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BIORTHOGONAL LAURENT POLYNOMIALS WITH BIORTHOGONAL DERIVATIVES

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ABSTRACT. If V_n and Y_n are monic polynomials of degree n with $V_n(0) \neq 0$ and $Y_n(0) \neq 0$ such that $\{V_n(z), z^{-n-1} \cdot Y_n(z)\}_{n=0}^{\infty}$ is a biorthogonal system (BOS) with respect to a moment functional Φ on the algebra of Laurent polynomials in z with complex coefficients, then $\{V_n(z), z^{-n-1}Y_n(z)\}_{n=0}^{\infty}$ is called a *regular* BOS.

It is shown that if $\{V_n(z), z^{-n-1}Y_n(z)\}_{n=0}^{\infty}$ and $\{(1/(n+1))V'_{n+1}(z), z^{-n-1}(1/(n+1))Y'_{n+1}(z)\}_{n=0}^{\infty}$ are regular BOSs, then $\{V_n(z), z^{-n-1}Y_n(z)\}_{n=0}^{\infty}$ is a so-called *classical* BOS, i.e., one of the systems of Examples 1–3 below. In this way we obtain a characterization of classical BOSs of Laurent polynomials, analogously to Hahn's [**3**] characterization of classical ordinary polynomials.

1. Introduction. It is well known that the derivatives of the classical polynomials, including the Bessel polynomials, are again orthogonal polynomials. In 1935, W. Hahn [**3**] showed that this property is characteristic for the classical orthogonal polynomials with positive weight function on a real interval. A few years later, in 1938, H.L. Krall [**6**] observed that Hahn's procedure also applies in the case of generalized orthogonality, i.e., orthogonality with respect to an eventually indefinite moment functional. Krall showed that the only generalized orthogonal polynomial systems with generalized orthogonal derivatives are, apart from a linear change of variable, the classical generalized orthogonal polynomials.

In the present paper we give a similar characterization of a certain class of biorthogonal systems of Laurent polynomials which is closely related to the class of the classical orthogonal Laurent polynomials as treated in [**4**], where a characterization of classical orthogonal Laurent polynomials in terms of second order differential equations is given (see [**2**] for ordinary polynomials).

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2. Preliminaries. A system of Laurent polynomials $\{Q_n\}_{n=0}^{\infty}$ is called *simple* if the Q_n are of the form

$$\begin{aligned} Q_{2n}(z) &= q_{-n}^{(2n)} z^{-n} + q_{-n+1}^{(2n)} z^{-n+1} + \cdots + q_n^{(2n)} z^n, \\ Q_{2n+1}(z) &= q_{-n-1}^{(2n+1)} z^{-n-1} + q_{-n}^{(2n+1)} z^{-n} + \cdots + q_n^{(2n+1)} z^n, \end{aligned}$$

with $q_i^{(j)} \in \mathbf{C}$, $q_{-n}^{(2n)} \neq 0$, $q_{-n-1}^{(2n+1)} \neq 0$ and $q_n^{(2n)} = q_n^{(2n+1)} = 1$, $n = 0, 1, 2, \dots$. The system of Laurent polynomials $\{Q_n\}_{n=0}^{\infty}$ is orthogonal with respect to a moment functional Φ, defined on the algebra $\mathcal{A}$ of Laurent polynomials with complex coefficients, if Φ is linear and $\Phi(Q_n Q_k) = 0$ as $n \neq k$ and $\Phi(Q_n^2) \neq 0$, $n, k = 0, 1, 2, \dots$. It follows easily from [**5**, Theorem 1.1] that a simple system $\{Q_n\}_{n=0}^{\infty}$ of Laurent polynomials is orthogonal with respect to a moment functional if and only if the Q_n satisfy the recurrence relations

$$\begin{aligned} Q_{2n+1} &= (1 - \alpha_{2n+1} z^{-1}) Q_{2n} - \beta_{2n+1} Q_{2n-1}, \\ Q_{2n+2} &= (z - \alpha_{2n+2}) Q_{2n+1} - \beta_{2n+2} Q_{2n}, \end{aligned} \tag{2.1}$$

with $\alpha_{2n+1}, \alpha_{2n+2}, \beta_{2n+1}, \beta_{2n+2} \neq 0$, $n = 0, 1, 2, \dots$, and $Q_{-1} = 0$ and $Q_0 = 1$.

