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ON TRANSFORMATIONS AND ZEROS OF POLYNOMIALS

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ABSTRACT. We survey certain transformations of the set $\pi_n[x]$ of n-th degree polynomials into themselves. These transformations share the property that polynomials with all their zeros in a certain real interval are mapped to polynomials with all their zeros in another real interval. Rich sources of such "zero-mapping" transformations can be found in the Laguerre-Pólya-Schur theory of multiplier sequences. Others follow from the theory of biorthogonal polynomials, by identifying them with a mapping from the parameter space. This identification leads to two general techniques for the generation of such transformations. As a consequence, we prove a result on the zeros of certain convolution orthogonal polynomials introduced by Al-Salam and Ismail.

1. Introduction. Let $\mathcal{T}$ be a transformation of the set of the n-th degree polynomials, forthwith denoted by $\pi_n[x]$, into itself. In general, even if it is known that all the zeros of u reside in a certain real interval, little can be said about the zeros' location of $\mathcal{T}u$. However, there exist many transformations that exhibit regularity in their "mapping" of zeros. Possibly the simplest nontrivial example is $\mathcal{T}u = u'$, which retains the property that all the zeros of u reside in a real interval (for a complex-plane version of this statement, the Gauss-Lucas theorem, cf. [**16**]). The themes of this paper are three general techniques to produce transformations that display interesting "zero-mapping" properties. The first is classical and its major elements can be traced back to the work of Laguerre, a century ago. Nonetheless, it deserves being better

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known and appreciated. The remaining two techniques are based on the theory of biorthogonal polynomials.

We commence with the classical theory of *multiplier sequences* [**18**]: the real sequence $\{a_n\}_0^\infty$ is a multiplier sequence of the first kind if, given any polynomial $\sum_{k=0}^n u_k x^k$ (of arbitrary degree n) with all its zeros real, the transformation

$$\mathcal{T}\left(\sum_{k=0}^{n} u_k x^k\right) = \sum_{k=0}^{n} a_k u_k x^k \tag{1.1}$$

maintains the reality of zeros. All such sequences can be characterized by means of certain analytic properties of the *generating function* $\Phi(z) := \sum_{n=0}^\infty (1/n!) a_n z^n$. This characterization will be surveyed in Section 2. Similar results for multiplier sequences of the second kind, which are defined similarly to (1.1) but, instead, map positive zeros to real zeros, can be found in [**18**].

It is, in general, a nontrivial task to generate interesting multiplier sequences. One technique, which is mentioned in Section 2, rests upon the connection between such sequences and the *Pólya frequency functions* [**12**]. A less general method originates in the theory of biorthogonal polynomials and it is explained in Section 3.

Another source of interesting transformations is the theory of *biorthogonal polynomials.* The authors were engaged recently in extending the perimeters of this theory, and the surveyed results follow from this work.

Let $\varphi(x, \mu)$ be a real distribution (as a function of $x \in (a, b)$) for all μ in the parameter interval (c, d). We say that $p_n(x; \mu_1, \ldots, \mu_n)$ is an n-th biorthogonal polynomial if $\mu_1, \ldots, \mu_n \in (c, d)$ are distinct, $p_n \in \pi_n[x]$ and

$$\int_a^b p_n(t; \mu_1, \ldots, \mu_n)\, d\varphi(t, \mu_k) = 0, \quad k = 1, \ldots, n. \tag{1.2}$$

It is obvious that biorthogonality generalizes "conventional" orthogonality, in the sense of $r_n(x) \equiv p_n(x; 0, 1, \ldots, n-1)$ being orthogonal with respect to $\tau(x)$, where $d\varphi(x, \mu) := x^\mu\, d\tau(x)$. Moreover, it can be easily checked that $r_n(x)$ coincides with $\lim_{\mu_1, \ldots, \mu_n \to 0} p_n(x; \mu_1, \ldots, \mu_n)$ where, now, $d\varphi(x, \mu) := \mu^x\, d\tau(x)$.

In Section 3 we survey elements of the theory of biorthogonal polynomials, mainly in connection to loci of their zeros and to the derivation of their explicit form. This theory is used in Section 4 to define a family of transformations by the relation

$$\mathcal{T}\left(\prod_{k=1}^{n}(x-\mu_k)\right) = p_n(x;\mu_1,\ldots,\mu_n). \tag{1.3}$$

Subject to further conditions on φ, the transformation $\mathcal{T}$ maps zeros in a predictable manner, from (c,d) to (a,b). We derive one such transformation in detail and list further transformations that "correspond," in a sense, to familiar distributions.

Section 5 is devoted to a second application of biorthogonality to the subject matter of this paper. By identifying the distribution $d\varphi(x,\mu)$ with $G(x,\mu)\,d\tau(x)$, where τ is a distribution and G a *generating function* of its orthogonal polynomials $\{r_n\}$, we obtain a transformation of the form

$$\mathcal{T}\left(\sum_{k=0}^{n} u_k x^k\right) = \sum_{k=0}^{n} \beta_k u_k r_k(x), \tag{1.4}$$

where the constants $\beta_0, \beta_1, \ldots$ depend on $\{r_n\}$. A condition identical to that of Section 3 ensures that (1.4) maps zeros in a predictable manner.

Finally, in Section 6, we consider transformations of the form

$$\mathcal{T}\left(\sum_{k=0}^{n} u_k x^k\right) = \sum_{k=0}^{n} b_k u_k (-x)_k,$$

where $(\alpha)_k$ is the *Pochhammer symbol*, defined by $(\alpha)_0 := 1$, $(\alpha)_k := (\alpha+k-1)(\alpha)_{k-1}$, $k = 1, 2, \ldots$. These transformations arise in the study of *discrete convolution-orthogonal polynomials* that have been introduced and analyzed by Al-Salam and Ismail [**1**]. For a suitable measure $d\tau$ and function $h(x)$, convolution-orthogonal polynomials $\{q_n\}$ formally satisfy the relation

$$\sum_{k,l=0}^{\infty} \eta_{k,l} q_m(k) q_n(l) = \lambda_m \delta_{m,n}, \quad m,n = 0,1,\ldots, \tag{1.5}$$

where $\lambda_m > 0$ and

$$\eta_{k,l} := \int_{-\infty}^{\infty} \frac{h^{(k)}(t)h^{(l)}(t)}{k!l!}(-t)^{k+l}\, d\tau(t).$$

For the case when $h(x)$ is the Laplace transform of a nonnegative function on $(0,\infty)$, Al-Salam and Ismail have stated that $q_n(x)$ has all its zeros in $(0,\infty)$. Unfortunately, this is not the case, as we demonstrate by providing a counter-example. We partially remedy this situation by giving a simple proof of a somewhat weaker statement, utilizing the composition of two transformations of biorthogonal type and the form (1.3).

