M_\lambda \oplus R) & h_3(M_\lambda \oplus R) & \cdots & h_m(M_\lambda \oplus R) \\ & 1 \oplus 1 & & & \\ & & 1 \oplus 1 & \mathrm{O} & \\ & \mathrm{O} & & \ddots & \\ & & & & 1 \oplus 1 \end{pmatrix}$$

($\in \mathcal{L}([\mathcal{A}(\gamma) \oplus \mathcal{Y}]^{(m)})$) is invertible, and both L and L^{-1} belong to $M_n[\mathcal{A}(M_\lambda \oplus R)]$; therefore (by [**12**, Proposition 1_n(vi)]), the coördinates of

$$L \begin{bmatrix} (f_1, y_1) \\ (f_2, y_2) \\ \cdot \\ \cdot \\ \cdot \\ (f_m, y_m) \end{bmatrix} = \begin{bmatrix} (e_0, z_1) \\ (f_2, y_2) \\ \cdot \\ \cdot \\ \cdot \\ (f_m, y_m) \end{bmatrix},$$

(where $z_1 = \sum_{j=1}^m h_j(R) y_j$) form a multicyclic m-tuple for $M_\lambda \oplus R$.

Similarly, the $m \times m$ operator matrix

$$N = \begin{pmatrix} 1 \oplus 1 & & & & \\ -f_2(M_\lambda \oplus R) & 1 \oplus 1 & & \mathrm{O} & \\ -f_3(M_\lambda \oplus R) & & 1 \oplus 1 & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & \mathrm{O} & & \cdot \\ -f_m(M_\lambda \oplus R) & & & & 1 \oplus 1 \end{pmatrix}$$

$(\in \mathcal{L}([\mathcal{A}(\gamma) \oplus \mathcal{Y}]^{(m)}))$ is invertible, and both N and N^{-1} belong to $M_n[\mathcal{A}(M_\lambda \oplus R)]$. Therefore the coördinates of

$$N \begin{bmatrix} (e_0, z_1) \\ (f_2, y_2) \\ (f_3, y_3) \\ \cdot \\ \cdot \\ \cdot \\ (f_m, y_m) \end{bmatrix} = \begin{bmatrix} (e_0, z_1) \\ (0, z_2) \\ (0, z_3) \\ \cdot \\ \cdot \\ \cdot \\ (0, z_m) \end{bmatrix}$$

(where $z_j = y_j - f_j(R)z_1, j = 2, 3, \dots, m$) form a multicyclic m-tuple for $M_\lambda \oplus R$. In other words,

$$\mathcal{A}(\gamma) \oplus \mathcal{Y} = \bigvee \{(M_\lambda \oplus R)^k (e_0, z_1); (M_\lambda \oplus R)^k (0, z_j) : j = 2, 3, \dots, m\}_{k=0}^{\infty}.$$

In particular, $(0, z_1) \in \mathcal{A}(\gamma) \oplus \mathcal{Y}$, and therefore there exist sequences $\{p_k^{(j)}\}_{k=1}^{\infty}$ of polynomials, $j = 1, 2, \dots, m$, such that

$$\left\| p_k^{(1)}(M_\lambda \oplus R)(e_0, z_1) + \sum_{j=2}^{m} p_k^{(j)}(M_\lambda \oplus R)(0, z_j) - (0, z_1) \right\|$$

$$= \left\| (p_k^{(1)}, \sum_{j=1}^{m} p_k^{(j)}(R) z_j) - (0, z_1) \right\| \to 0,\ k \to \infty.$$

It readily follows that $p_k^{(1)}(\lambda) \to 0, k \to \infty$, uniformly on $\hat{\gamma}$, and therefore $\|p_k^{(1)}(R)\| \to 0, k \to \infty$, and

$$\left\| \sum_{j=2}^{m} p_k(R) z_j - z_1 \right\| \to 0,\ k \to \infty.$$

Since $\mathcal{Y} = \bigvee \{R^k z_j : j = 1, 2, \dots, m\}_{k=0}^{\infty}$, we conclude that

$$\mathcal{Y} = \bigvee \{R^k z_j : j = 2, 3, \dots, m\}_{k=0}^{\infty}$$

(because this last closed span actually contains z_1!).

Hence, $\mu(R) \leq m - 1 = \mu(M_\lambda \oplus R) - 1$, whence the result follows. $\square$

PROOF OF THEOREM 5.3. Clearly, $m = \mathrm{nul}(\lambda - T)^* \leq \mu(T) \leq m, \lambda \in \gamma$, so that $\mu(T) = m$; moreover, $\mu(T \oplus R) \leq \mu(T) + \mu(R) = \mu(R) + m$.