If $q_{-n}^{(2n)} = 0$ or $q_n^{(2n+1)} = 0$ for some n while $\{Q_n\}_{n=0}^{\infty}$ is still orthogonal, it may happen that the Q_n do not satisfy three term recurrence relations as in (2.1). For this reason we only consider simple orthogonal systems of Laurent polynomials.

Using the *corresponding ordinary polynomials*

$$V_{2n}(z) = z^n Q_{2n}(z), \quad V_{2n+1}(z) = z^{n+1} Q_{2n+1}(z), \quad n = 0, 1, 2, \dots,$$

we see that the simple system $\{Q_n\}_{n=0}^{\infty}$ is orthogonal if and only if the V_n satisfy

$$V_n = (z - \alpha_n) V_{n-1} - \beta_n z V_{n-2}, \quad n = 1, 2, \dots, V_{-1} = 0, V_0 = 1, \tag{2.2}$$

with $\alpha_n \neq 0$, $n = 1, 2, \dots$ and $\beta_n \neq 0$, $n = 2, 3, \dots$.

A pair of sequences of Laurent polynomials $\{A_n, B_n\}_{n=0}^{\infty}$ is said to be a *biorthogonal system* (BOS) with respect to a moment functional Ω on $\mathcal{A}$ if Ω is linear and $\Omega(A_n B_k) = 0$ if $n \neq k$ and $\Omega(A_n B_n) \neq 0$, $n, k = 0, 1, 2, \dots$.

If $\{Q_n\}_{n=0}^{\infty}$ is a simple system of Laurent polynomials, orthogonal with respect to Φ, and V_n are the corresponding ordinary polynomials, then (see [**5**])

$$\Phi(z^{-k}V_n)\begin{cases} = 0, & \text{if } k = 1, 2, \dots, n \\ \neq 0, & \text{if } k = 0 \text{ or } k = n+1, \end{cases}$$

and it is easily seen that $\{V_n, z^{-n-1}(V_{n+1} - zV_n)\}_{n=0}^{\infty}$ is a BOS with respect to Φ.

Conversely, let $\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}$ be a BOS with respect to Φ, where V_n and Y_n are ordinary polynomials with $\deg V_n = n$, $\deg Y_n \leq n$, V_n monic, $V_n(0) \neq 0$, $Y_n(0) \neq 0$, $n = 0, 1, 2, \dots$. Then, clearly,

$$\Phi(z^{-k}V_n) = 0, \quad k = 1, 2, \dots, n, \quad n = 0, 1, 2, \dots$$
$$\Phi(z^{-n-1}V_n) \neq 0, \quad n = 0, 1, 2, \dots.$$

Let $n \geq 2$. Then there is an $\alpha_n \in \mathbf{C}$ such that $V_n(0) + \alpha_n V_{n-1}(0) = 0$, so there is also a $\beta_n \in \mathbf{C}$ such that $\deg(V_n - zV_{n-1} + \alpha_n V_{n-1} + \beta_n z V_{n-2}) \leq n-2$. But then

$$W := V_n - zV_{n-1} + \alpha_n V_{n-1} + \beta_n z V_{n-2} \in \text{span}\,\{z, z^2, \dots, z^{n-2}\},$$

so there are $\lambda_0, \dots, \lambda_{n-3}$ such that $z^{-1}W = \lambda_0 V_0 + \lambda_1 V_1 + \dots + \lambda_{n-3}V_{n-3}$. Since $\Phi(z^{-k-1}z^{-1}W) = 0$, $k = 0, 1, \dots, n-3$, it follows from the biorthogonality that $\lambda_0 = \dots = \lambda_{n-3} = 0$. Hence,