An alternative formulation of the transformations that are considered in this paper follows by virtue of linearity of each $\mathcal{T}u$ in the coefficients of u, as long as we represent both in "correct" bases. Since this representation generally depends on n, we find for every $n = 0, 1, \ldots$ two bases of $\pi_n[x]$, the *range basis* $\xi_{n,0}, \xi_{n,1}, \ldots, \xi_{n,n}$ and the *image basis* $v_{n,0}, v_{n,1}, \ldots, v_{n,n}$, such that the action of $\mathcal{T}$ on $u \in \pi_n[x]$ is fully described by

$$\mathcal{T} : \xi_{n,k}(x) \mapsto v_{n,k}(x), \quad k = 0, 1, \ldots, n. \tag{1.6}$$

For example, the multiplier transformation (1.1) is, simply, $\mathcal{T} : x^k \mapsto a_k x^k$, and (1.4) corresponds to $\mathcal{T} : x^k \mapsto \beta_k r_k(x)$. Note that all these four bases form *Newton systems*—each $\xi_{n,k}$ and $v_{n,k}$ is independent of n. This does not reflect the full generality of the transformations in this paper.

Applications of "zero-mapping" transformations are outside the scope of this survey. Multiplier sequences were already used in a large number of fields, e.g., in theory of delay equations [**6**], analysis of numerical algorithms [**17**] and approximation theory [**10**]. Moreover, the definition (and the range) of underlying "zero-mapping" transformations can be extended to entire functions, leading to the *Pólya-Laguerre class* [**7**]. We believe that the more novel forms of transformations are likely to be of equally wide-ranging applicability.

A word about notational conventions in this paper: We deal here with three different concepts of orthogonality, a whole host of distributions and generating functions of diverse definition and form and several distinct transformations. To keep the exposition comprehensible, we consistently adhere to Table 1.

TABLE 1. Notational conventions.

$\pi_n[x]$:	The set of all n-th degree polynomials with real coefficients.
$u \equiv \sum_0^n u_k x^k$	An arbitrary polynomial in $\pi_n[x]$.
$\mathcal{T}$:	A mapping of $\pi_n[x]$ into itself.
$\{\xi_{n,k}\}$:	A range sequence.
$\{v_{n,k}\}$:	An image sequence.
$\{a_k\}$:	A multiplier sequence.
$\{b_k\}$:	An Al-Salam and Ismail sequence.
Φ:	The generating function of $\{a_k\}$.
τ:	A distribution in x.
$\{r_n\}$:	Polynomials orthogonal with respect to the distribution τ.
G:	A generating function of $\{r_n\}$.
$\{q_n\}$:	Discrete convolution-orthogonal polynomials of Al-Salam and Ismail.
φ:	A distribution in $x \in (a, b)$ for all $\mu \in (c, d)$.
$\{I_n\}$:	Generalized moments of φ with respect to the monic sequence $\{\rho_n\}$.
$\{p_n\}$:	Polynomials biorthogonal with respect to the distribution φ and parameters $\mu_1, \dots, \mu_n \in (c, d)$.
$\{s_n\}$:	The parameter polynomials $\prod_{k=1}^n (x - \mu_k)$.

2. Multiplier sequences. Multiplier sequences of the first kind have already been defined in the introduction. To recap, we define the transformation $\mathcal{T}$ by (1.1),

$$\mathcal{T}\left(\sum_{k=0}^{n} u_k x^k\right) = \sum_{k=0}^{n} a_k u_k x^k,$$

or, in the alternative formulation of range and image sequences, $\xi_{n,k}(x) = x^k$, $v_{n,k}(x) = a_k x^k$, and say that $\{a_n\}$ is a *multiplier sequence of the first kind* if $\mathcal{T}$ takes polynomials with real zeros to polynomials with real zeros. Such sequences were already characterized by Laguerre [**15**] and, in a more streamlined and modern form, by Pólya and Schur [**18**]:

Theorem 1 (Pólya and Schur). *Let* $\Phi(z) := \sum_{n=0}^{\infty}(1/n!)a_n z^n$. *Then* $\{a_n\}$ *is a multiplier sequence of the first kind if and only if* Φ *is a real entire function and either* $\Phi(z)$ *or* $\Phi(-z)$ *possesses the factorization*

$$c_0 e^{c_1 z} z^M \prod_{k=1}^{N} \left(1 + \frac{z}{\zeta_k}\right),$$

where $c_0 \in \mathbf{R}\backslash\{0\}$, $c_1 \geq 0$, $\zeta_1, \zeta_2, \ldots > 0$, $M \in \{0, 1, 2, \ldots\}$, $0 \leq N \leq \infty$, *and* $\sum_{k=1}^{N} \zeta_k^{-1} < \infty$.

An equivalent characterization, again due to Pólya and Schur [**18**], is in terms of the *Jensen polynomials* $g_n(x) := \sum_{k=0}^{n} \binom{n}{k} a_k x^k$, $n = 0, 1, \ldots$:

Theorem 2 (Pólya and Schur). $\{a_n\}$ *is a multiplier sequence of the first kind if and only if all Jensen polynomials have only real zeros, all of the same sign.*

A classical (and useful!) example of a multiplier sequence is $\{1/(1+\omega)_n\}$, where $\omega > -1$ and $(\alpha)_n$ is the *Pochhammer symbol* [**20**]. We have

$$\begin{aligned}\Phi(z) &= \sum_{n=0}^{\infty} \frac{1}{n!(1+\omega)_n} z^n \\ &= {}_0F_1 \left[\begin{matrix} —; & \\ & z \\ 1+\omega; & \end{matrix}\right] = \frac{\Gamma(1+\omega)}{z^{\frac{1}{2}\omega}} I_\omega(2\sqrt{z}),\end{aligned}$$

where ${}_pF_q$ stands for a generalized hypergeometric function [**20**] and I_ω is the modified Bessel function, $I_\omega(z) := i^{-\omega} J_\omega(iz)$. It is elementary that the Bessel function J_ω, normalized by $z^{-\omega}$, is entire, of order 1 and with real zeros [**23**] and it readily follows that Φ obeys the conditions of Theorem 1 (with positive ζ_k's) and $\{1/(1+\omega)_n\}$ is, indeed, a multiplier sequence. An alternative proof is by means of Theorem 2: The Jensen polynomial is

$$\begin{aligned} g_n(z) &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{(1+\omega)_k} x^k \\ &= {}_1F_1 \left[\begin{matrix} -n; \\ \\ 1+\omega; \end{matrix} \; -x \right] = \frac{n!}{(1+\omega)_n} L_n^{(\omega)}(-x), \end{aligned}$$

where $L_n^{(\alpha)}$ is a Laguerre polynomial [**20**]. Thus, all the zeros of g_n are negative.

