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EXISTENCE OF TRANSVERSAL HOMOCLINIC POINTS IN A DEGENERATE CASE

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Introduction. Let F be a diffeomorphism on a compact manifold. Smale [**8, 9**] shows that if F has a transversal homoclinic point there is a Cantor-like set on which some iterate of F is invariant and isomorphic to the Bernoulli shift on a finite number of symbols. In Palmer [**5, 7**] it was shown how this result could be simply deduced from the shadowing lemma for hyperbolic sets.

Also, in [**5**] (see also Gruendler [**2**]), a periodic differential equation

$$\dot{x} = g(x) + \mu h(t, x, \mu), \quad x \in \mathbf{R}^k, \tag{1}$$

was considered, where the unperturbed system

$$\dot{x} = g(x) \tag{2}$$

has a saddle point and an associated homoclinic connection $\phi(t)$ such that, up to a scalar multiple, $\phi'(t)$ is the unique bounded solution of the variational equation

$$\dot{x} = g'(\phi(t))x. \tag{3}$$

Under this condition the equation adjoint to (3) also has, up to a scalar multiple, a unique bounded solution $\psi(t)$, and, if the Melnikov function

$$\Delta(\alpha) = -\int_{-\infty}^{\infty} \psi^*(t+\alpha)h(t, \phi(t+\alpha), 0)\, dt$$

has a simple zero, it turns out for $\mu \neq 0$ sufficiently small that the period map for equation (1) has a transversal homoclinic point.

Note that the condition on the variational equation (3) is equivalent to the requirement that the tangent spaces to the stable and unstable manifolds of the saddle point have a one-dimensional intersection along the homoclinic orbit $\phi(t)$. In this paper we want to relax this condition, assuming a two-dimensional intersection or, equivalently, that the subspace of bounded solutions of equation (3) has dimension 2. Then $\phi(t + \alpha)$ may or may not be a one-parameter subfamily of a two-parameter family of homoclinic orbits. If it is, a certain scalar quantity is zero.

First, we consider the case where $\phi(t + \alpha)$ *is* a one-parameter subfamily of a two-parameter family of homoclinic orbits and show that if a certain vector function consisting of two Melnikov functions has a simple zero, then the period map for equation (1) has a transversal homoclinic point for $\mu \neq 0$ sufficiently small. A similar result was obtained by Gruendler [**2**] under three additional conditions: (a) "uniformly transverse perturbation," (b) the number of parameters is at least two and (c) rank $A(\xi) = 2$ (cf. [**2**] for notation).

Second, we consider the case where $\phi(t + \alpha)$ *is not* a one-parameter subfamily of a two-parameter family. This is ensured by the nonvanishing of the aforementioned scalar quantity. (We have not considered the even more degenerate case where this scalar quantity vanishes but $\phi(t+\alpha)$ is still not a one-parameter subfamily of a two-parameter family.) To handle this case, we prove an abstract theorem related to one of Hale and Táboas [**3**]. We consider an equation $f(x, \mu) = 0$ which, for $\mu = 0$, has a one-parameter family of solutions $\zeta(\alpha)$ such that $f_x(\zeta(\alpha), 0)$ is Fredholm of index zero with $\dim \mathcal{N}(f_x(\zeta(\alpha), 0)) = 2$. (The case $\dim \mathcal{N}(f_x(\zeta(\alpha), 0)) = 1$ was dealt with in Theorem 4.1 in Palmer [**5**].) We impose a nondegeneracy condition corresponding to the nonvanishing of the aforementioned scalar quantity. Then if one of our two Melnikov functions has a simple zero and the other is nonzero at that point, the equation $f(x, \mu) = 0$ has two solutions for μ on one side of zero and none on the other side. When we apply the theorem to equation (1), these solutions correspond to transversal homoclinic points of the period map.

Finally, I wish to thank J. Moser for suggesting consideration of this problem to me.

2. The case of a two-parameter family of homoclinic orbits. Let $g : \mathbf{R}^k \to \mathbf{R}^k$ be a C^2 function such that

(A1) system (2) has an equilibrium point x_0 such that the eigenvalues of $g'(x_0)$ have nonzero real parts;

(A2) system (2) has a family of solutions $\phi(t,\beta)$, $\beta \in J$, J an interval, such that $|\phi(t,\beta) - x_0| \to 0$ as $|t| \to \infty$; $\phi(t,\beta)$ is a C^2 function of its arguments and is bounded together with its derivatives; also $\phi_t(t,\beta)$ and $\phi_\beta(t,\beta)$ are linearly independent for all (t,β);

(A3) for each β the variational equation

$$\dot{x} = g'(\phi(t,\beta))x \tag{4}$$

has a two-dimensional subspace of bounded solutions.

Since $|\phi(t,\beta) - x_0| \to 0$ as $|t| \to \infty$ and the eigenvalues of $g'(x_0)$ have nonzero real parts, it follows from the roughness theorem for exponential dichotomies (Coppel [**1**, p. 34]) that system (4) has an exponential dichotomy on $[0,\infty)$ and $(-\infty,0]$ and the ranks of the corresponding projections $P(\beta)$, $Q(\beta)$ are equal. Then if we take the Banach spaces $\mathcal{E} = C^1(\mathbf{R},\mathbf{R}^k)$, $\mathcal{F} = C^0(\mathbf{R},\mathbf{R}^k)$, it follows from Lemma 4.2 in Palmer [**5**] that the linear operator from $\mathcal{E}$ into $\mathcal{F}$ defined by $x(\cdot) \to x'(\cdot) - g'(\phi(t,\beta))x(\cdot)$ is Fredholm of index zero and a function $p \in \mathcal{F}$ is in its range if and only if

$$\int_{-\infty}^{\infty} \psi^*(t)p(t)\,dt = 0$$

for all bounded solutions $\psi(t)$ of the adjoint equation

$$\dot{x} = -g'(\phi(t,\beta))^* x. \tag{5}$$

Now the subspace of bounded solutions of (5) consists of those solutions with initial values in $\mathcal{N}(P(\beta)^*) \cap \mathcal{R}(Q(\beta)^*)$. This is just the nullspace of the matrix $\begin{bmatrix} P(\beta)^* \\ I - Q(\beta)^* \end{bmatrix}$, which has constant rank $k-2$ and, by Proposition 2.3 in Palmer [**6**], is C^1 in β. So we can find a C^1 basis $\psi_1(\beta)$, $\psi_2(\beta)$ for its nullspace. Then it follows from the proof of Proposition 2.3 in Palmer [**6**] that if $\psi_1(t,\beta)$, $\psi_2(t,\beta)$ are the solutions of (5) with initial values $\psi_1(\beta)$, $\psi_2(\beta)$, then $\beta \to \psi_i(\cdot,\beta)$, $i = 1,2$, are C^1 as functions into $C^0(\mathbf{R},\mathbf{R}^k)$ and $\psi_i(t,\beta)$, $\psi_{i\beta}(t,\beta) \to 0$ exponentially as $|t| \to \infty$.