$$V_n = (z - \alpha_n)V_{n-1} - \beta_n z V_{n-2}, \quad n \geq 2.$$

With $V_{-1} = 0$ this relation is trivially true for $n = 1$. Since $V_n(0) \neq 0$, we have $\alpha_n \neq 0$, $n = 1, 2, \dots$, and $0 = \Phi(z^{-n}V_n) = \Phi(z^{-n+1}V_{n-1}) - \alpha_n \Phi(z^{-n}V_{n-1}) - \beta_n \Phi(z^{-n+1}V_{n-2})$, $n = 2, 3, \dots$, implies $\beta_n \neq 0$, $n = 2, 3, \dots$. By the Favard-type theorem [**5**, Theorem 1.1], it follows that $\{Q_n\}_{n=0}^{\infty}$, defined by $Q_{2n} = z^{-n}V_{2n}$, $Q_{2n+1} = z^{-n-1}V_{2n+1}$ is a simple orthogonal system of Laurent polynomials. Hence, there is a one-to-one correspondence between BOSs $\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}$ with V_n, Y_n polynomials such that $\deg V_n = n$, $V_n(0) \neq 0$, $\deg Y_n \leq n$, $Y_n(0) \neq 0$, V_n and Y_n monic, and the simple orthogonal systems of Laurent polynomials $\{Q_n\}_{n=0}^{\infty}$. Consequently, if $\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}$ is such a BOS, then $Y_n = K_n(V_{n+1} - zV_n)$ for some nonzero constant K_n, $n = 0, 1, 2, \ldots$.

In this paper we only consider BOSs in $\mathcal{A}$ of the form $\{A_n, z^{-n-1}B_n\}_{n=0}^{\infty}$, where A_n and B_n are ordinary polynomials. Such a BOS $\{A_n, z^{-n-1}B_n\}_{n=0}^{\infty}$ is called *regular* if $\deg A_n = \deg B_n = n$, A_n and B_n are monic and $A_n(0) \neq 0$ and $B_n(0) \neq 0$, $n = 0, 1, 2, \ldots$.

The collection of all the regular BOSs $\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}$ such that $\{(1/(n+1))V'_{n+1}, z^{-n-1}(1/(n+1))Y'_{n+1}\}_{n=0}^{\infty}$ is again a regular BOS, will be referred to as class $\mathcal{D}$. Sometimes $\{(1/(n+1))V'_{n+1}, z^{-n-1}(1/(n+1))Y'_{n+1}\}_{n=0}^{\infty}$ will be called the derivative of the system $\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}$.

As we wish to determine all the systems belonging to class $\mathcal{D}$, we first derive some properties of systems of $\mathcal{D}$. To this end, we assume $\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty} \in \mathcal{D}$;

(2.2)
$$V_n = (z - \alpha_n)V_{n-1} - \beta_n z V_{n-2}, \quad n = 1, 2, \ldots, \quad V_{-1} = 0, V_0 = 1,$$

with $\alpha_n, \beta_n \in \mathbf{C}$, $\alpha_n \neq 0$, $n = 1, 2, \ldots$, and $\beta_n \neq 0$, $n = 2, 3, \ldots$, and $\beta_1 = 0$;

$$\text{(2.3)} \qquad \frac{1}{n}V'_n = (z - \alpha'_n)\frac{1}{n-1}V'_{n-1} - \beta'_n z \frac{1}{n-2}V'_{n-2}, \quad n = 2, 3, \ldots$$

$((1/(n-2))V'_{n-2} = 0$ if $n = 2)$, with $\alpha'_n, \beta'_n \in \mathbf{C}$, $\alpha'_n \neq 0$, $n = 2, 3, \ldots$, and $\beta'_n \neq 0$, $n = 3, 4, \ldots$, and $\beta'_2 = 0$.

Because $Y_n = K_n(V_{n+1} - zV_n) = -K_n(\alpha_{n+1}V_n + \beta_{n+1}zV_{n-1})$ and Y_n is monic, we have $\alpha_{n+1} + \beta_{n+1} \neq 0$ and

(2.4)
$$Y_n = \frac{-1}{\alpha_{n+1} + \beta_{n+1}}(V_{n+1} - zV_n) = \frac{\alpha_{n+1}V_n + \beta_{n+1}zV_{n-1}}{\alpha_{n+1} + \beta_{n+1}}, \quad n = 0, 1, \ldots .$$

In the same way we find $\alpha'_{n+2} + \beta'_{n+2} \neq 0$ and

$$\text{(2.5)} \qquad \begin{aligned} \frac{1}{n+1}Y'_{n+1} &= \frac{-1}{\alpha'_{n+2} + \beta'_{n+2}}\left(\frac{1}{n+2}V'_{n+2} - z\frac{1}{n+1}V'_{n+1}\right) \\ &= \frac{\alpha'_{n+2}\frac{1}{n+1}V'_{n+1} + \beta'_{n+2}z\frac{1}{n}V'_n}{\alpha'_{n+2} + \beta'_{n+2}}, \quad n = 0, 1, \ldots . \end{aligned}$$

In the sequel we write $\sigma_n = \alpha_n + \beta_n$, $n = 1, 2, \ldots$, and $\sigma'_n = \alpha'_n + \beta'_n$, $n = 2, 3, \ldots$. Note that $\sigma_1 = \alpha_1$ and $\sigma'_2 = \alpha'_2$.