According to [**11**, Theorem 1] (see also [**12**, Theorem 1], [**15**, Theorem 1], or [**1**, Chapter 11]), conditions (i) and (ii) are actually equivalent to the existence of an intertwining mapping $X : \mathcal{X} \to \mathcal{A}(\gamma)^{(m)}$ with dense range such that $Xy_j = (0, 0, \ldots, 0, e_0(j\text{-th coordinate}), 0, \ldots, 0)$, $j = 1, 2, \ldots, m$, and $XT = M_\lambda^{(m)} X$.

Define $Y : \mathcal{X} \oplus \mathcal{Y} \to \mathcal{A}(\gamma)^{(m)} \oplus \mathcal{Y}$ by $Y = X \oplus 1_{\mathcal{Y}}$; then

$$Y(T \oplus R) = (M_\lambda^{(m)} \oplus R)Y.$$

Since Y has dense range,

$$\mu(M_\lambda^{(m)} \oplus R) \leq \mu(T \oplus R) \leq \mu(R) + m$$

(See [**11**, Proposition 1(vi)]). Thus, it suffices to show that $\mu(M_\lambda^{(m)} \oplus R) \geq \mu(R) + m$.

If $m = 1$, this follows from Lemma 5.4. If $m \geq 2$, then we construct m Jordan curves $\gamma_1 = \gamma, \gamma_2, \gamma_3, \ldots, \gamma_m$, such that $\sigma(R) \subset$ interior (γ_m) and $\gamma_j \subset$ interior (γ_{j-1}) for $j = 2, 3, \ldots, m$.

The "restriction operator" $C_j : \mathcal{A}(\gamma) \to \mathcal{A}(\gamma_j)$ (defined by $C_j f = f|\hat{\gamma}_j, f \in \mathcal{A}(\gamma)$) is an injective mapping with dense range and satisfies

$$C_j M_\lambda\,(\text{on}\,\mathcal{A}(\gamma)) = M_\lambda\,(\text{on}\,\mathcal{A}(\gamma_j))C_j,\ j = 1, 2, \ldots, m$$

($C_1 =$ identity on $\mathcal{A}(\gamma)$). Therefore

$$\Big(\bigoplus_{j=1}^{m} C_j\Big) M_\lambda^{(m)}\ (\text{on}\,\mathcal{A}(\gamma)^{(m)}) = \Big[\bigoplus_{j=1}^{m} M_\lambda\ (\text{on}\,\mathcal{A}(\gamma_j))\Big]\Big(\bigoplus_{j=1}^{m} C_j\Big)$$

and

$$\mu(M_\lambda^{(m)} \oplus R) \geq \mu([\oplus_{j=1}^{m} M_\lambda\ (\text{on}\,\mathcal{A}(\gamma_j))] \oplus R).$$

Since

$$\sigma\Big(\Big[\bigoplus_{j=h}^{m} M_\lambda(\text{on}\ \mathcal{A}(\gamma_j))\Big] \oplus R\Big) = \hat{\gamma}_h, \quad h = 1, 2, \ldots, m,$$

and $\hat{\gamma}_h \subset \text{interior}(\gamma_{h-1})$ for $h = 2, 3, \ldots, m$, by repeated use of Lemma 5.4, we obtain

$$\begin{aligned}\mu(T \oplus R) &\geq \mu(M_\lambda^{(m)} \oplus R) \geq \mu([\bigoplus_{j=1}^{m} M_\lambda(\text{on } \mathcal{A}(\gamma_j))] \oplus R)\\ &= \mu([\bigoplus_{j=2}^{m} M_\lambda(\text{on } \mathcal{A}(\gamma_j))] \oplus R) + 1\\ &= \mu([\bigoplus_{j=3}^{m} M_\lambda(\text{on } \mathcal{A}(\gamma_j))] \oplus R) + 2\\ &= \cdots = \mu([M_\lambda(\text{on } \mathcal{A}(\gamma_m))] \oplus R) + (m-1) = \mu(R) + m.\end{aligned}$$

Hence, $\mu(T \oplus R) = \mu(T) + \mu(R)$. □

COROLLARY 5.5. *For each $k \geq 1$, the sequences*

(B_k) $\{nk+1\}_{n=1}^{\infty}$,

(C_k) $\{nk+2\}_{n=1}^{\infty}$,

(D_k) $\{k+1, 2k, 3k, 4k, 5k, 6k, \ldots\}$

are attainable.