THEOREM 1. *Let* $g : \mathbf{R}^k \to \mathbf{R}^k$ *be a* C^2 *function satisfying* (A1), (A2), (A3), *and let* $h : \mathbf{R} \times \mathbf{R}^k \times \mathbf{R} \to \mathbf{R}^k$ *be a* C^2 *function, bounded* *together with its derivatives and T-periodic in its first variable. Define* $\Delta(\alpha,\beta) = (\Delta_1(\alpha,\beta), \Delta_2(\alpha,\beta))$, *where, for* $i = 1, 2$,

$$\Delta_i(\alpha,\beta) = -\int_{-\infty}^{\infty} \psi_i^*(t+\alpha,\beta)h(t,\phi(t+\alpha,\beta),0)\,dt. \tag{6}$$

Suppose, for some (α_0,β_0),

$$\Delta(\alpha_0,\beta_0) = 0, \quad \det \Delta'(\alpha_0,\beta_0) \neq 0.$$

Then if $\mu \neq 0$ *is sufficiently small, the period map for system* (1) *has a unique transversal homoclinic point with orbit near that of* $\phi(\alpha_0,\beta_0)$.

PROOF. We use Theorem 4.1 in [**5**]. Note that conditions (ii) and (iii) in that theorem can be reformulated as follows (cf. Palmer [**6**, p. 341]). Referring to the notation of Theorem 4.1, let $\dim \mathcal{N}(L) = d$ and suppose $\mathcal{M}$ is parameterized as $\zeta(\alpha)$, where α is in a neighborhood of α_0 in $\mathbf{R}^d$. Then, for α near $\alpha_0(\zeta(\alpha_0) = 0)$, $f_x(\zeta(\alpha),0)$ is also Fredholm of index zero and its nullspace has dimension d. Let $\psi_1(\alpha),\dots,\psi_d(\alpha)$ be linear functionals in $\mathcal{F}^*$ such that

$$\mathcal{R}(f_x(\zeta(\alpha),0)) = \bigcap_{i=1}^{d} \mathcal{N}(\psi_i(\alpha)).$$

These can be chosen to be C^1 in α by a result in the Appendix. We define

$$\Delta(\alpha) = (\Delta_1(\alpha),\dots,\Delta_d(\alpha)),$$

where

$$\Delta_i(\alpha) = \psi_i(\alpha)(f_\mu(\zeta(\alpha),0)).$$

Then conditions (ii) and (iii) of Theorem 4.1 are equivalent to the requirement that

$$\Delta(\alpha_0) = 0, \quad \det \Delta'(\alpha_0) \neq 0. \tag{7}$$

Here we take $\mathcal{E} = C^1(\mathbf{R},\mathbf{R}^k)$, $\mathcal{F} = C^0(\mathbf{R},\mathbf{R}^k)$ and define $f : \mathcal{E}\times\mathbf{R} \to \mathcal{F}$ by

$$\begin{aligned}[f(z,\mu)](t) = \dot z(t) - \{&g(\phi(t+\alpha_0,\beta_0)+z(t)) - g(\phi(t+\alpha_0,\beta_0)) \\ &+ \mu h(t,\phi(t+\alpha_0,\beta_0)+z(t),\mu)\}.\end{aligned}$$

It is clear that f is C^2 and that, for all y in $\mathcal{E}$,

$$[f_z(z,\mu)y](t) = \dot{y}(t) - \{g'(\phi(t+\alpha_0,\beta_0)+z(t)) \\ + \mu h_x(t,\phi(t+\alpha_0,\beta_0)+z(t),\mu)\}y(t).$$

We see also that $f(\zeta(\alpha,\beta),0)=0$, where

$$\zeta(\alpha,\beta) = \phi(\cdot+\alpha,\beta) - \phi(\cdot+\alpha_0,\beta_0),$$

and that

$$[f_z(\zeta(\alpha,\beta),0)y](t) = \dot{y}(t) - g'(\phi(t+\alpha,\beta))y(t).$$

It follows from (A3) and the remarks before the theorem that $f_z(\zeta(\alpha,\beta), 0)$ is Fredholm of index zero with $\dim \mathcal{N}(f_z(\zeta(\alpha,\beta),0)) = 2$. With $\mathcal{M}$ as the two-dimensional manifold parametrized as $\zeta(\alpha,\beta)$, we see that $T_{\zeta(\alpha,\beta)}\mathcal{M} \subset \mathcal{N}(f_z(\zeta(\alpha,\beta),0))$. Since both these subspaces have dimension 2, they must be equal. Also, it follows from the remarks before the theorem that, for the application of Theorem 4.1, we may take $\psi_i(\alpha,\beta) \in \mathcal{F}^*$, $i=1,2$, as the linear functional defined by

$$\psi_i(\alpha,\beta)(p) = \int_{-\infty}^{\infty} \psi_i^*(t+\alpha,\beta)p(t)\,dt.$$

Note also that

$$f_\mu(\zeta(\alpha,\beta),0) = -h(t,\phi(t+\alpha,\beta),0),$$

so that, in this case, $\Delta_i(\alpha,\beta)$ is defined by (6) for $i=1,2$ and we may apply Theorem 4.1 in [**5**] to deduce the existence of $\sigma > 0$ such that, for $\mu \neq 0$ sufficiently small, the equation

$$f(z,\mu) = 0$$

has a unique solution $z(\mu)$ satisfying $\|z(\mu)\| \le \sigma$. Moreover, $z(0)=0$, $z(\mu)$ is C^1 and, when $\mu \neq 0$, $f_z(z(\mu),\mu)$ is invertible. Then, as in Palmer [**5**], we deduce that, for $\mu \neq 0$, $x(t,\mu) = \phi(t+\alpha_0,\beta_0) + z(\mu)(t)$ is the unique solution of (1) such that, for all t, $|x(t,\mu) - \phi(t+\alpha_0,\beta_0)| \le \sigma$. Moreover, its variational equation has an exponential dichotomy on

$(-\infty, \infty)$. As in [**5**] again, it follows that the period map for equation (1) has $x(0, \mu)$ as a transversal homoclinic point for $\mu \neq 0$.

Finally, a standard Gronwall lemma argument shows the existence of $\sigma_0 > 0$, $\mu_0 > 0$ such that if $0 < |\mu| < \mu_0$ and $x(t)$ is a solution of (1) satisfying $|x(nT) - \phi(nT + \alpha_0, \beta_0)| \leq \sigma_0$ for all n, then $|x(t) - \phi(t + \alpha_0, \beta_0)| \leq \sigma$ for all t. Hence, if $\mu \neq 0$ is sufficiently small, $x(0, \mu)$ is the unique point whose orbit $\{x(nT, \mu)\}$ under the period map satisfies $|x(nT, \mu) - \phi(nT + \alpha_0, \beta_0)| \leq \sigma_0$ for all n. □

3. An abstract theorem. This theorem is related to a theorem of Hale and Táboas [**3**]. They applied their theorem to the existence of periodic solutions.

Let $\mathcal{E}, \mathcal{F}$ be Banach spaces and let $f : \mathcal{E} \times \mathbf{R} \to \mathcal{F}$ be a C^5 function such that

$$f(\zeta(\alpha), 0) = 0,$$

where $\zeta(\alpha)$ is a C^4 function defined for real α near α_0 with $\zeta'(\alpha_0) \neq 0$. We write $L(\alpha) = f_x(\zeta(\alpha), 0)$ and make the following two hypotheses:

(H1) $L(\alpha)$ is Fredholm of index zero with $\dim \mathcal{N}(L(\alpha)) = 2$;

(H2) if $w \in \mathcal{N}(L(\alpha))$ and $f_{xx}(\zeta(\alpha), 0)ww \in \mathcal{R}(L(\alpha))$, then w is a multiple of $\zeta'(\alpha)$.