Differentiation of (2.4) yields

$$\begin{aligned} \frac{1}{n} Y'_n &= -\frac{V'_{n+1} - zV'_n - V_n}{n\sigma_{n+1}} \\ &= \frac{\alpha_{n+1} V'_n + \beta_{n+1} z V'_{n-1} + \beta_{n+1} V_{n-1}}{n\sigma_{n+1}}, \quad n = 1, 2, \ldots . \end{aligned} \tag{2.6}$$

With n replaced by $n-1$, (2.5) reads

$$\begin{aligned} \frac{1}{n} Y'_n &= -\frac{1}{\sigma'_{n+1}} \left(\frac{1}{n+1} V'_{n+1} - z \frac{1}{n} V'_n \right) \\ &= \frac{1}{\sigma'_{n+1}} \left(\alpha'_{n+1} \frac{1}{n} V'_n + \beta'_{n+1} z \frac{1}{n-1} V'_{n-1} \right), \quad n = 1, 2, \ldots , \end{aligned} \tag{2.7}$$

$((1/(n-1))V'_{n-1} = 0$ as $n = 1)$.

From (2.6) and (2.7) we obtain, by elimination of Y'_n,

$$V_n = \left(1 - \frac{n}{n+1} \frac{\sigma_{n+1}}{\sigma'_{n+1}}\right) V'_{n+1} - \left(1 - \frac{\sigma_{n+1}}{\sigma'_{n+1}}\right) z V'_n, \quad n = 1, 2, \ldots , \tag{2.8}$$

and

$$V_n = \left(\frac{\alpha'_{n+2}}{\beta_{n+2}} \frac{\sigma_{n+2}}{\sigma'_{n+2}} - \frac{\alpha_{n+2}}{\beta_{n+2}} \right) V'_{n+1} - \left(1 - \frac{n+1}{n} \frac{\beta'_{n+2}}{\beta_{n+2}} \frac{\sigma_{n+2}}{\sigma'_{n+2}} \right) z V'_n, \quad n = 1, 2, \ldots . \tag{2.9}$$

Since $V_n(0) \neq 0$ and $V'_{n+1}(0) \neq 0$, (2.8) and (2.9) must be identical, so

$$\frac{\sigma'_{n+1}}{\sigma_{n+1}} = \frac{n}{n-1} \frac{\beta'_{n+1}}{\beta_{n+1}} \frac{\sigma'_n}{\sigma_n}, \quad n = 2, 3, \ldots . \tag{2.10}$$

Now we partially follow Hahn's method [**3**] to show that $\beta'_n/(n-2) = \beta_n/(n-1)$, $n = 3, 4, \ldots$. Differentiation of (2.2) gives

$$V'_n = (z - \alpha_n) V'_{n-1} - \beta_n z V'_{n-2} + V_{n-1} - \beta_n V_{n-2}, \quad n = 1, 2, \ldots , \tag{2.11}$$

and also, by (2.2),

$$zV_n' = (z-\alpha_n)zV_{n-1}' - \beta_n z^2 V_{n-2}' + V_n + \alpha_n V_{n-1}, \quad n = 1, 2, \dots . \tag{2.12}$$

Elimination of V_{n-2}' from (2.3) and (2.12) gives

$$\left(\frac{\beta_n'}{n-2} - \frac{\beta_n}{n}\right) zV_n' = \left[\frac{\beta_n'}{n-2}(z-\alpha_n) - \frac{\beta_n}{n-1}(z-\alpha_n')\right] zV_{n-1}' + \frac{\beta_n'}{n-2}V_n + \frac{\alpha_n\beta_n'}{n-2}V_{n-1}, \quad n = 3, 4, \dots , \tag{2.13}$$

and elimination of V_n' from (2.3) and (2.11) gives, after replacing n by $n+1$,

$$\left[\frac{z-\alpha_{n+1}}{n+1} - \frac{z-\alpha_{n+1}'}{n}\right] V_n' = \left(\frac{\beta_{n+1}}{n+1} - \frac{\beta_{n+1}'}{n-1}\right) zV_{n-1}' - \frac{1}{n+1}V_n + \frac{\beta_{n+1}}{n+1}V_{n-1}, \quad n = 1, 2, \dots . \tag{2.14}$$