PROOF. (B_k). Apply Theorem 5.3 with $T = S^{(k)}$ (S = unilateral shift) $R = E$ as in §3 and $\gamma = \{\lambda \in \mathcal{C} : |\lambda| = 1/2\}$. (Observe that $\sigma(E) = \{0\}$.)

(C_k). Apply Theorem 5.3 with $T = S^{(k)}, R = F$ as in Corollary 4.1 and $\gamma = \{\lambda \in \mathcal{C} : |\lambda| = 1/2\}$.

(D_K). If $k = 1$, this is the result of Corollary 5.2. If $k \geq 2$, apply Proposition 5.1 to T_1 satisfying (A_k) and T_2 satisfying (C_{k-1}):

$$\max[(k-1)+2, k] = k+1,$$

but

$$\max[n(k-1)+2, nk] = nk \quad \text{for all } n \geq 2. \; \square$$

REMARK 5.6. The following simple criterion (somehow related to the proof of Theorem 5.3) can be used to estimate multiplicities of certain operators: suppose $T \in \mathcal{L}(\mathcal{X}), \mu(T) = m$ and $\mathrm{nul}(\lambda_0 - T)^* = p\,(\leq m)$ for some $\lambda_0 \in \mathcal{C}$. If $(y_1, y_2, \ldots, y_m)$ is a multicyclic m-tuple for $\mathcal{T}$ and $\mathcal{S} = \bigvee\{y_1, y_2, \ldots, y_m\}$, then $\mathcal{S} = \mathcal{R} + \mathcal{L}$, where $\mathcal{L} = \mathcal{S} \cap [\mathrm{ran}(\lambda_0 - T)]^-$, $\dim \mathcal{R} = p, \mathcal{X} = \mathcal{R} + [\mathrm{ran}(\lambda_0 - T)]^-$ and $\mathcal{R} \cap [\mathrm{ran}(\lambda_0 - T)]^- = \mathcal{R} \cap \mathcal{L} = \{0\}$. Clearly, we can directly assume that $\mathcal{R} = \bigvee\{y_1, y_2, \ldots y_p\}$.

Let $\mathcal{M} = \bigvee\{T^k y_j : j = 1, 2, \ldots, p\}_{k=0}^{\infty} = \bigvee\{(\lambda_0 - T)^k y_j : j = 1, 2, \ldots, p\}_{k=0}^{\infty}$, and let $T^0 \in \mathcal{L}(\mathcal{X}/\mathcal{M})$ be the operator induced by T on the quotient space (defined by $T^0(x^0) = (Tx)^0$, where $x^0 = x + \mathcal{M}$); then

$$\mu(T^0) = m - p.$$

Indeed, it is easily seen that $\mu(T|\mathcal{M}) \leq p$ and

$$m = \mu(T) \leq \mu(T|\mathcal{M}) + \mu(T^0).$$

On the other hand, $(y_j)^0 = y_j + \mathcal{M} = \mathcal{M}$ for $j = 1, 2, \ldots, p$, and therefore $\mathcal{X} = \bigvee\{T^k y_j : j = 1, 2, \ldots, m\}_{k=0}^{\infty}$ implies that

$$\mathcal{X}/\mathcal{M} = \bigvee\{(T^0)^k (y_j)^0 : j = p+1, p+2, \ldots, m\}_{k=0}^{\infty},$$

whence we obtain $\mu(T^0) \leq m - p$.

Thus $m \leq \mu(T|\mathcal{M}) + \mu(T^0) \leq p + (m - p) = m$, and therefore $\mu(T|\mathcal{M}) = p, \mu(T^0) = m - p$ and $\mu(T) = \mu(T|\mathcal{M}) + \mu(T^0)$.

6. Rational multiplicity, etc.

(1) Clearly, the multiplicity of the algebra $\mathcal{A}(T)$ coincides with $\mu(T)$. Similarly, if $\mathcal{A}(T)^{(n)} = \{A^{(n)} : A \in \mathcal{A}(T)\}$, then $\mu[\mathcal{A}(T)^{(n)}] = \mu(T^{(n)}), n = 1, 2, \ldots$.