Another example of a multiplier sequence, likewise derivable from elementary principles, is $\{q^{n^2}\}$, where $-1 < q < 1$ [**15**]. In Section 4 we produce alternative proofs for both sequences, based on biorthogonal polynomials.

Functions Φ that obey the conditions of Theorem 1 are called the *type* I *functions in the Pólya-Laguerre class* [**12**]. They are precisely all the real entire functions that can be obtained as uniform limits (on compact subsets of the complex plane) of polynomials with only real zeros, all of which have identical sign [**4**]. Such functions have obvious significance in approximation theory.

Theorems 1 and 2 notwithstanding, it is quite complicated to produce nontrivial examples of multiplier sequences—the two above examples reoccur time and again in literature. Fortunately, a fruitful mechanism for generation of multiplier sequences is provided in terms of *Pólya frequency functions*. We say that a function, f, defined on the real line, is a Pólya frequency function if the kernel $f(x-y)$ is *strictly totally positive*. It is a *Pólya frequency density* if, in addition, it is integrable in $(-\infty,\infty)$. Deferring a discussion of strict total positivity, in the more general framework of *strict sign consistency*, to Section 3, we just quote an important result of Schoenberg [**21**].

Theorem 3 (Schoenberg). *Φ is a type* I *function in the Pólya-Laguerre class if and only if it is of the form $\Phi(z) = C/\Psi(z)$, where $C \in \mathbf{R}\backslash\{0\}$ and Ψ is a Laplace transform of a Pólya frequency density.*

We refer the reader to [**12**] for an extensive exposition of Pólya frequency functions and to [**2**] for some of their applications to rational approximation theory.

Another means for derivation of multiplier sequences is the theory of biorthogonal polynomials. Unlike the three preceding theorems, it falls short of characterization. Nonetheless, it provides a useful and easy technique for the problem at hand. We discuss this approach in the next section.

We mention in passing the generalization of multiplier sequences from the real line to portions of the complex plane in [**13, 14, 4**].

3. Biorthogonal polynomials. Although biorthogonal polynomials were introduced and applied to various problems, more notably interpolation [**5**], their treatment in [**8**] and subsequent papers differs in a crucial detail; instead of assuming a discrete set of distributions that generate underlying biorthogonal polynomials, we stipulate a distribution that depends on a continuous parameter. The discrete set of distributions that is required to define a specific biorthogonal polynomial is produced by sampling the parameter interval the requisite number of times. This dependence on continuous parameters yields the mechanism for generation of transformations.

Let $\varphi(x,\mu)$ be a distribution in $x \in (a,b)$ for every $\mu \in (c,d)$. Given distinct $\mu_1,\dots,\mu_n \in (c,d)$, we say that $p_n(x) \equiv p_n(x;\mu_1,\dots,\mu_n)$ is the n-th *biorthogonal polynomial* (BOP) if it obeys (1.2), i.e.,

$$\int_a^b p_n(t)\,d\varphi(t,\mu_k) = 0, \quad k = 1,\dots,n.$$

The theory of BOP's is surveyed in [**8**]. It is proved there that the BOP system $\{p_n\}_{n=0}^{\infty}$ exists and is unique, subject to

$$\det \begin{bmatrix} \int_a^b d\varphi(t,\mu_1) & \int_a^b t\,d\varphi(t,\mu_1) & \cdots & \int_a^b t^{n-1}\,d\varphi(t,\mu_1) \\ \int_a^b d\varphi(t,\mu_2) & \int_a^b t\,d\varphi(t,\mu_2) & \cdots & \int_a^b t^{n-1}\,d\varphi(t,\mu_2) \\ \vdots & \vdots & & \vdots \\ \int_a^b d\varphi(t,\mu_n) & \int_a^b t\,d\varphi(t,\mu_n) & \cdots & \int_a^b t^{n-1}\,d\varphi(t,\mu_n) \end{bmatrix} \neq 0$$

for all $n = 1,2,\dots$ and distinct $\mu_1,\dots,\mu_n \in (c,d)$. Distributions that obey the inequality are said to be *regular*.

Given a sequence $\rho_0, \rho_1, \dots$, where each ρ_n is a monic polynomial of degree n, we set

$$I_n(\mu) := \int_a^b \rho_n(t)\, d\varphi(t,\mu), \quad n = 0, 1, \dots .$$

Then, up to a multiplicative normalization constant,

$$p_n(x; \mu_1, \dots, \mu_n) = \det \begin{bmatrix} I_0(\mu_1) & I_1(\mu_1) & \cdots & I_n(\mu_1) \\ I_0(\mu_2) & I_1(\mu_2) & \cdots & I_n(\mu_2) \\ \vdots & \vdots & & \vdots \\ I_0(\mu_n) & I_1(\mu_n) & \cdots & I_n(\mu_n) \\ \rho_0(x) & \rho_1(x) & \cdots & \rho_n(x) \end{bmatrix}. \tag{3.1}$$

Unlike their orthogonal counterparts, zeros of biorthogonal polynomials are not necessarily confined to the interval (a, b). However, let us suppose that $d\varphi(x, \mu) = \omega(x, \mu)\, d\psi(x)$, where the distribution ψ is independent of μ, whereas ω is a C^1 function of $\mu \in (c, d)$, which is *strictly sign consistent* (SSC): Given any $n = 1, 2, \dots$ and monotone sequences $a < x_1 < x_2 < \cdots < x_n < b$, $c < \mu_1 < \mu_2 < \cdots < \mu_n < d$, the determinant

$$\det \begin{bmatrix} \omega(x_1,\mu_1) & \omega(x_2,\mu_1) & \cdots & \omega(x_n,\mu_1) \\ \omega(x_1,\mu_2) & \omega(x_2,\mu_2) & \cdots & \omega(x_n,\mu_2) \\ \vdots & \vdots & & \vdots \\ \omega(x_1,\mu_n) & \omega(x_2,\mu_n) & \cdots & \omega(x_n,\mu_n) \end{bmatrix}$$

is nonzero and of a sign which depends only on n, but not on the choice of the monotone sequences [**12**].

Lemma 4 (Iserles and Nørsett). *If either φ or ω is C^1 in μ and* SSC, *then all the zeros of the* BOP p_n *reside in (a, b) and are distinct.*

The proof of the lemma follows readily from strict sign consistency, since it implies that $\{\varphi(x,\mu_1), \dots, \varphi(x,\mu_n)\}$ or $\{\omega(x,\mu_1), \dots, \omega(x,\mu_n)\}$ is a Chebyshev set.