As proved below, (H2) implies that the closed subspace V_α generated by $\mathcal{R}(L(\alpha))$ and $\{f_{xx}(\zeta(\alpha), 0)ww : w \in \mathcal{N}(L(\alpha))\}$ has co-dimension 1 and there exist $\psi_1(\alpha)$, $\psi_2(\alpha)$ in $\mathcal{F}^*$, which are C^4 in α, such that $\mathcal{R}(L(\alpha)) = \mathcal{N}(\psi_1(\alpha)) \cap \mathcal{N}(\psi_2(\alpha))$ and $V_\alpha = \mathcal{N}(\psi_1(\alpha))$.

Note that (H2) is a differential condition implying that $\zeta(\alpha)$ is *not* a one-parameter subfamily of a two-parameter family of solutions of $f(x, 0) = 0$.

THEOREM 2. *Let $\mathcal{E}, \mathcal{F}$ be Banach spaces, and let $f : \mathcal{E} \times \mathbf{R} \to \mathcal{F}$ be a C^5 function such that $f(\zeta(\alpha), 0) = 0$, where $\xi(\alpha)$ is a C^4 function defined for real α near α_0 with $\zeta'(\alpha_0) \neq 0$.*

Suppose (H1) *and* (H2) *are satisfied, and define*

$$\Delta_i(\alpha) = \psi_i(\alpha)(f_\mu(\zeta(\alpha), 0)), \quad i = 1, 2,$$
$$\lambda_0 = -2\Delta_2(\alpha_0)/\psi_2(\alpha_0)(f_{xx}(\zeta(\alpha_0), 0)v(\alpha_0)v(\alpha_0)),$$

where $\{\zeta'(\alpha_0), v(\alpha_0)\}$ *form a basis for* $\mathcal{N}(L(\alpha_0))$. *Then if*

$$\Delta_1(\alpha_0) = 0, \quad \Delta_1'(\alpha_0) \neq 0, \quad \Delta_2(\alpha_0) \neq 0, \tag{8}$$

the equation

$$f(x, \mu) = 0 \tag{9}$$

has exactly two solutions $x_1(\mu), x_2(\mu)$ *near* $\zeta(\alpha_0)$ *for* $\mu > 0$ *(if* $\lambda_0 > 0$, $\mu < 0$ *if* $\lambda_0 < 0$*) sufficiently small and no such solution for* $\mu < 0$ *(respectively,* $\mu > 0$*) sufficiently small. Moreover,* $x_1(\mu), x_2(\mu)$ *are* C^2 *functions of* $\sqrt{\mu}$ *(respectively* $\sqrt{-\mu}$ *) with* $x_1(0) = x_2(0) = \zeta(\alpha_0)$ *and* $f_x(x_i(\mu), \mu)$, $i = 1, 2$, *is invertible for* $\mu \neq 0$.

PROOF. *Preliminaries.* We choose a projection $P(\alpha)$ which is C^4 in α and such that $\mathcal{R}(P(\alpha)) = \mathcal{N}(L(\alpha))$. (See the appendix for a proof of this statement.) Then we can find $v(\alpha) \in \mathcal{N}(L(\alpha))$ depending C^4 on α such that $\mathcal{N}(L(\alpha))$ is generated by $\zeta'(\alpha)$ and $v(\alpha)$—for example, we can take $\zeta'(\alpha_0)$ and v_0 as a basis for $\mathcal{N}(L(\alpha_0))$ and then choose $v(\alpha) = P(\alpha)v_0$. As proved in the appendix, we can also find bounded linear functionals $\psi_1(\alpha)$, $\psi_2(\alpha)$, which are C^4 in α, such that $\mathcal{R}(L(\alpha)) = \mathcal{N}(\psi_1(\alpha)) \cap \mathcal{N}(\psi_2(\alpha))$ and a projection $Q(\alpha)$, which is C^4 in α, such that $\mathcal{R}(Q(\alpha)) = \mathcal{R}(L(\alpha))$.

Differentiating the equations

$$f_x(\zeta(\alpha), 0)\zeta'(\alpha) = 0, \quad f_x(\zeta(\alpha), 0)v(\alpha) = 0$$

with respect to α, we find that $f_{xx}(\zeta(\alpha), 0)\zeta'(\alpha)\zeta'(\alpha)$ and $f_{xx}(\zeta(\alpha), 0) \cdot\zeta'(\alpha)v(\alpha)$ are both in $\mathcal{R}(L(\alpha))$. Now (H2) implies that $w(\alpha) = f_{xx}(\zeta(\alpha), 0)v(\alpha)v(\alpha) \notin \mathcal{R}L(\alpha)$. So the subspace V_α generated by $\mathcal{R}(L(\alpha))$ and $\{f_{xx}(\zeta(\alpha), 0)ww : w \in \mathcal{N}(L(\alpha))\}$ is a closed subspace of $\mathcal{F}$ of codimension 1. Since $w(\alpha) \notin \mathcal{R}L(\alpha)$, we can assume without loss of generality that, for α near α_0, $\psi_2(\alpha)(w(\alpha)) \neq 0$. Then $(\psi_1(\alpha) + \beta\psi_2(\alpha))(w(\alpha)) = 0$ if $\beta = -\psi_1(\alpha)(w(\alpha))/\psi_2(\alpha)(w(\alpha))$. So if we replace $\psi_1(\alpha)$ by $\psi_1(\alpha) + \beta\psi_2(\alpha)$, we see that $V_\alpha = \mathcal{N}(\psi_1(\alpha))$. Clearly, $\psi_1(\alpha)$ is unique up to a scalar multiple.

Existence. We look for solutions to equation (9) of the form $x = \zeta(\alpha) + \beta v(\alpha) + w$, where $\beta \in \mathbf{R}$ and $w \in \mathcal{N}(P(\alpha_0))$. So we want to solve the equation

$$f(\zeta(\alpha) + \beta v(\alpha) + w, \mu) = 0$$

for $\alpha, \beta \in \mathbf{R}$ and $w \in \mathcal{N}(P(\alpha_0))$. For $|\alpha - \alpha_0| < \delta_1$, this is equivalent to the solution of the pair of equations

$$Q(\alpha_0)f(\zeta(\alpha) + \beta v(\alpha) + w, \mu) = 0 \tag{10}$$

$$[I - Q(\alpha)]f(\zeta(\alpha) + \beta v(\alpha) + w, \mu) = 0. \tag{11}$$

We can write the left side of (10) as $G(\alpha, \beta, w, \mu)$, where G is a mapping of $I \times \mathbf{R} \times \mathcal{N}(P(\alpha_0)) \times \mathbf{R}$ into $\mathcal{R}(L(\alpha_0))$ (I is an interval around α_0). Note that G is C^4, $G(\alpha, 0, 0, 0) = 0$ and $G_w(\alpha_0, 0, 0, 0) = Q(\alpha_0)L(\alpha_0) : \mathcal{N}(P(\alpha_0)) \to \mathcal{R}(L(\alpha_0))$. The latter operator is invertible and so, by the implicit function theorem, there exist $\delta_2 > 0$, $\sigma_2 > 0$ such that equation (10) has a unique solution $w = w(\alpha, \beta, \mu)$ in $|w| < \sigma_2$ when $|\alpha - \alpha_0| < \delta_2$, $|\beta| < \delta_2$, $|\mu| < \delta_2$. $w(\alpha, \beta, \mu)$ is a C^4 function and, by uniqueness, $w(\alpha, 0, 0) = 0$.