In addition to $\mathcal{A}(T)$, we can consider the other three algebras naturally associated with $T : \mathcal{A}^a(T)$ (mentioned in §5), $\mathcal{A}'(T) = \{A \in \mathcal{L}(\mathcal{X}) : AT = TA\}$ (= the commutant of T) and $\mathcal{A}''(T) = \{B \in \mathcal{L}(\mathcal{X}) : BA = AB$ for all $A \in \mathcal{A}'(T)\}$(= the double commutant of T). We always have $\mathcal{A}(T) \subset \mathcal{A}^a(T) \subset \mathcal{A}''(T) \subset \mathcal{A}'(T)$, and therefore

$$\mu[\mathcal{A}(T)^{(n)}] \geq \mu[\mathcal{A}^a(T)^{(n)}] \geq \mu[\mathcal{A}''(T)^{(n)}] \geq \mu[\mathcal{A}'(T)^{(n)}]$$

for all $n = 1, 2, \ldots$.

What can be said about the sequences $\{\mu[\mathcal{A}^a(T)^{(n)}]\}_{n=1}^{\infty}, \{\mu[\mathcal{A}''(T)^{(n)}]\}_{n=1}^{\infty}$ and $\{\mu[\mathcal{A}'(T)^{(n)}]\}_{n=1}^{\infty}$?

Of course, the three of them satisfy the inequalities $\mu_{\max[m,n]} \leq \mu_{m+n} \leq \mu_m + \mu_n, m, n \geq 1$; moreover, each of the examples $A_k = S^{(k)}, E, F = E \oplus E^*, S^{(k)} \oplus E, S^{(k)} \oplus F$ satisfy $\mathcal{A}(T) = \mathcal{A}^a(T)$ (because the spectra have no holes), whence it readily follows that each of the sequences $(A_k), (B_k), (C_k), (D_k), (E), (F)$ and (G) are attainable for $\mathcal{A}^a(T)$.

On the other hand, by using the examples of [**19**], we can easily check that (A_k) can be attained by $\mathcal{A}''(T)$ and $\mathcal{A}'(T)$, for each $k \geq 1$.

(2) Theorem 2.1 remains true if $\mathcal{A}(T)$ is replaced by $\mathcal{A}^a(T)$ (same proof, with polynomials replaced by rational functions with poles outside $\sigma(T)$).

The "rational version" of Lemma 5.4 follows by the same argument by using [**5**, Theorem 3.1], [**6**]: let Ω be a bounded open subset of $\mathcal{C}$ whose boundary consists of finitely many pairwise disjoint Jordan curves, let $\mathcal{A}(\Omega) = \{f : f$ is continuous on Ω^- and analytic on $\Omega\}$, and let $R \in \mathcal{L}(\mathcal{Y})$ be an operator such that $\sigma(R) \subset \Omega$; then $\mu(M_\lambda \oplus R) = \mu(R) + 1$. By using this result, we obtain the "rational version" of Theorem 5.3.

THEOREM 5.3^a. *Suppose* $T \in \mathcal{L}(\mathcal{X}), R \in \mathcal{L}(\mathcal{Y}), \mathcal{X} = \bigvee\{Ay_j : A \in \mathcal{A}^a(T), j = 1, 2, \ldots, m\}$ *and there exist* Ω *as above such that* $\partial\Omega \subset \{\lambda \in \mathcal{C} : \mathrm{nul}(\lambda - T)^* = m\}$ *and a function* $\phi : \partial\Omega \to \mathcal{X}^{*(m)}, \phi = (\phi_1, \phi_2, \ldots, \phi_m)$*, such that*

(i) $\ker(\lambda - T)^* = \bigvee\{\phi_i(\lambda)\}_{i=1}^m$ *for each* $\lambda \in \partial\Omega$;

(ii) $\|\phi(\lambda)\| \leq C$ *and* $|\det(\phi_i(\lambda)(y_j))_{i,j=1}^m| \geq \delta > 0$ (*for some positive constants* C, δ) *for all* $\lambda \in \partial\Omega$;

(iii) *each bounded component of* $\mathcal{C} \setminus \Omega^-$ *includes a bounded component of* $\mathcal{C} \setminus \sigma(T)$*; and*

(iv) $\sigma(R) \subset \Omega$.

Then

$$\mu[A^a(T \oplus R)] = \mu[A^a(T)] + \mu[A^a(R; \Omega)] = m + \mu[A^a(R; \Omega)],$$

where $\mathcal{A}^a(R; \Omega)$ *is the weak closure of the rational functions in* R *with poles outside* Ω^-.

(3) The final section of A. Atzmon's article [**2**] contains some interesting results about the sequence $\{\mu(T^{(n)})\}_{n=1}^{\infty}$. We hope that his technique of multilinear mappings will shed some light on our problem.