Note that strict sign consistency generalizes the more familiar concept of strict total positivity, whereby the determinant is always positive [**12**].

The explicit form of biorthogonal polynomials is frequently easy to derive, either by (3.1) or by other techniques. It is often linked to an expansion of the *parameter polynomials* $s_n(x) := \prod_{k=1}^{n}(x-\mu_k)$ and, as in the preceding section, the choice of basis is crucial.

An easy example is

$$\varphi(x,\mu) = \psi\left(\frac{x}{\mu}\right), \quad (a,b) = (c,d) = (0,\infty). \tag{3.2}$$

We choose $\rho_n(x) = x^n$, therefore

$$I_n(\mu) = \int_0^\infty t^n \, d\psi\left(\frac{t}{\mu}\right) = \mu^n \int_0^\infty t^n \, d\psi(t) = c_n\mu^n,$$

where $c_n > 0$ is the n-th moment of ψ. Let $s_n(x) = \sum_{k=0}^{n} s_{n,k}x^k$. Then

$$p_n(x;\mu_1,\dots,\mu_n) = \sum_{k=0}^{n}\frac{s_{n,k}}{c_k}x^k \tag{3.3}$$

[**8**], since

$$\int_0^\infty \sum_{k=0}^{n}\frac{s_{n,k}}{c_k}t^k \, d\psi\left(\frac{t}{\mu_l}\right) = \sum_{k=0}^{n} s_{n,k}\mu_l^k = s_n(\mu_l) = 0, \quad l = 1,\dots,n.$$

Note that the transformation $s_n \mapsto p_n$ is nothing else but (1.1) with $a_k = 1/c_k$, $k = 0,1,\dots,n$.

Considerably more fruitful examples of biorthogonal polynomials being explicitly known are obtained by taking any combination of a φ, together with $\{\rho_n\}$, such that

$$\frac{I_{n+1}(\mu)}{I_n(\mu)} = \frac{g_n(\mu)}{h_n(\mu)}, \quad n = 0,1,\dots, \tag{3.4}$$

where both g_n and h_n are linear functions, such that $h_n, g_nh_k' - g_n'h_k \neq 0$ on (c,d) for all $n = 0,1,\dots, k = 0,1,\dots,n$. Normalizing $I_0(\mu) \equiv 1$, we obtain

$$I_n(\mu) = \prod_{k=0}^{n-1}\frac{g_k(\mu)}{h_k(\mu)}.$$

A special case is provided by the distributions that lead to *Hahn-type orthogonal polynomials* [**3**].

A long and technical proof of regularity is given in [**9**]. Taking regularity for granted, the explicit form is easy to verify: Present the parameter polynomial in the basis

$$s_n(x) = \sum_{k=0}^{n} \tilde{s}_{n,k} \prod_{j=0}^{k-1} g_j(x) \prod_{j=k}^{n-1} h_j(x).$$

Then

$$\int_{-\infty}^{\infty} \sum_{k=0}^{n} \tilde{s}_{n,k} \rho_k(t)\, d\varphi(t, \mu_l) = \sum_{k=0}^{n} \tilde{s}_{n,k} I_k(\mu_l) = \frac{1}{\prod_{k=0}^{n-1} h_k(\mu_l)} s_n(\mu_l) = 0$$

for all $l = 1, \dots , n$. Thus, by regularity,

$$p_n(x; \mu_1, \dots , \mu_n) = \sum_{k=0}^{n} \tilde{s}_{n,k} \rho_k(x). \tag{3.5}$$

The crucial observation that allows biorthogonality to be used to construct "zero-mapping" transformations is encapsulated in the following theorem.

Theorem 5 (Iserles and Nørsett). *Let the conditions of Lemma 4 be satisfied. Then the transformation*

$$\mathcal{T} s_n(x) = p_n(x) \tag{3.6}$$

maps polynomials with all their zeros in (c, d) *into polynomials with all zeros in* $[a, b]$.

Proof. Given any polynomial with *distinct* zeros in (c, d), we can always identify it with a parameter polynomial s_n and the transformation (3.6) is well defined. Moreover, by Lemma 4, all the zeros of $\mathcal{T} s_n$ are in (a, b) and distinct. The theorem follows by allowing confluent zeros by a limiting argument. □

In principle, (3.6) does not define a mapping from $\pi_n[x]$ into itself. Fortunately, in practice it is trivial to extend its range to all of $\pi_n[x]$.

To illustrate the last theorem, we consider (3.2). First we choose $d\psi(x) = (1/\Gamma(\omega+1))x^\omega e^{-x}\,dx$, where $\omega > -1$. Thus,

$$c_n = \frac{1}{\Gamma(\omega+1)} \int_0^\infty t^{\omega+n} e^{-t}\,dt = \frac{\Gamma(\omega+n+1)}{\Gamma(\omega+1)} = (\omega+1)_n$$

and (3.3) gives $p_n(x) = \sum_{k=0}^{n} (s_{n,k}/(1+\omega)_k)x^k$. A classical result [**12**] is

Lemma 6. *The function x^y is* SSC *for all $0 < x < \infty$, $y \in \mathbf{R}$.*

Thus, all the conditions of Lemma 4 can be easily verified and Theorem 5 yields an alternative proof that $\{1/(\omega+1)_n\}$ is a multiplier sequence.

Our next example is the Stieltjes-Wigert distribution

$$d\psi(x) = \frac{\sigma}{\sqrt{\pi}} e^{-\sigma^2(\log x)^2} dx,$$

where $\sigma > 0$. The moments are $c_n = q^{-(n+1)^2}$, $n = 0, 1, \ldots,$ with $q := \exp(-1/(4\sigma^2)) \in (0,1)$ [**3**]. Since $\omega(x,\mu) = Cf(x)f(\mu)x^{2\sigma^2 \log \mu}$, where $C > 0$, $f(y) = e^{-\sigma^2(\log y)^2} > 0$, Lemma 6 ensures strict sign consistency. A trivial change of variable proves Laguerre's result that $\{q^{n^2}\}$ is a multiplier sequence (cf. Section 2).

4. Biorthogonal transformations. In this section we use Theorem 5, in tandem with distributions (3.4), to present several transformations with predictable behavior of zeros. The presentation follows a fixed pattern: we specify a distribution and a sequence $\{\rho_n\}$, verify (3.4) and use Theorem 5 on the explicit form (3.5). This yields a transformation which, frequently, needs to be further recast for greater clarity. This work is based on [**9**], except for the recast transformations that lead to image sequences of orthogonal polynomials which appear in [**11**]. We make no effort to address ourselves to the full generality of results in these references.