We substitute $w = w(\alpha, \beta, \mu)$ into (11) and obtain the equation

$$[I - Q(\alpha)]f(\zeta(\alpha) + \beta v(\alpha) + w(\alpha, \beta, \mu), \mu) = 0.$$

This is equivalent to the equation

$$h(\alpha, \beta, \mu) = (h_1(\alpha, \beta, \mu), h_2(\alpha, \beta, \mu)) = 0, \tag{12}$$

where

$$h_i(\alpha, \beta, \mu) = \psi_i(\alpha)[f(\zeta(\alpha) + \beta v(\alpha) + w(\alpha, \beta, \mu), \mu)].$$

Now h is C^4 and it is easy to verify that, for $i = 1, 2$,

$$\begin{gathered} h(\alpha, 0, 0) = 0, \quad h_\beta(\alpha, 0, 0) = 0 \\ h_{i\mu}(\alpha, 0, 0) = \Delta_i(\alpha), \quad h_{i\beta\beta}(\alpha, 0, 0) = \psi_i(\alpha)(w(\alpha)). \end{gathered} \tag{13}$$

Then, by Taylor's theorem, we may write

$$\begin{aligned} h(\alpha, \beta, \mu) = {} & h_\mu(\alpha, 0, 0)\mu + \frac{1}{2}h_{\beta\beta}(\alpha, 0, 0)\beta^2 + h_{\beta\mu}(\alpha, 0, 0)\beta\mu \\ & + \frac{1}{2}h_{\mu\mu}(\alpha, 0, 0)\mu^2 + p(\alpha, \beta, \mu)\beta^2 \\ & + q(\alpha, \beta, \mu)\beta\mu + r(\alpha, \beta, \mu)\mu^2, \end{aligned} \tag{14}$$

where $p = (p_1, p_2)$, $q = (q_1, q_2)$ and $r = (r_1, r_2)$ are C^2 and zero at $(\alpha, 0, 0)$.

Now we solve equation (12). There are two cases.

Case 1. $\lambda_0 = -2\Delta_2(\alpha_0)/\psi_2(\alpha_0)(w(\alpha_0)) > 0$. In this case we set $\mu = \epsilon^2$, $\beta = \epsilon\gamma$ and define

$$H(\alpha,\gamma,\epsilon) = \begin{cases} \epsilon^{-2} h(\alpha, \epsilon\gamma, \epsilon^2), & \epsilon \neq 0 \\ h_\mu(\alpha,0,0) + \frac{1}{2} h_{\beta\beta}(\alpha,0,0)\gamma^2, & \epsilon = 0. \end{cases}$$

It follows from (14) that, for $\epsilon \neq 0$,

$$\begin{aligned} H(\alpha,\gamma,\epsilon) =& h_\mu(\alpha,0,0) + \frac{1}{2} h_{\beta\beta}(\alpha,0,0)\gamma^2 + \epsilon h_{\beta\mu}(\alpha,0,0)\gamma \\ &+ \frac{1}{2}\epsilon^2 h_{\mu\mu}(\alpha,0,0) + p(\alpha,\epsilon\gamma,\epsilon^2)\gamma^2 \\ &+ \epsilon q(\alpha,\epsilon\gamma,\epsilon^2) r + \epsilon^2 r(\alpha,\epsilon\gamma,\epsilon^2). \end{aligned}$$

Thus, $H = (H_1, H_2)$ is C^2 with

$$\begin{aligned} H_i(\alpha,\gamma,0) &= \Delta_i(\alpha) + \frac{1}{2}\psi_i(\alpha)(w(\alpha))\gamma^2, \\ H_{i\alpha}(\alpha,\gamma,0) &= \Delta_i'(\alpha) + \frac{1}{2}\frac{d}{d\alpha}\{\psi_i(\alpha)(w(\alpha))\}\gamma^2, \\ H_{i\gamma}(\alpha,\gamma,0) &= \psi_i(\alpha)(w(\alpha))\gamma \end{aligned}$$

for $i = 1, 2$. So $H(\alpha,\gamma,0) = 0$ for $\alpha = \alpha_0$, $\gamma = \pm\sqrt{\lambda_0}$ and, using the fact that $\psi_1(\alpha)(w(\alpha)) = 0$, we find that the determinant of the Jacobian matrix of $H(\alpha,\gamma,0)$ at $(\alpha_0, \pm\sqrt{\lambda_0})$ is

$$\pm\Delta_1'(\alpha_0)\sqrt{\lambda_0}\psi_2(\alpha_0)(w(\alpha_0)) \neq 0.$$

Then, by the implicit function theorem, there exist $\epsilon_3 > 0$, $\delta_3 > 0$ such that, when $|\epsilon| < \epsilon_3$, the equation

$$H(\alpha,\gamma,\epsilon) = 0 \tag{15}$$

has a unique solution $\alpha = \alpha_1(\epsilon)$, $\gamma = \gamma_1(\epsilon)$ (respectively $\alpha_2(\epsilon), \gamma_2(\epsilon)$) in $|\alpha - \alpha_0| < \delta_3$, $|\gamma - \sqrt{\lambda_0}| < \delta_3$ (respectively $|\alpha - \alpha_0| < \delta_3$,

$|\gamma + \sqrt{\lambda_0}| < \delta_3$). Moreover, $\alpha_i(\epsilon), \gamma_i(\epsilon)$, $i = 1, 2$, are C^2 functions with $\alpha_1(0) = \alpha_2(0) = \alpha_0$, $\gamma_1(0) = \sqrt{\lambda_0}$, $\gamma_2(0) = -\sqrt{\lambda_0}$.

Then, for $i = 1, 2$,

$$x_i(\epsilon) = \zeta(\alpha_i(\epsilon)) + \epsilon\gamma_i(\epsilon)v(\alpha_i(\epsilon)) + w(\alpha_i(\epsilon), \epsilon\gamma_i(\epsilon), \epsilon^2), \quad \epsilon = \sqrt{\mu},$$

is a C^2 (in $\epsilon = \sqrt{\mu}$) solution of equation (9) with $x_i(0) = \zeta(\alpha_0)$. To see that $x_1(\epsilon) \neq x_2(\epsilon)$ for $\epsilon \neq 0$, note that, for $i = 1, 2$,

$$x_i'(0) = \alpha_i'(0)\zeta'(\alpha_0) + (-1)^{i+1}\sqrt{\lambda_0}[v(\alpha_0) + w_\beta(\alpha_0, 0, 0)].$$

If we differentiate $f(x_i(\epsilon), \epsilon^2) = 0$ with respect to ϵ and set $\epsilon = 0$, we obtain

$$f_x(\zeta'(\alpha_0), 0)x_i'(0) = 0.$$

This means $x_i'(0) \in \mathcal{N}(L(\alpha_0)) = \mathcal{R}(P(\alpha_0))$. But $w_\beta(\alpha_0, 0, 0) \in \mathcal{N}(P(\alpha_0))$ and so it must be zero. Thus, for $i = 1, 2$,

$$x_i'(0) = \alpha_i'(0)\zeta'(\alpha_0) + (-1)^{i+1}\sqrt{\lambda_0}v(\alpha_0). \tag{16}$$

Hence, $x_1(\epsilon) \neq x_2(\epsilon)$ for $\epsilon \neq 0$ sufficiently small.