(4) If the sequence $\mu_n \equiv m$ is attainable for all $m \geq 1$, then so is every convex sequence satisfying $\mu_n \geq n$ for all $n = 1, 2, \ldots$: use Proposition 5.1 and Theorem 5.3 as in the proof of Corollary 5.5. Indeed, a sequence like this satisfies $\mu_n \leq n\mu_1$ (for all $n \geq 1$), and therefore $\mu_n = nk + m$, for some $k \geq 1$, for all n large enough. It is not difficult to deduce that $\mu_n = \max[nk_j + m_j : j = 1, 2, \ldots, p]$ for some finite family with $m_1 > m_2 > \cdots > m_p = m$ and $k_1 < k_2 < \cdots < k_p = k$.

ADDED IN PROOF: Professor N. K. Nikol'skiĭ wrote several articles about sufficient conditions for $\mu(S \oplus T) = \mu(S) + \mu(T)$, and related problems. We list these articles below with the hope that they will help to complete the analysis begun in the present article:

1) Selected problems of weighted approximation and spectral analysis, Proc. Steklov Inst. Math. 120 (1974); English transl. Amer. Math. Soc., Providence, R.I., 1976. (See especially Section 3.4.)

2) Methods for calculating the spectral multiplicity of orthogonal sums (Russian), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 126 (1983), 150-158.

3) Ha-plitz operators: a survey of some recent results, Operators and Function Theory (Lancaster, 1984), NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci. 153, Reidel, Dordrecht-Boston, Mass., 1985, pp. 87-137.

REFERENCES

1. C. Apostol, L.A. Fialkow, D.A. Herrero and D. Voiculescu, *Approximation of Hilbert space operators.* Volume II, Research Notes in Mathematics 102, Pitman Adv. Books Program, Boston-London-Melbourne, 1984.

2. A. Atzmon, *Multilinear mappings and estimates of multiplicity*, Integral Equations Operator Theory **10** (1987), 1–16.

3. S. Caradus, W. E. Pfaffenberger and B. Yood, *Calkin algebras and algebras of operators on Banach spaces,* Lect. Notes in Pure and Appl. Math. **9**, Marcel Dekker, Inc., New York, 1974.

4. J. B. Conway, *Subnormal operators,* Research Notes in Math. **51**, Pitman Adv. Books Program, Boston-London-Melbourne, 1981.

5. G. Corach and F.D. Suárez, *Stable rank in holomorphic function algebras*, Illinois J. Math. **29** (1985), 627–639.

6. ——— and ———, *Extension problems and stable rank in commutative Banach algebras*, Topology Appl. **21** (1985), 1–8.

7. J. A. Deddens, *On $A \oplus A^*$*, Preprint, not for publication, 1972.

8. T. W. Gamelin, *Uniform algebras*, Prentice Hall, Englewood Cliffs, New Jersey, 1969.

9. P. R. Halmos, *A Hilbert space problem book* (2nd ed.) revised and enlarged, Springer-Verlag, New York-Heidelberg-Berlin, 1982.

10. D.A. Herrero, *Eigenvectors and cyclic vectors for bilateral weighted shifts*, Rev. Un. Mat. Argentina **26** (1972), 24–41.

11. ———, *On multicyclic operators*, Integral Equations Operator Theory **1** (1978), 57–102.

12. ———, *Possible structures for the set of cyclic vectors*, Indiana Univ. Math. J. **28** (1980), 913–926.

13. ———, *On the multiplicities of the powers of a Banach space operator*, Proc. Amer. Math. Soc. **94** (1985), 239–243.

14. ———, *The Fredholm structure of an n-multicyclic operator*, Indiana Univ. Math. J. **36** (1987), 549–566.

15. ———, and L. Rodman, *The multicyclic n-tuples of an n-multicyclic operator, and analytic structures on its spectrum*, Indiana Univ. Math. J. **34** (1985), 619–629.

16. P. W. Jones, D. Marshall and T. Wolff, *Stable rank of the disc algebra*, Proc. Amer. Math. Soc. **96** (1986), 603–604.

17. N. K. Nikol'skiĭ and V.I. Vasjunin, *Control subspaces of minimal dimension and root vectors*, Integral Equations Operator Theory **6** (1983), 274–311.

18. W.R. Wogen, *On some operators with cyclic vectors*, Indiana Univ. Math. J. **27** (1978), 163–171.

19. ———, *On cyclicity of commutants*, Integral Equations Operator Theory **5** (1982), 141–143.

Department of Mathematics, Arizona State University, Tempe, AZ 85287

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27514