Example I. We set $d\varphi(x,\mu) := (1/\Gamma(\mu))x^{\mu-1}e^{-x}dx$, where $(a,b) = (c,d) = (0,\infty)$. Strict sign consistency follows at once from Lemma 6. Moreover, selecting $\rho_n(x) = x^n$, we have $I_n(\mu) = (\mu)_n$, $n = 0,1,\ldots$, hence (3.4) holds with $g_n(\mu) = n+\mu$ and $h_n(\mu) \equiv 1$. Application of Theorem 5 to the explicit form (3.5) yields at once (in the terminology of range and image sequences which we use whenever possible)

Theorem 7 (Iserles and Nørsett). *The transformation*

$$\mathcal{T} : (x)_n \mapsto x^n \tag{4.1}$$

maps polynomials with positive zeros into polynomials with positive zeros.

Note that in the proof of the theorem it is necessary to rule out zeros migrating to the origin—a trivial task.

To recast (4.1) we invoke a technical lemma from [**11**]:

Lemma 8 (Iserles and Saff). *Given constants* $v_0,\ldots,v_n \in \mathbf{R}$ *and* $\beta_0,\ldots,\beta_n > -1$, *the identity*

$$\sum_{k=0}^{n}(x)_k v_k = \sum_{k=0}^{n}\frac{(\beta_k+1-x)_k}{k!}u_k, \quad x \in \mathbf{R}, \tag{4.2}$$

implies that

$$v_k = (-1)^k \sum_{l=k}^{n}\binom{l}{k}\frac{(\beta_l+k+1)_{l-k}}{l!}u_l, \quad k = 0,1,\ldots,n.$$

Theorem 7A (Iserles and Saff). *The transformation*

$$\mathcal{T} : \frac{(\beta_n+1-x)_n}{n!} \mapsto L_n^{(\beta_n)}(x),$$

where $L_n^{(\alpha)}$ *is a Laguerre polynomial* [**20**], *maps positive zeros to positive zeros.*

Proof. We change range base from $\{(x)_n\}$ to $\{(\beta_n+1-x)_n/n!\}$, using (4.2). Thus, by Lemma 8,

$$\begin{aligned}\mathcal{T}\bigg(\sum_{k=0}^{n}\frac{(\beta_k+1-x)_k}{k!}u_k\bigg) &= \mathcal{T}\bigg(\sum_{k=0}^{n}(x)_k v_k\bigg)\\ &= \sum_{k=0}^{n} v_k x^k = \sum_{k=0}^{n}\frac{(\beta_k+1)_k u_k}{k!}\,{}_1F_1\left[\begin{matrix}-k; \\ \beta_k+1;\end{matrix}\; x\right]\\ &= \sum_{k=0}^{n} u_k L_k^{(\beta_k)}(x),\end{aligned}$$

exploiting the hypergeometric form of Laguerre polynomials [**20**]. This proves the theorem. □

Example II. Let $d\varphi(x,\mu) = (1-\mu/\lambda)\mu^x d\psi(x)$, where $\lambda > 0$, $(a,b) = (0,\infty)$, $(c,d) = (0,\lambda)$ and ψ is a step function with jumps of $(\lambda)_m/(m!\lambda^m)$ at $m = 0,1,\ldots$. Clearly, ω is SSC by Lemma 6.

We choose $\rho_n(x) := (-1)^n(-x)_n$. Therefore

$$I_n(\mu) = \left(\frac{\lambda}{\lambda-\mu}\right)^{\lambda-1}\left(\frac{\mu}{\lambda-\mu}\right)^n(\lambda)_n, \quad n = 0,1,\ldots$$

and (3.4) holds with $g_n(\mu) = (1+n/\lambda)\mu$, $h_n(\mu) = 1-\mu/\lambda$. Invoking Theorem 5, we have

Theorem 9 (Iserles and Nørsett). *The transformation*

$$\mathcal{T}: x^k(\lambda-x)^{n-k} \mapsto \frac{(-1)^k}{(\lambda)_k}(-x)_k, \quad k = 0,1,\ldots,n, \tag{4.3}$$

maps polynomials with zeros in $(0,\lambda)$ *into polynomials with positive zeros.*

Again, the proof easily rules out zeros migrating to the origin —hence the zeros of the image polynomial are in $(0,\infty)$ rather than $[0,\infty)$.

The limiting case $\lambda = \infty$ of Example II will be of importance in our later discussion of discrete convolution-orthogonal polynomials. Hence, we state it as

Theorem 9A. *The transformation*

$$\mathcal{T} : x^k \mapsto (-1)^k(-x)_k, \quad k = 0, 1, \dots, n, \tag{4.4}$$

maps polynomials with all zeros in $(0, \infty)$ *into polynomials with all zeros in* $(0, \infty)$.

It is possible to recast (4.3) into two different forms:

Theorem 9B (Iserles and Nørsett). *The transformation*

$$\mathcal{T} : x^k \mapsto (-1)^k(-x)_k(\lambda + x)_{n-k}\lambda^k$$

maps polynomials with zeros in $(0, \lambda)$ *into polynomials with positive zeros.*

Theorem 9C (Iserles and Saff). *Let* $c \in (0, 1)$ *be given. The transformation*

$$\mathcal{T} : x^k \mapsto (-1)^k(x + \lambda)_{n-k}(c\lambda)^k m_k(x; \lambda + n - k, c), \quad k = 0, 1, \dots, n,$$

maps polynomials with zeros in $(c\lambda, (c + 1)\lambda)$ *into polynomials with positive zeros, where* m_n *denotes here the Meixner polynomial of the first kind* [**3**].

Example III. Many transformations in [**9**] involve *basic hypergeometric series* [**22**]. A typical example is

$$d\varphi(x, \mu) = \frac{1}{(-q^{-1}\mu; q^{-1})_\infty} x^{\frac{\log \mu}{\log q}} d\psi(x), \quad (a, b) = (1, \infty), (c, d) = (0, \infty),$$

where $q > 1$ and ψ is a step function with jumps of $(-1)^m/[q]_m$ at q^m, $m = 0, 1, \dots$. Here $(\alpha; q)_n$ is the *q-rising factorial*

$$\begin{aligned} (\alpha; q)_0 &:= 1, \\ (\alpha; q)_n &:= (1 - q^{n-1}\alpha)(\alpha; q)_{n-1}, \quad n = 1, 2, \dots, \\ (\alpha; q)_\infty &:= \lim_{n\to\infty} (\alpha; q)_n, \end{aligned}$$

whereas $[q]_n := (q;q)_n$. Strict sign consistency is valid by Lemma 6. Moreover, setting $\rho_n(x) := x^n$,

$$I_n(\mu) = \frac{1}{(-q^{-1}\mu;q^{-1})_\infty}\,{}_0\phi_0\left[\begin{matrix}-; & \\ & q, -q^n\mu \\ -; & \end{matrix}\right],$$

where

$$f(z) := {}_0\phi_0\left[\begin{matrix}-; & \\ & q, z \\ -; & \end{matrix}\right]$$

is a basic hypergeometric function. The function f is entire and

$$f(q^{-1}z) - f(z) = \sum_{k=1}^{\infty} \frac{q^{-k}z^k}{[q]_{k-1}} = q^{-1}zf(q^{-1}z).$$