Case 2. $\lambda_0 = -2\Delta_2(\alpha_0)/\psi_2(\alpha_0)(w(\alpha_0)) < 0$. In this case we set $\mu = -\epsilon^2$, $\beta = \epsilon\gamma$ and consider

$$H(\alpha, \gamma, \epsilon) = \begin{cases} \epsilon^{-2}h(\alpha, \epsilon\gamma, -\epsilon^2), & \epsilon \neq 0 \\ -h_\mu(\alpha, 0, 0) + \frac{1}{2}h_{\beta\beta}(\alpha, 0, 0)\gamma^2, & \epsilon = 0. \end{cases}$$

We proceed as in Case 1 with obvious modifications to get C^2 solutions $\alpha_i(\epsilon)$, $\gamma_i(\epsilon)$ of $H(\alpha, \gamma, \epsilon) = 0$ with $\alpha_i(0) = \alpha_0$, $\gamma_i(0) = (-1)^{i+1}\sqrt{-\lambda_0}$. Again, we get two distinct solutions $x_i(\epsilon)$ of (9) with $x_i(0) = \zeta(\alpha_0)$.

Uniqueness. Assuming $\lambda_0 > 0$ (the other case can be similarly dealt with), we show there exists $\delta_4 > 0$ such that if (α, β, μ) is a solution of equation (12) with $|\alpha - \alpha_0| < \delta_4$, $|\beta| < \delta_4$, $0 < |\mu| < \delta_4$, then $\mu > 0$ and $\alpha = \alpha_i(\sqrt{\mu})$, $\beta = \sqrt{\mu}\gamma_i(\sqrt{\mu})$ for $i = 1$ or 2.

Using (13) and (14), we may write, for $i = 1, 2$,

$$h_i(\alpha, \beta, \mu) = \Delta_i(\alpha)\mu + \left[\frac{1}{2}\psi_i(\alpha)(w_i(\alpha)) + p_i(\alpha, \beta, \mu)\right]\beta^2 + g_i(\alpha, \beta, \mu)\mu,$$

where $g_i(\alpha, \beta, \mu)$ is C^2 with $g_i(\alpha, 0, 0) = 0$.

Suppose $h(\alpha,\beta,\mu)=0$ where $|\alpha-\alpha_0|<\delta_4$, $|\beta|<\delta_4$, $0\le|\mu|<\delta_4$ (δ_4 satisfies $\delta_4<\delta_3,\epsilon_3$ and other conditions to be mentioned below). Then if we assume δ_4 is so small that $|p_2(\alpha,\beta,\mu)|<\frac{1}{2}|\psi_2(\alpha)(w(\alpha))|$, we can solve $h_2(\alpha,\beta,\mu)=0$ to get

$$\frac{\beta^2}{\mu}=\frac{-[\Delta_2(\alpha)+g_2(\alpha,\beta,\mu)]}{\frac{1}{2}\psi_2(\alpha)(w(\alpha))+p_2(\alpha,\beta,\mu)}.$$

Hence, if δ_4 is sufficiently small, β^2/μ has the same sign as λ_0 and $|\beta/\sqrt{\mu}-(\pm\sqrt{\lambda_0})|\le\delta_3$. So if we put $\epsilon=\sqrt{\mu}$ and $\gamma=\beta/\sqrt{\mu}$, (α,γ,ϵ) is a solution of equation (12) satisfying $|\alpha-\alpha_0|<\delta_3$, $|\gamma-(\pm\sqrt{\lambda_0})|<\delta_3$, $|\epsilon|<\epsilon_3$. Then, by the uniqueness above, $\alpha=\alpha_i(\epsilon)$ and $\beta=\beta_i(\epsilon)$ for $i=1$ or 2, and so the assertion follows.

Now we show there exists $\eta>0$ such that if $|x-\zeta(\alpha_0)|<\eta$, $x=\zeta(\alpha)+\beta v(\alpha)+w$ where $w\in\mathcal{N}(P(\alpha_0))$ and $|\alpha-\alpha_0|<\min\{\delta_1,\delta_2,\delta_4\}$, $|\beta|<\min\{\delta_2,\delta_4\}$, $|w|<\sigma_2$. To do this, consider the C^4 mapping $g: I\times\mathbf{R}\times\mathcal{N}(P(\alpha_0))\to\mathcal{E}$ defined by

$$g(\alpha,\beta,w)=\zeta(\alpha)+\beta v(\alpha)+w.$$

Also, $g(\alpha_0,0,0)=\zeta(\alpha_0)$ and the derivative at $(\alpha_0,0,0)$ is $[\zeta'(\alpha_0)\ \ v(\alpha_0)\ \ J]$ where $J:\mathcal{N}(P(\alpha_0))\to\mathcal{E}$ is the inclusion. The derivative has a bounded inverse sending $y\in\mathcal{E}$ into $(\gamma,\sigma,[I-P(\alpha_0)]y)$, where γ,σ are the unique solutions of $\gamma\zeta'(\alpha_0)+\sigma v(\alpha_0)=P(\alpha_0)y$. So, by the inverse function theorem, there exist open neighborhoods U and V of $(\alpha_0,0,0)$ and $\zeta(\alpha_0)$, respectively, such that $g:U\to V$ is a diffeomorphism. The existence of η follows immediately.

Finally, let x be a solution of equation (9) with $|x-\zeta(\alpha_0)|<\eta$, $0<|\mu|<\min\{\delta_2,\delta_4\}$. Then $x=\zeta(\alpha)+\beta v(\alpha)+w$ where $|\alpha-\alpha_0|<\min\{\delta_1,\delta_2,\delta_4\}$, $|\beta|<\min\{\delta_2,\delta_4\}$, $|w|<\sigma_2$, $w\in\mathcal{N}(P(\alpha_0))$, and α,β,w,μ is a solution of the two equations (10), (11). By the uniqueness there, $w=w(\alpha,\beta,\mu)$ and (α,β,μ) is a solution of equation (12) with $|\alpha-\alpha_0|<\delta_4$, $|\beta|<\delta_4$, $0<|\mu|<\delta_4$. Then it follows that $\mu>0$ and that $\alpha=\alpha_i(\sqrt{\mu})$, $\beta=\sqrt{\mu}\gamma_i(\sqrt{\mu})$, for $i=1$ or 2, so that $x=x_i(\sqrt{\mu})$ for $i=1$ or 2.

Invertibility. Finally, we must show that $\Lambda(\epsilon)=f_x(x(\epsilon),\epsilon^2)$ ($f_x(x(\epsilon),-\epsilon^2)$ in Case 2), where $x(\epsilon)=x_1(\epsilon)$ or $x_2(\epsilon)$, is invertible for $\epsilon\ne0$ sufficiently small. First we observe that $\Lambda(0)=f_x(\zeta(\alpha_0),0)$

is Fredholm of index zero with $\mathcal{N}(\Lambda(0))$ spanned by $\phi'(\alpha_0)$ and $v(\alpha_0)$. Also

$$\Lambda'(0) = f_{xx}(\zeta(\alpha_0), 0)x'(0), \tag{17}$$

where, in Case 1, $x'(0)$ is given by (16) and, in Case 2, by the same formula with $-\lambda_0$ instead of λ_0. Thus we have the situation that $\Lambda'(0)\zeta'(\alpha_0) \in \mathcal{R}(\Lambda(0))$ but $\Lambda'(0)v(\alpha_0) \notin \mathcal{R}(\Lambda(0))$, and so need the following lemma.