It is easily verified that $\beta_n'/(n-2) = \beta_n/(n-1)$ if (2.13) and (2.14) are identical. If (2.13) and (2.14) are not identical, then, by elimination of V_n, we obtain the nontrivial relation

$$\begin{aligned}
&\left[\frac{n-1}{n(n+1)}\left(\frac{\beta_n'}{n-2} - \frac{\beta_n}{n-1}\right) z + \frac{\beta_n'}{n-2}\left(\frac{\alpha_{n+1}'}{n} - \frac{\alpha_{n+1}}{n+1}\right)\right] V_n' \\
&\quad = \Bigg[\frac{1}{n+1}\left(\frac{\beta_n'}{n-2} - \frac{\beta_n}{n-1}\right) z \\
&\qquad - \left\{\frac{1}{n+1}\left(\frac{\alpha_n\beta_n'}{n-2} - \frac{\alpha_n'\beta_n}{n-1}\right) + \frac{\beta_n'}{n-2}\left(\frac{\beta_{n+1}'}{n-1} - \frac{\beta_{n+1}}{n+1}\right)\right\}\Bigg] zV_{n-1}' \\
&\qquad + \frac{(\alpha_n + \beta_{n+1})\beta_n'}{(n-2)(n+1)}V_{n-1}, \quad n = 3, 4, \dots .
\end{aligned} \tag{2.15}$$

Comparing the nontrivial relation (2.15) with (2.8) with n replaced by $n-1$ and observing that the V_n' do not have common zeros since the V_n' satisfy (2.3), it follows that again $\beta_n'/(n-2) = \beta_n/(n-1)$. Hence,

$$\frac{\beta_n'}{n-2} = \frac{\beta_n}{n-1}, \quad n = 3, 4, \dots . \tag{2.16}$$

In order to derive further relations between $\alpha_n, \beta_n, \alpha'_n, \beta'_n$, we write

$$V_n(z) = \xi_0^{(n)} + \xi_1^{(n)} z + \cdots + \xi_{n-1}^{(n)} z^{n-1} + \xi_n^{(n)} z^n,$$

$\xi_j^{(n)} \in \mathbf{C}$, $\xi_n^{(n)} = 1$, $j = 0, 1, \dots n$, $n = 0, 1, \dots$. It follows easily from (2.2) and (2.3) that

$$\xi_0^{(n)} = (-1)^n \alpha_1 \alpha_2 \cdots \alpha_n, \quad n = 0, 1, \dots, \tag{2.17}$$

$$\frac{1}{n} \xi_1^{(n)} = (-1)^{n-1} \alpha'_2 \alpha'_3 \cdots \alpha'_n, \quad n = 1, 2, \dots, \tag{2.18}$$

and

$$\xi_1^{(n)} = \xi_0^{(n-1)} - \alpha_n \xi_1^{(n-1)} - \beta_n \xi_0^{(n-2)}, \quad n = 1, 2, \dots . \tag{2.19}$$

Elimination of $\xi_0^{(n)}, \xi_0^{(n-1)}, \xi_1^{(n)}, \xi_1^{(n-1)}$ from (2.17)–(2.19) gives

$$\begin{aligned} &\alpha'_n(\alpha_{n-1} + \beta_n)[(n+1)\alpha'_{n+1} - n\alpha_{n+1}] \\ &= \alpha_{n-1}(\alpha_n + \beta_{n+1})[n\alpha'_n - (n-1)\alpha_n], \quad n = 2, 3, \dots . \end{aligned} \tag{2.20}$$

Comparing the coefficients of z^{n-1} in (2.2) and coefficients of z^{n-2} in (2.3) we get

$$\xi_{n-1}^{(n)} - \xi_{n-2}^{(n-1)} + \sigma_n = 0 \quad \text{and} \quad \frac{n-1}{n} \xi_{n-1}^{(n)} - \frac{n-2}{n-1} \xi_{n-2}^{(n-1)} + \sigma'_n = 0,$$
$$n = 2, 3, \dots,$$