Thus, by induction,

$$f(z) = (q^{-1}z;q^{-1})_m f(q^{-m}z), \quad m = 0,1,\ldots .$$

Since $q > 1$, letting m tend to infinity yields

$$f(z) = (q^{-1}z;q^{-1})_\infty;$$

thus,

$$I_n(\mu) = (-\mu;q)_n, \quad n = 0,1,\ldots .$$

We are within the framework of (3.4) with $g_n(\mu) = 1+q^n\mu$, $h_n(\mu) \equiv 1$. As migration of zeros to the endpoints can be, again, ruled out, we have

Theorem 10 (Iserles and Nørsett). *The transformation*

$$\mathcal{T} : (-x;q)_n \mapsto x^n$$

maps polynomials with positive zeros into polynomials with zeros in $(1,\infty)$.

Thirteen further examples are listed in [**9**], but the underlying principle should be, by now, clear. Equally clear should be the gap between a schematic application of Theorem 5 and the final form of transformations—a gap that is filled by a great deal of careful formulae manipulation.

5. Expansions in orthogonal polynomials. Let ψ be an arbitrary distribution, defined for $x \in (a,b)$, and $\{r_n\}$ the corresponding orthogonal polynomials. We set $f_n := \int_a^b r_n^2(t)\, d\psi(t)$. Further, we stipulate the convergence and strict sign consistency of the *generating function*

$$G(x,\mu) := \sum_{n=0}^{\infty} d_n r_n(x)\mu^n, \quad d_0, d_1, \ldots \neq 0,$$

for all $x \in (a,b)$ and $\mu \in (c,d)$ (typically, by virtue of convergence, $c = -d$, but sometimes strict sign consistency "takes over" and further reduces the interval).

Following [**11**], we set $d\varphi(x,\mu) := G(x,\mu)\, d\psi(x)$, i.e., identify G with ω, and consider the underlying biorthogonal polynomials. Let $\rho_n(x) := (1/g_n) r_n(x)$, where g_n is the coefficient of x^n in r_n—hence ρ_n is monic. Then

$$\begin{aligned} I_n(\mu) &= \int_a^b \rho_n(t) G(t,\mu)\, d\psi(t) \\ &= \sum_{k=0}^{\infty} \frac{d_k}{g_n} \int_a^b r_n(t) r_k(t)\, d\psi(t) \mu^k \\ &= \frac{d_n f_n}{g_n} \mu^n, \quad n = 0, 1, \ldots, \end{aligned}$$

by virtue of orthogonality of $\{r_n\}$. The explicit form of $\{p_n\}$ is obtained by an argument identical to that in Section 3: Let the parameter polynomial be $s_n(x) := \sum_{k=0}^n s_{n,k} x^k$. Then

$$\begin{aligned} &\int_a^b \sum_{k=0}^{n} \frac{g_k s_{n,k}}{d_k f_k} \rho_k(t) G(t,\mu_l)\, d\psi(t) \\ &\qquad = \sum_{j=0}^{\infty} d_j g_j \left(\sum_{k=0}^{n} \frac{g_k s_{n,k}}{d_k f_k} \int_a^b \rho_k(t)\rho_j(t)\, d\psi(t) \right) \mu_l^j \\ &\qquad = \sum_{j=0}^{n} s_{n,j} \mu_l^j = s_n(\mu_l) = 0, \quad l = 1, 2, \ldots, n. \end{aligned}$$

Consequently,

$$p_n(x;\mu_1,\dots,\mu_n) = \sum_{k=0}^{n} \frac{g_k s_{n,k}}{d_k f_k} \rho_k(x) = \sum_{k=0}^{n} \frac{s_{n,k}}{d_k f_k} r_k(x),$$

and Theorem 5 leads to the main result of this section:

Theorem 11 (Iserles and Saff). *Provided that G is* SSC *for all $x \in (a,b)$ and $\mu \in (c,d)$, the transformation*

$$\mathcal{T} : x^n \mapsto \frac{1}{d_n f_n} r_n(x) \tag{5.1}$$

maps polynomials with all zeros in (c,d) into polynomials with all their zeros in $[a,b]$.

It is frequently easy to exclude zeros from the endpoints, as in Section 4, replacing $[a,b]$ with (a,b) in the last theorem.

The main difficulty in implementing (5.1) typically rests in verifying strict sign consistency of a generating function. A wide range of well-known orthogonal polynomial systems and their generating functions were analyzed in [**11**]. We select three examples from that reference.

Example I. $d\psi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} dx$, $(a,b) = \mathbf{R}$. This is the *Hermite distribution* and $r_n \equiv H_n, f_n = 2^n n!$. The choice $d_n = 1/n!$ leads to the classical generating function

$$G(x,\mu) = e^{x\mu - \mu^2}$$

[**20**] which is convergent and, by virtue of Lemma 6, SSC for all $\mu \in \mathbf{R}$. We have

Theorem 12 (Iserles and Saff). *The transformation*

$$\mathcal{T} : x^n \mapsto H_n(x)$$

maps polynomials with real zeros into themselves.

Example II. $d\psi(x) = x^\alpha e^{-x}dx$, $\alpha > -1$, $(a,b) = (0,\infty)$. The underlying *Laguerre polynomials* $L_n^{(\alpha)}$ admit the generating function

$$G(x,\mu) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\mu^n = (1-\mu)^{-1-\alpha}e^{-\frac{x\mu}{1-\mu}},$$

which is convergent and SSC for all $0 < x$, $|\mu| < 1$. Moreover, $f_n = \Gamma(\alpha+n+1)/n!$ [**20**] and Theorem 11 yields

Theorem 13 (Iserles and Saff). *The transformation*

$$\mathcal{T} : x^n \mapsto \frac{n!}{(\alpha+1)_n} L_n^{(\alpha)}(x)$$

maps polynomials with zeros in $(-1,1)$ *into polynomials with nonnegative zeros.*

The last result can be somewhat sharpened for $\alpha = 0$. Let $s_n(x) = \sum_{k=0}^n s_{n,k}x^k$ have all its zeros in $(-1,1)$. Then $(1)_k = k!$, $L_k^{(0)}(0) = 1$ imply that

$$\mathcal{T}s_n(0) = \sum_{k=0}^{n} s_{n,k} = s_n(1) \neq 0$$

and the zeros of $\mathcal{T}s_n$ stay clear of the origin.