LEMMA . *Let $\mathcal{E}, \mathcal{F}$ be Banach spaces and, for ϵ near 0, let $\Lambda(\epsilon) : \mathcal{E} \to \mathcal{F}$ be Fredholm of index zero with $\mathcal{N}(\Lambda(0)) = \operatorname{span}\{\phi_1, \phi_2\}$. Suppose also that $\Lambda(\epsilon)$ is C^2 in ϵ and that $\Lambda'(0)\phi_1 \in \mathcal{R}(\Lambda(0))$ but $\Lambda'(0)\phi_2 \notin \mathcal{R}(\Lambda(0))$. Choose linear functionals ψ_1, ψ_2 in $\mathcal{F}^*$ such that $\mathcal{R}(\Lambda(0)) = \mathcal{N}(\psi_1) \cap \mathcal{N}(\psi_2)$ and $\psi_1(\Lambda'(0)\phi_2) = 0$, and let $w \in \mathcal{E}$ be such that $\Lambda(0)w = -\Lambda'(0)\phi_1$. Then if*

$$\psi_1(2\Lambda'(0)w + \Lambda''(0)\phi_1) \neq 0, \tag{18}$$

$\Lambda(\epsilon)$ is invertible for $\epsilon \neq 0$ sufficiently small.

PROOF. Let w_1, w_2 satisfy $\Lambda(0)w_1 = \Lambda(0)w_2 = -\Lambda'(0)\phi_1$. Then $w_1 - w_2 \in \mathcal{N}(\Lambda(0))$, and so $w_1 - w_2 = \gamma_1\phi_1 + \gamma_2\phi_2$ for some scalars γ_1, γ_2. Then

$$\psi_1(2\Lambda'(0)(w_1 - w_2)) = 2\sum_{i=1}^{2} \gamma_i \psi_1(\Lambda'(0)\phi_i) = 0.$$

So, the quantity $\psi_1(2\Lambda'(0)w + \Lambda''(0)\phi_1)$ is independent of the w chosen. Also, since ψ_1 is determined up to a scalar multiple, condition (18) is independent of the ψ_1 chosen also.

Now let P and Q be projections such that $\mathcal{R}(P) = \mathcal{N}(\Lambda(0))$, $\mathcal{R}(Q) = \mathcal{R}(\Lambda(0))$. All we have to show is that $\Lambda(\epsilon)$ is one to one for $\epsilon \neq 0$ sufficiently small. If $x \in \mathcal{E}$ we can write $x = \gamma_1\phi_1 + \gamma_2\phi_2 + w$, where γ_1, γ_2 are scalars and $w \in \mathcal{N}(P)$, and we want to solve

$$\Lambda(\epsilon)[\gamma_1\phi_1 + \gamma_2\phi_2 + w] = 0$$

for γ_1, γ_2, w. This is equivalent to the equations

$$Q\Lambda(\epsilon)[\gamma_1\phi_1 + \gamma_2\phi_2 + w] = 0 \tag{19}$$

$$\psi_i(\Lambda(\epsilon)(\gamma_1\phi_1 + \gamma_2\phi_2 + w)) = 0, \quad i = 1, 2. \tag{20}$$

We can write (19) as

$$K(\epsilon)w = -Q\Lambda(\epsilon)(\gamma_1\phi_1 + \gamma_2\phi_2), \tag{21}$$

where $K(\epsilon) = Q\Lambda(\epsilon) : \mathcal{N}(P) \to \mathcal{R}(\Lambda(0))$. $K(0)$ is invertible and, for ϵ sufficiently small, so also is $K(\epsilon)$. Hence, we may solve (21) for

$$w = -K(\epsilon)^{-1}Q\Lambda(\epsilon)(\gamma_1\phi_1 + \gamma_2\phi_2) = \gamma_1 w_1(\epsilon) + \gamma_2 w_2(\epsilon),$$

where, for $i = 1, 2$,

$$w_i(\epsilon) = -K(\epsilon)^{-1}Q\Lambda(\epsilon)\phi_i.$$

Substituting this back into (20) obtains

$$\sum_{j=1}^{2} \psi_i(\Lambda(\epsilon)(\phi_j + w_j(\epsilon))\gamma_j = 0, \quad i = 1, 2. \tag{22}$$

Now $x = \sum_{i=1}^{2} \gamma_i(\phi_i + w_i(\epsilon))$ where $w_i(0) = 0$. Thus, if ϵ is sufficiently small, $\phi_1 + w_1(\epsilon)$ and $\phi_2 + w_2(\epsilon)$ are linearly independent, and so $x \neq 0$ if and only if not both γ_1, γ_2 are zero. Hence, if we can show that the 2×2 matrix

$$[a_{ij}(\epsilon)] = [\psi_i(\Lambda(\epsilon)(\phi_j + w_j(\epsilon))]$$

is invertible for $\epsilon \neq 0$ sufficiently small, it will follow that (22) has only the solution $\gamma_1 = \gamma_2 = 0$ and the lemma will be proved.

Write $d(\epsilon) = \det[a_{ij}(\epsilon)]$. Since $a_{ij}(0) = 0$ for all i, j, $d(0) = d'(0) = 0$. Also

$$d''(0) = 2[a'_{11}(0)a'_{22}(0) - a'_{12}(0)a'_{21}(0)] = 0,$$

since $a'_{i1}(0) = \psi_i(\Lambda'(0)\phi_1) = 0$ for $i = 1, 2$. Then

$$\begin{aligned} d'''(0) &= 3[a''_{11}(0)a'_{22}(0) - a'_{12}(0)a''_{21}(0)] \\ &= 3\psi_1(\Lambda''(0)\phi_1 + 2\Lambda'(0)w'_1(0))\ \psi_2(\Lambda'(0)\phi_2). \end{aligned}$$

Since $\Lambda'(0)\phi_2 \notin \mathcal{R}(\Lambda(0))$ but $\psi_1(\Lambda'(0)\phi_2) = 0$, we must have $\psi_2(\Lambda'(0)\phi_2) \neq 0$. To calculate $w_1'(0)$, observe that

$$Q\Lambda(\epsilon)w_1(\epsilon) = -Q\Lambda(\epsilon)\phi_1.$$

Hence,

$$Q\Lambda(0)w_1'(0) = -Q\Lambda'(0)\phi_1.$$

Since $\Lambda'(0)\phi_1 \in \mathcal{R}(\Lambda(0)) = \mathcal{R}(Q)$, it follows that $\Lambda(0)w_1'(0) = -\Lambda'(0)\phi_1$. Then, by hypothesis, $\psi_1(\Lambda''(0)\phi_1 + 2\Lambda'(0)w_1'(0)) \neq 0$. Thus, $d'''(0) \neq 0$ and $d(\epsilon) \neq 0$ for ϵ sufficiently small. So the proof of the lemma is complete. □

We now apply the lemma to $\Lambda(\epsilon) = f_x(x(\epsilon), \epsilon^2)$ (or $f_x(x(\epsilon), -\epsilon^2)$). We may take $\phi_1 = \zeta'(\alpha_0)$, $\phi_2 = v(\alpha_0)$, $\psi_1 = \psi_1(\alpha_0)$, $\psi_2 = \psi_2(\alpha_0)$. Note that $\psi_1(\alpha)$ was chosen so that $\psi_1(\alpha_0)(\Lambda'(0)\phi_2)$ is a multiple of $\psi_1(\alpha_0)(f_{xx}(\zeta(\alpha_0), 0)v(\alpha_0)v(\alpha_0))$ which is 0. We have to find w such that $\Lambda(0)w = -\Lambda'(0)\phi_1$. Differentiating the identities

$$f_x(\zeta(\alpha), 0)\zeta'(\alpha) = 0, \quad f_x(\zeta(\alpha), 0)v(\alpha) = 0$$

with respect to α and setting $\alpha = \alpha_0$, we obtain

$$\begin{aligned} \Lambda(0)\zeta''(\alpha_0) &= -f_{xx}(\zeta(\alpha_0), 0)\zeta'(\alpha_0)\zeta'(\alpha_0), \\ \Lambda(0)v'(\alpha_0) &= -f_{xx}(\zeta(\alpha_0), 0)\zeta'(\alpha_0)v(\alpha_0) \end{aligned}$$

so that, using (17),

$$\Lambda(0)[\alpha'(0)\zeta''(\alpha_0) + \gamma(0)v'(\alpha_0)] = -\Lambda'(0)\zeta'(\alpha_0),$$

where $\alpha(0) = \alpha_i(0), \gamma(0) = \gamma_i(0)$. So, if we define $q(\alpha) = \alpha'(0)\zeta'(\alpha) + \gamma(0)v(\alpha)$, then $x'(0) = q(\alpha_0)$ and we may take $w = q'(\alpha_0)$.