and elimination of $\xi_{n-2}^{(n-1)}, \xi_{n-1}^{(n)}, \xi_n^{(n+1)}$ from

$$\left\{ \begin{array}{cccccccc} \xi_n^{(n+1)} & - & \xi_{n-1}^{(n)} & & & + & \sigma_{n+1} & = 0 \\ \frac{n}{n+1}\xi_n^{(n+1)} & - & \frac{n-1}{n}\xi_{n-1}^{(n)} & & & + & \sigma'_{n+1} & = 0 \\ & & \xi_{n-1}^{(n)} & - & \xi_{n-2}^{(n-1)} & + & \sigma_n & = 0 \\ & & \frac{n-1}{n}\xi_{n-1}^{(n)} & - & \frac{n-2}{n-1}\xi_{n-2}^{(n-1)} & + & \sigma'_n & = 0 \end{array} \right. ,$$

$$n = 2, 3, \dots,$$

yields $(n+1)\sigma'_{n+1} - (n-1)\sigma'_n = n\sigma_{n+1} - (n-2)\sigma_n$, $n = 2, 3, \dots$. From (2.2) and (2.3) with $n = 2$ we easily get $2\alpha'_2 = \alpha_1 + \alpha_2 + \beta_2$, so $2\sigma'_2 = \sigma_1 + \sigma_2$. Thus,

$$(2.21)\quad (n+1)\sigma'_{n+1} - (n-1)\sigma'_n = n\sigma_{n+1} - (n-2)\sigma_n, \quad n = 1, 2, \dots .$$

As Y_n and $(1/(n+1))Y'_{n+1}$ satisfy three-term recurrence relations similar to (2.2) and (2.3), there are formulae similar to (2.20) and (2.21) corresponding to the polynomials Y_n and $(1/(n+1))Y'_{n+1}$. Indeed, it follows from (2.4) that

$$(2.22)\quad Y_n = (z - a_n)Y_{n-1} - b_n z Y_{n-2}, \quad n = 1, 2, \dots, Y_{-1} = 0, Y_0 = 1,$$

with $a_n = (\sigma_n/\sigma_{n+1})\alpha_{n+1}$, $n = 1, 2, \dots$, and $b_n = (\sigma_{n-1}/\sigma_n)\beta_n$, $n = 2, 3, \dots$. Since $\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}$ has the same structure as its derivative $\{(1/(n+1))V'_{n+1}, z^{-n-1}(1/(n+1))Y'_{n+1}\}_{n=0}^{\infty}$, we also have

$$(2.23)\quad \frac{1}{n+1}Y'_{n+1} = (z - \alpha'_{n+1})\frac{1}{n}Y'_n - b'_{n+1} z \frac{1}{n-1}Y'_{n-1}, \quad n = 2, 3, \dots,$$

$((1/(n-1))Y'_{n-1} = 0$ if $n = 1)$ with $a'_{n+1} = (\sigma'_{n+1}/\sigma'_{n+2})\alpha'_{n+2}$, $n = 1, 2, \dots$, and $b'_{n+1} = (\sigma'_n/\sigma'_{n+1})\beta'_{n+1}$, $n = 2, 3, \dots$.

Only the analog of (2.20) is needed in this paper. With the just mentioned expressions for a_n, a'_n, b_n, b'_n, and with (2.16), we get

$$(2.24)\quad \alpha'_{n+1}\sigma_{n+1}[(n+1)\alpha'_{n+2} - n\alpha_{n+2}] = \alpha_n\sigma_{n+2}[n\alpha'_{n+1} - (n-1)\alpha_{n+1}], \quad n = 2, 3, \dots .$$

The following examples of BOSs belonging to class $\mathcal{D}$ can be derived from the examples in [**4**].

Example 1.

$$V_n = \frac{(c)_n}{(-a)_n}\,{}_2F_1(-n, -a; -c-n+1; z), \quad n = 0, 1, \dots,$$
$$Y_n = \frac{(c+1)_n}{(-a-1)_n}\,{}_2F_1(-n, -a-1; -c-n; z), \quad n = 0, 1, \dots,$$
$$a+1, -c, a-c \neq 0, 1, 2, \dots,$$

(Notice that ${}_2F_1(p+1,q;r;z) - {}_2F_1(p,q+1;r;z) = \frac{q-p}{r} z\, {}_2F_1(p+1,q+1;r+1;z)$),

$$\frac{1}{n+1} V'_{n+1} = \frac{(c)_n}{(-a+1)_n}\, {}_2F_1(-n,-a+1;-c-n+1;z), \quad n = 0,1,\dots,$$
$$\frac{1}{n+1} Y'_{n+1} = \frac{(c+1)_n}{(-a)_n}\, {}_2F_1(-n,-a;-c-n;z), \quad n = 0,1,\dots,$$