Example III. $d\psi(x) = (1-x^2)^\alpha dx$, $(a,b) = (-1,1)$. This is the *ultraspherical distribution* (or, under different normalization, the *Gegenbauer distribution*) [**20**] and $r_n \equiv P_n^{(\alpha,\alpha)}$. It is known that

$$G(x,\mu) := \sum_{n=0}^{\infty} \frac{(2\alpha+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\alpha)}(x)\mu^n = \frac{1}{(1-2x\mu+\mu^2)^{\alpha+\frac{1}{2}}};$$

$$\int_{-1}^{1}(1-x^2)^\alpha [P_n^{(\alpha,\alpha)}(t)]^2 dt = 2^{2\alpha+1}\frac{[\Gamma(\alpha+n+1)]^2}{n!(2\alpha+2n+1)\Gamma(2\alpha+n+1)}$$

[**20**], thus

$$\frac{1}{d_n f_n} = C\frac{n!(\alpha+n+\frac{1}{2})}{(\alpha+1)_n} \tag{5.2}$$

where $C = \Gamma(\alpha+\frac{1}{2})/(\sqrt{\pi}\Gamma(\alpha+1))$.

Strict sign consistency is considerably more complicated to examine than in previous examples and results are available at present only for the range $\alpha > -1/2$. They depend on a criterion for strict total positivity from [**19**]:

Lemma 14 (Pólya and Szegö). *Let two functions f and g be given in $(0,\infty)$, such that f is positive and nonincreasing and g is monotone throughout the range. Then the function*

$$(g(x)+g(y)+1)\int_0^\infty t^{g(x)+g(y)} f(t)\,dt \tag{5.3}$$

is strictly totally positive.

To bring the generating function to the form required by Lemma 14, we set $y := (2\mu)/(1+\mu^2)$ and transform

$$x \mapsto \frac{x-1}{x+1}, \quad y \mapsto \frac{y-1}{y+1}.$$

Setting $\beta := \alpha + 1/2 > 0$, it follows that G is SSC for $x \in (-1,1)$ and $\mu > 0$ if and only if

$$\left(\frac{1}{2}\frac{(x+1)(y+1)}{x+y}\right)^\beta$$

is SSC for all $x, y > 0$. This, in turn, is equivalent to $(x+y)^{-\beta}$ being SSC for $x, y > 0$. We now invoke Lemma 14 with

$$f(x) := \begin{cases} \frac{(-\log x)^\beta}{\Gamma(\beta+1)}, & 0 < x < 1, \\ 0, & 1 \le x, \end{cases}$$

and $g(x) := x - 1/2$. Conditions of Lemma 14 are satisfied and substitution in (5.3) yields

$$\begin{aligned} \frac{x+y}{\Gamma(\beta+1)}\int_0^1 t^{x+y-1}(-\log t)^\beta\,dt &= \frac{x+y}{\Gamma(\beta+1)}\int_0^\infty e^{-(x+y)t} t^\beta\,dt \\ &= \frac{1}{\Gamma(\beta+1)(x+y)^\beta}\int_0^\infty e^{-t}t^\beta\,dt = \frac{1}{(x+y)^\beta}, \end{aligned}$$

and strict sign consistency (in fact, the stronger strict total positivity) follows. Consequently, we can now substitute (5.2) into Theorem 11:

Theorem 15 (Iserles and Saff). *Let $\alpha > -1/2$. The transformation*

$$\mathcal{T} : x^n \mapsto \frac{n!(\alpha + n + \frac{1}{2})}{(\alpha+1)_n} P_n^{(\alpha,\alpha)}(x)$$

maps polynomials with all their zeros in $(-1, 1)$ into polynomials with all zeros in $[-1, 1]$.

The value $\alpha = -1/2$ falls just outside the scope of the theorem. This is only to be expected, since the generating function reduces to a constant. There is some experimental evidence that $-1 < \alpha < -1/2$ brings about strict sign consistency (but not strict total positivity!) in the generating function, but no proofs exist.

The border-line case is interesting since it corresponds to Chebyshev polynomials. In that instance, elementary considerations are sufficient to ascertain that the transformation

$$\mathcal{T} : x^n \mapsto T_n(x) \tag{5.4}$$

maps real polynomials with zeros in $\{z \in \mathcal{C} : |z| < 1\}$ into polynomials with zeros in $(-1, 1)$ [**11**]: Let the real polynomial $s_n(x) = \sum_{k=0}^{n} s_{n,k} x^k$ have all its zeros inside the complex unit disc. We apply the argument principle to s_n along $|z| = 1$. Now s_n has exactly n zeros inside, and none on the perimeter of the unit disc; hence, the argument varies by $2\pi n$. Therefore,

$$\mathcal{T}\{s_n\}(\cos\theta) = \sum_{k=0}^{n} s_{n,k} \cos k\theta = \operatorname{Re} s_n(e^{i\theta})$$

has precisely n zeros in $\theta \in [0, \pi)$. This proves our assertion concerning (5.4).

6. Transformations of Al-Salam and Ismail. Let $\{\alpha_n\}$ be a sequence of real numbers with the property that $\alpha_0 = 1$ and $\alpha_k \neq 0$ for

$k \geq 1$. Al-Salam and Ismail [**1**] introduced and studied the sequence-to-function transformation $L_\alpha[f](x)$ defined by

$$L_\alpha[f](x) := \sum_{k=0}^{\infty} (-1)^k \alpha_k \Delta^k f(0) x^k,$$

where $\Delta f(x) := f(x+1) - f(x)$, $\Delta^k f(x) = \Delta(\Delta^{k-1} f(x))$. This transformation maps the set of sequences to a set of formal power series. In particular, if j is a fixed nonnegative integer and

$$f_j(k) := \frac{(-k)_j}{j!\alpha_j}, \quad k = 0, 1, \dots, \tag{6.1}$$

then it is easy to verify that

$$L_\alpha[f_j](x) = x^j. \tag{6.2}$$

Consequently, for an arbitrary polynomial $u(x) = \sum_{k=0}^{n} u_k x^k$,

$$L_\alpha\left[\sum_{j=0}^{n} u_j f_j\right](x) = u(x). \tag{6.3}$$

Let $\{r_n\}$ be polynomials that are orthogonal (in the conventional meaning of this phrase) with respect to the distribution τ. Then, in view of (6.3), for each $n \geq 0$, there is a polynomial q_n of degree n such that

$$L_\alpha[q_n(k)](x) = r_n(x),$$

where L_α acts on the sequence $\{q_n(k)\}_{k=0}^{\infty}$. The polynomials $\{q_n\}$ are called *discrete convolution-orthogonal polynomials* because, if

$$h(x) := \sum_{j=0}^{\infty} \alpha_j x^j, \tag{6.4}$$

then, at least formally,

$$\sum_{k,l=0}^{\infty} \eta_{k,l} q_m(k) q_n(l) = \lambda_m \delta_{m,n}, \quad m, n = 0, 1, \dots,$$

where

$$\eta_{k,l} := \int_{-\infty}^{\infty} \frac{h^{(k)}(t)h^{(l)}(t)}{k!l!}(-t)^{k+l}\, d\tau(t).$$