Then

$$\begin{aligned} &2\Lambda'(0)w + \Lambda''(0)\phi_1 \\ &\quad = 2f_{xx}(\zeta(\alpha_0), 0)q(\alpha_0)q'(\alpha_0) \\ &\qquad + f_{xxx}(\zeta(\alpha_0), 0)q(\alpha_0)q(\alpha_0)\zeta'(\alpha_0) \\ &\qquad + f_{xx}(\zeta(\alpha_0), 0)x''(0)\zeta'(\alpha_0) \pm 2f_{x\mu}(\zeta(\alpha_0), 0)\zeta'(\alpha_0) \\ &\quad = p'(\alpha_0), \end{aligned}$$

where

$$p(\alpha) = f_x(\zeta(\alpha), 0)x''(0) + f_{xx}(\zeta(\alpha), 0)q(\alpha)q(\alpha) \pm 2f_\mu(\zeta(\alpha), 0) \tag{23}$$

and we have + in Case 1 and − in Case 2.

All we have to show is that $\psi_1(\alpha_0)(p'(\alpha_0)) \neq 0$. First note that differentiating $f(x(\epsilon), \pm\epsilon^2) = 0$ twice with respect to ϵ, and setting $\epsilon = 0$, gives

$$f_x(\zeta(\alpha_0), 0)x''(0) + f_{xx}(\zeta(\alpha_0), 0)x'(0)x'(0) \pm 2f_\mu(\zeta(\alpha_0), 0) = 0.$$

Thus $p(\alpha_0) = 0$.

Hence,

$$\psi_1(\alpha_0)(p'(\alpha_0)) = \frac{d}{d\alpha}(\psi_1(\alpha)(p(\alpha))|_{\alpha=\alpha_0}.$$

But, using (23) and observing that $\mathcal{R}(f_x(\zeta(\alpha), 0)) \subset \mathcal{N}(\psi_1(\alpha))$ and $\psi_1(\alpha)(f_{xx}(\zeta(\alpha), 0)q(\alpha)q(\alpha)) = 0$,

$$\psi_1(\alpha)(p(\alpha)) = \pm 2\psi_1(\alpha)(f_\mu(\zeta(\alpha), 0)) = \pm 2\Delta_1(\alpha)$$

so that $\psi_1(\alpha_0)(p'(\alpha_0)) = \pm 2\Delta_1'(\alpha_0) \neq 0$.

So, by the lemma, $f_x(x(\epsilon), \pm\epsilon^2)$ is invertible for $\epsilon \neq 0$ sufficiently small and the proof of Theorem 2 is completed. □

4. The case of a single homoclinic orbit with a two-dimensional nullspace. Let $g : \mathbf{R}^k \to \mathbf{R}^k$ be a C^5 function such that (A1) is satisfied and such that

(A4) System (2) has a solution $\phi(t)$ with $|\phi(t) - x_0| \to 0$ as $|t| \to \infty$;

(A5) The variational equation (3) has a two-dimensional subspace of bounded solutions;

(A6) If the subspace of bounded solutions of (3) is spanned by $\phi'(t)$ and $v(t)$, then the equation

$$\dot{x} = g'(\phi(t))x + g''(\phi(t))v(t)v(t)$$

has no bounded solution.

As in Section 2, it follows from (A4) and (A5) that the adjoint equation

$$\dot{x} = -g'(\phi(t))^* x \tag{24}$$

has a two-dimensional subspace of bounded solutions. Let this subspace be spanned by $\psi_1(t), \psi_2(t)$. It follows from (A6) and Lemma 4.2 in Palmer [**5**] that

$$\int_{-\infty}^{\infty} \psi_i^*(t) g''(\phi(t)) v(t) v(t)\, dt \neq 0 \tag{25}$$

for $i = 1$ or 2. Assume without loss of generality that $i = 2$. Then, if we replace $\psi_1(t)$ by $\psi_1(t) + \beta\psi_2(t)$, where

$$\beta = -\int_{-\infty}^{\infty} \psi_1^*(t) g''(\phi(t)) v(t) v(t)\, dt \Big/ \int_{-\infty}^{\infty} \psi_2^*(t) g''(\phi(t)) v(t) v(t)\, dt,$$

we see that

$$\int_{-\infty}^{\infty} \psi_1^*(t) g''(\phi(t)) v(t) v(t)\, dt = 0. \tag{26}$$

THEOREM 3. *Let* $g : \mathbf{R}^k \to \mathbf{R}^k$ *be a* C^5 *function satisfying* (A1), (A4), (A5), (A6), *and let* $\psi_1(t), \psi_2(t)$ *be linearly independent bounded solutions of* (24) *such that* (26) *is satisfied. Let* $h : \mathbf{R} \times \mathbf{R}^k \times \mathbf{R}$ *be a* C^5 *function, bounded together with its derivatives and* T*-periodic in its first variable. Define* $\Delta_i : \mathbf{R} \to \mathbf{R}$, $i = 1, 2$, *by*

$$\Delta_i(\alpha) = -\int_{-\infty}^{\infty} \psi_i^*(t+\alpha) h(t, \phi(t+\alpha), 0)\, dt.$$

Suppose, for some α_0,

$$\Delta_1(\alpha_0) = 0, \quad \Delta_1'(\alpha_0) \neq 0, \quad \Delta_2(\alpha_0) \neq 0.$$

Then if $\Delta_2(\alpha_0)/\int_{-\infty}^{\infty} \psi_2^*(t) g''(\phi(t)) v(t) v(t) dt > 0$ (*respectively* < 0) *sufficiently small, the period map of equation* (1) *has exactly two transversal homoclinic points whose orbits under the period map lie* *near that of $\phi(\alpha_0)$ and, for $\mu < 0$ (respectively > 0) sufficiently small, it has none.*

PROOF. We apply Theorem 2, taking $\mathcal{E} = C^1(\mathbf{R}, \mathbf{R}^k)$, $\mathcal{F} = C^0(\mathbf{R}, \mathbf{R}^k)$ and defining $f : \mathcal{E} \times \mathbf{R} \to \mathcal{F}$ by

$$[f(x,\mu)](t) = x'(t) - g(x(t)) - \mu h(t, x(t), \mu).$$

It is clear that f is C^5 and that, for y in $\mathcal{E}$,

$$[f_x(x,\mu)y](t) = y'(t) - [g'(x(t)) + \mu h_x(t, x(t), \mu)]y(t).$$

We see also that $f(\zeta(\alpha), 0) = 0$, where $\zeta(\alpha)(t) = \phi(t+\alpha)$,

$$[L(\alpha)y](t) = [f_x(\zeta(\alpha), 0)y](t) = y'(t) - g'(\phi(t+\alpha))y(t).$$

As in the proof of Theorem 1, we can show that $L(\alpha)$ is Fredholm of index zero with $\dim \mathcal{N}(L(\alpha)) = 2$. Also, we see that $\mathcal{N}(L(\alpha))$ is spanned by $\phi'(\cdot + \alpha)$, $v(\cdot + \alpha)$ and $p(\cdot) \in \mathcal{R}(L(\alpha))$ if and only if $\int_{-\infty}^{\infty} \psi_i^*(t+\alpha)p(t)\,dt = 0$ for $i = 1, 2$. So, if we define $\psi_i(\alpha) \in \mathcal{F}^*$ for $i = 1, 2$ by $\psi_i(\alpha)(p) = \int_{-\infty}^{\infty} \psi_i^*(t+\alpha)p(t)\,dt$, ψ_i depends C^5 on α and $\mathcal{R}(L(\alpha)) = \mathcal{N}(\psi_1(\alpha)) \cap \mathcal{N}(\psi_2(\alpha))$.