$\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}$ is biorthogonal with respect to

$$\Phi(z^{-n}) = \frac{(a)_n}{(c)_n}, \quad n \in \mathbf{Z},$$

$\{\frac{1}{n+1}V'_{n+1}, z^{-n-1}\frac{1}{n+1}Y'_{n+1}\}_{n=0}^{\infty}$ is biorthogonal with respect to

$$\Theta(z^{-n}) = \frac{(a-1)_n}{(c)_n}, \quad n \in \mathbf{Z}.$$

Moreover, (2.2), (2.3), (2.22), (2.23) are valid with

$$\alpha_n = -\frac{c+n-1}{-a+n-1}, \qquad \beta_n = \frac{(n-1)(c-a+n-2)}{(-a+n-2)(-a+n-1)},$$
$$\alpha'_{n+1} = -\frac{c+n-1}{-a+n}, \qquad \beta'_{n+1} = \frac{(n-1)(c-a+n-1)}{(-a+n-1)(-a+n)},$$
$$a_n = -\frac{c+n}{-a+n-2}, \qquad b_n = \frac{(n-1)(c-a+n-2)}{(-a+n-3)(-a+n-2)},$$
$$a'_{n+1} = -\frac{c+n}{-a+n-1}, \qquad b'_{n+1} = \frac{(n-1)(c-a+n-1)}{(-a+n-2)(-a+n-1)},$$
$$n = 1,2,\dots.$$

Example 2.

$$V_n = (-1)^n (c)_n\, {}_1F_1(-n;-c-n+1;-z), \quad n = 0,1,\dots,$$
$$Y_n = (-1)^n (c+1)_n\, {}_1F_1(-n;-c-n;-z), \quad n = 0,1,\dots,$$
$$-c \neq 0,1,2,\dots,$$

(Notice that ${}_1F_1(p;r;-z) - {}_1F_1(p+1;r;-z) = (1/r)z\,{}_1F_1(p+1;r+1;-z)$),

$$\frac{1}{n+1}V'_{n+1} = V_n, \quad n=0,1,\dots,$$
$$\frac{1}{n+1}Y'_{n+1} = Y_n, \quad n=0,1,\dots,$$

$\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}$ is biorthogonal with respect to

$$\Phi(z^{-n}) = \frac{1}{(c)_n}, \quad n \in \mathbf{Z},$$

$$\left\{\frac{1}{n+1}V'_{n+1}, z^{-n-1}\frac{1}{n+1}Y'_{n+1}\right\}_{n=0}^{\infty} = \{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}.$$

In (2.2), (2.3), (2.22), (2.23) we have

$$\alpha_n = \alpha'_{n+1} = c+n-1, \quad \beta_n = \beta'_{n+1} = -n+1,$$
$$a_n = a'_{n+1} = c+n, \qquad b_n = b'_{n+1} = -n+1,$$
$$n = 1,2,\dots.$$

Example 3.

$$V_n = \frac{1}{(-a)_n}\,{}_2F_0(-n,-a;-z), \quad n=0,1,\dots,$$
$$Y_n = \frac{1}{(-a-1)_n}\,{}_2F_0(-n,-a-1;-z), \quad n=0,1,\dots,$$
$$a \neq -1,0,1,2,\dots,$$

$({}_2F_0(p,q+1;-z) - {}_2F_0(p+1,q;-z) = (q-p)z\,{}_2F_0(p+1,q+1;-z))$,

$$\frac{1}{n+1}V'_{n+1} = \frac{1}{(-a+1)_n}\,{}_2F_0(-n,-a+1;-z), \quad n=0,1,\dots,$$
$$\frac{1}{n+1}Y'_{n+1} = \frac{1}{(-a)_n}\,{}_2F_0(-n,-a;-z), \quad n=0,1,\dots,$$