What can be said about the zeros of such polynomials q_n? Here we give a simple answer for the case when

$$\alpha_j = \frac{(-1)^j}{j!}c_j, \quad c_j := \int_0^{\infty} t^j\, d\psi(t), \tag{6.5}$$

where ψ is a real distribution on $(0, \infty)$ and $\{c_j^{-1}\}$ is a multiplier sequence.

Theorem 16. *Suppose that α_j is of the form (6.5) with $c_0 = \int_0^\infty d\psi(t) = 1$ and $c_j > 0$, $j = 1, 2, \ldots, n$. Moreover, assume that $\{c_j^{-1}\}$ is a multiplier sequence. If r_n is a polynomial of degree n having all its zeros lying in $(0, \infty)$, then the same is true of $q_n := L_\alpha^{-1}[r_n]$.*

Proof. From the definition of multiplier sequences, positive zeros are mapped into positive zeros and the transformation

$$\mathcal{T}_1 : x^j \mapsto \frac{x^j}{c_j}$$

maps r_n to a polynomial R_n having zeros in $(0, \infty)$. Furthermore, by Theorem 9A, the transformation

$$\mathcal{T}_2 : x^j \mapsto (-1)^j(-x)_j$$

maps R_n to a polynomial with zeros in $(0, \infty)$. But

$$(\mathcal{T}_2 \circ \mathcal{T}_1)(r_n) = q_n$$

since (cf. (6.1) and (6.2))

$$\begin{aligned} L_\alpha[(\mathcal{T}_2 \circ \mathcal{T}_1)x^j|_{x=k}](x) &= L_\alpha\left[\frac{(-1)^j(-k)_j}{c_j}\right](x) \\ &= L_\alpha\left[\frac{(-k)_j}{j!\alpha_j}\right] = x^j. \end{aligned}$$

Hence, q_n has all its zeros positive and simple. $\square$

Note that if $h(x)$ in (6.4) is the Laplace transform of a nonnegative weight $w(t)\,dt = d\psi(t)$, i.e.,

$$h(x) = \int_0^\infty e^{-xt} d\psi(t) \tag{6.6}$$

where

$$\int_0^\infty d\psi(t) = 1, \tag{6.7}$$

then

$$\alpha_j = \frac{h^{(j)}(0)}{j!} = \frac{(-1)^j}{j!} \int_0^\infty t^j \, d\psi(t) = \frac{(-1)^j}{j!} c_j, \tag{6.8}$$

provided the integrals and derivatives exist. Theorem 5.1 in Al-Salam and Ismail [**1**] states that if $\{r_n\}$ is a polynomial set orthogonal on $(0,\infty)$ and $h(x)$ is the Laplace transform of a nonnegative function, then $q_n = L_\alpha^{-1}[r_n]$ has all its zeros simple and lying in $(0,\infty)$. This theorem, in its full generality, is incorrect: for example, let

$$\psi(x) := \begin{cases} x, & 0 \le x < 1, \\ 3, & 1 \le x, \end{cases}$$

and

$$h(x) = \int_0^\infty e^{-xt} d\psi(t) = 2e^{-x} + \frac{1 - e^{-x}}{x}.$$

Thus, h is a Laplace transform of a nonnegative weight. We have $\alpha_k = (-1)^k (2k+1)/(k+1)!$, hence $(-1)^k / c_k = (-1)^k (k+1)/(2k+1)$. Every polynomial with all its zeros positive and distinct is orthogonal with respect to some distribution supported on $(0,\infty)$ [**24**]. We choose $r_3(x) = (x-1)(x-2)(x-3)$. Easy calculation verifies that

$$q_3(x) = -6 + \frac{166}{35} x - \frac{186}{35} x^2 + \frac{4}{7} x^3.$$

q_3 has a complex conjugate pair of zeros, providing a counter-example to Theorem 5.1 in [**1**]. The incorrectness of the proof of that theorem can be traced to its use of an unjustified version of a "generalized variation diminishing property" of Laplace transforms.

Theorem 16 demonstrates that some convolution-orthogonal polynomials have all their zeros in $(0, \infty)$. The treatment given here suggests a connection between discrete convolution-orthogonal polynomials and the composition of biorthogonal maps which is worthy of further investigation.

We shall say that $\{b_k\}$ is an *Al-Salam and Ismail sequence* (ASIS) if

$$b_k = \frac{1}{h^{(k)}(0)}, \quad k = 0, 1, \dots ,$$

where h satisfies (6.6–8) and $\{(-1)^k b_k\}$ is a multiplier sequence. Then the result of Theorem 16 yields

Theorem 17. *Let $\{b_j\}$ be an ASIS sequence. Then the transformation*

$$\mathcal{T} : \left\{ \sum_{k=0}^{n} u_k x^k \right\} := \sum_{k=0}^{n} b_k u_k (-x)_k$$

maps polynomials with all zeros positive to polynomials with all zeros positive.

Example I. $h(x) = (1 + x)^{-\alpha}$, $\alpha > 0$. Since

$$h(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-(1+x)t} dt,$$

h is the Laplace transform of a nonnegative function. It produces the ASIS

$$b_k = \frac{(-1)^k}{(\alpha)_k}, \quad k = 0, 1, \dots$$

(we use Section 2 to argue that $\{1/(\alpha)_k\}$ is a multiplier sequence) and the transformation

$$\mathcal{T} : x^k \mapsto (-1)^k \frac{(-x)^k}{(\alpha)_k}, \quad \alpha > 0.$$

Example II. $h(x) = (1 - e^{-x})/x$. h is the Laplace transform of the piecewise-constant function that equals 1 on $(0,1]$ and vanishes in $(1,\infty)$. It produces the ASIS $b_k = (-1)^k(k+1)$, $k = 0,1,\dots$, and the transformation

$$\mathcal{T} : x^k \mapsto (-1)^k(k+1)(-x)_k.$$

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