From the proof of Theorem 2, we see that $f_{xx}(\zeta(\alpha), 0)w_1w_2 \in \mathcal{R}(L(\alpha))$ if $w_1 = \zeta'(\alpha)$ and $w_2 = \zeta'(\alpha)$ or $v(\cdot + \alpha)$. But (A6) tells us that

$$\psi_2(\alpha)(f_{xx}(\zeta(\alpha), 0)v(\cdot+\alpha)v(\cdot+\alpha)) = -\int_{-\infty}^{\infty} \psi_2^*(t)g''(\phi(t))v(t)v(t)\,dt \neq 0,$$

so that $f_{xx}(\zeta(\alpha), 0)v(\cdot + \alpha)v(\cdot + \alpha) \notin \mathcal{R}(L(\alpha))$. Moreover, it follows from (26) that $\psi_1(\alpha)(f_{xx}(\zeta(\alpha), 0)v(\cdot + \alpha)v(\cdot + \alpha)) = 0$.

Now

$$[f_\mu(\zeta(\alpha), 0)](t) = -h(t, \phi(t+\alpha), 0).$$

So, for $i = 1, 2$,

$$\psi_i(\alpha)[f_\mu(\zeta(\alpha), 0)] = -\int_{-\infty}^{\infty} \psi_i^*(t+\alpha)h(t, \phi(t+\alpha), 0)\,dt.$$

Then, by Theorem 2, if

$$2\Delta_2(\alpha_0)\Big/\int_{-\infty}^{\infty}\psi_2^*(t)g''(\phi(t))v(t)v(t)\,dt > 0 \quad (\text{respectively } < 0),$$

for $\mu > 0$ (respectively < 0) sufficiently small, equation (9) has exactly two solutions $x_1(\mu), x_2(\mu)$ near $\zeta(\alpha_0)$ and for $\mu < 0$ (respectively > 0) it has no such solutions. Moreover, $f_x(x_i(\mu),\mu)$ is invertible for $\mu \neq 0$. Then the conclusions of the theorem follow as in the proof of Theorem 1. □

Appendix

Let $\mathcal{E}, \mathcal{F}$ be Banach spaces, and, for α near α_0, let $L(\alpha) : \mathcal{E} \to \mathcal{F}$ be a Fredholm operator with index and dimension of nullspace independent of α. Suppose also that $L(\alpha)$ is C^k.

Let P_0, Q_0 be projections such that

$$\mathcal{R}(P_0) = \mathcal{N}(L(\alpha_0)), \mathcal{R}(Q_0) = \mathcal{R}(L(\alpha_0)).$$

Now $x \in \mathcal{N}(L(\alpha))$ if and only if

$$L(\alpha_0)x = \{L(\alpha_0) - L(\alpha)\}x. \tag{27}$$

We can write $x = u + v$ where $u \in \mathcal{R}(P_0)$, $v \in \mathcal{N}(P_0)$. Then equation (27) is equivalent to the two equations

$$\begin{aligned} Q_0L(\alpha_0)v &= Q_0\{L(\alpha_0) - L(\alpha)\}(u+v) \\ (I - Q_0)\{L(\alpha_0) &- L(\alpha)\}(u+v) = 0. \end{aligned} \tag{28}$$

Now $Q_0L(\alpha_0) : \mathcal{N}(P_0) \to \mathcal{R}(Q_0)$ is invertible, and so $M(\alpha) = Q_0L(\alpha_0) - Q_0\{L(\alpha_0) - L(\alpha)\} : \mathcal{N}(P_0) \to \mathcal{R}(Q_0)$ is also invertible if α is near α_0. Moreover, $M(\alpha)^{-1}$ is C^k in α. Hence, we may solve (28) to get $v = M(\alpha)^{-1}Q_0\{L(\alpha_0) - L(\alpha)\}u = N(\alpha)u$. This shows that $\mathcal{N}(L(\alpha)) \subset \mathcal{N}(L(\alpha_0)) + N(\alpha)(\mathcal{N}(L(\alpha_0)))$. But, if α is near α_0, $|N(\alpha)| < 1$ and the subspace on the right has the same dimension as $\mathcal{N}(L(\alpha_0))$. Since $\dim \mathcal{N}(L(\alpha) = \dim \mathcal{N}(L(\alpha_0))$, we must have equality.

Then $P(\alpha) = (I + N(\alpha))P_0$ *is a projection such that* $\mathcal{R}(P(\alpha)) = \mathcal{N}(L(\alpha))$ *and* $P(\alpha)$ *is* C^k *in* α. Note this also enables us to choose a C^k basis for $\mathcal{N}(L(\alpha))$.

It follows, from Kato [**4**, p. 234], that $L(\alpha)^* : \mathcal{F}^* \to \mathcal{E}^*$ is Fredholm with $\mathcal{N}(L(\alpha)^*)$ equal to the annihilator of $\mathcal{R}(L(\alpha))$. Also $L(\alpha)^*$ is C^k in α. By what we have just proved, we can choose a C^k basis $\psi_i(\alpha)$ for $\mathcal{N}(L(\alpha)^*)$ so that $\mathcal{R}(L(\alpha)) = \cap_i \mathcal{N}(\psi_i(\alpha))$.

Now $\mathcal{F} = \mathcal{R}(L(\alpha)) \oplus \mathcal{N}(Q_0)$ if α is near α_0. Let $Q(\alpha)$ be the projection with $\mathcal{R}(Q(\alpha)) = \mathcal{R}(L(\alpha))$ and $\mathcal{N}(Q(\alpha)) = \mathcal{N}(Q_0)$. We show $Q(\alpha)$ is C^k. Let $v_1, v_2, \dots$ be a basis for $\mathcal{N}(Q_0)$. Then we may write

$$Q(\alpha)y = y - \sum_j \gamma_j(\alpha)(y)v_j$$

where $\gamma_j(\alpha) \in \mathcal{F}^*$. All we have to show is that each $\gamma_j(\alpha)$ is C^k. Now, for all i and y,

$$0 = \psi_i(\alpha)(y) - \sum_j \psi_i(\alpha)(v_j)\gamma_j(\alpha)(y).$$

So, for all i and α,

$$\sum_j \psi_i(\alpha)(v_j)\gamma_j(\alpha) = \psi_i(\alpha).$$

Now the matrix $[\psi_i(\alpha)(v_j)]$ is invertible since $\mathcal{R}(L(\alpha)) \cap \mathcal{N}(Q_0) = \{0\}$ and everything is C^k in α. So $\gamma_j(\alpha)$ is also C^k in α.

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