$\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}$ is biorthogonal with respect to

$$\Phi(z^{-n}) = (a)_n, \quad n \in \mathbf{Z},$$

$\{\frac{1}{n+1}V'_{n+1}, z^{-n-1}\frac{1}{n+1}Y'_{n+1}\}_{n=0}^{\infty}$ is biorthogonal with respect to

$$\Theta(z^{-n}) = (a-1)_n, \quad n \in \mathbf{Z}.$$

In (2.2), (2.3), (2.22), (2.23) we have

$$\begin{aligned}
\alpha_n &= -\frac{1}{-a+n-1}, & \beta_n &= \frac{n-1}{(-a+n-2)(-a+n-1)},\\
\alpha'_{n+1} &= -\frac{1}{-a+n}, & \beta'_{n+1} &= \frac{n-1}{(-a+n-1)(-a+n)},
\end{aligned}$$

$$\begin{aligned}
a_n &= -\frac{1}{-a+n-2}, & b_n &= \frac{n-1}{(-a+n-3)(-a+n-2)},\\
a'_{n+1} &= -\frac{1}{-a+n-1}, & b'_{n+1} &= \frac{n-1}{(-a+n-2)(-a+n-1)},
\end{aligned}$$

$$n = 1, 2, \dots .$$

The Pochhammer symbol $(x)_n$ occurring in the above examples is defined as $(x)_n = \Gamma(x+n)/\Gamma(x)$ for $x \in \mathbf{C}\backslash\mathbf{N}$ and $n \in \mathbf{Z}$.

3. The characterization. In this section it will be shown that the only BOSs which belong to class $\mathcal{D}$ are essentially the BOSs of Examples 1–3. Therefore, the BOSs of Examples 1–3 will be called *classical biorthogonal systems*. The classical BOSs correspond to a large subclass of the classical orthogonal Laurent polynomials as treated in [**4**]. Only the classical orthogonal Laurent polynomials of ${}_2F_1$-type with $a = -1$ and of ${}_2F_0$-type with $a = -1$ do not correspond to classical BOSs.

From (2.10) and (2.16) we get $\sigma_{n+1}/\sigma'_{n+1} = \sigma_n/\sigma'_n$, $n = 2, 3, \dots$, so there is a $\rho \in \mathbf{C}\backslash\{0\}$ such that

$$\frac{\sigma_n}{\sigma'_n} = \rho, \quad n = 2, 3, \dots . \tag{3.1}$$

Hence, (2.21) can be written as

$$(n+1)\frac{\sigma_{n+1}}{\sigma_n} - n + 1 = \rho\left(n\frac{\sigma_{n+1}}{\sigma_n} - n + 2\right), \quad n = 2, 3, \dots ,$$

and it follows that

$$\frac{\sigma_{n+1}}{\sigma_n} = \frac{n-1-\rho(n-2)}{n+1-\rho n}, \quad n = 2, 3, \dots .$$

From (2.21) with $n = 1$ we get $\sigma_2 + \sigma_1 = 2\sigma_2'$, so $\sigma_2 + \sigma_1 = (2/\rho)\sigma_2$ by (3.1), and we also have

$$\frac{\sigma_2}{\sigma_1} = \frac{\rho}{2-\rho}.$$

Hence,

$$\frac{\sigma_{n+1}}{\sigma_n} = \frac{n-1-\rho(n-2)}{n+1-\rho n}, \quad n = 1, 2, \dots . \tag{3.2}$$

The case $\rho \neq 1$. Then, with $a = 1/(\rho - 1)$, (3.2) can be rewritten as

$$\frac{\sigma_{n+1}}{\sigma_n} = \frac{-a+n-2}{-a+n}, \quad n = 1, 2, \dots ,$$

so $a \neq -1, 0, 1, \dots$ since $\sigma_n \neq 0$, $n = 1, 2, \dots$. Moreover,

$$\begin{aligned}
\sigma_n &= \frac{\sigma_n}{\sigma_{n-1}} \cdot \frac{\sigma_{n-1}}{\sigma_{n-2}} \cdots \frac{\sigma_2}{\sigma_1} \cdot \sigma_1 \\
&= \frac{-a+n-3}{-a+n-1} \cdot \frac{-a+n-4}{-a+n-2} \cdots \frac{-a}{-a+2} \cdot \frac{-a-1}{-a+1} \cdot \alpha_1 \\
&= \frac{-a(-a-1)\alpha_1}{(-a+n-2)(-a+n-1)}, \quad n = 1, 2, \dots ,
\end{aligned}$$

so

$$\sigma_n = \frac{-a(-a-1)\alpha_1}{(-a+n-2)(-a+n-1)}, \quad n = 1, 2, \dots . \tag{3.3}$$

Elimination of β_n from

$$\begin{gathered}
\alpha_n' + \frac{n-2}{n-1}\beta_n = \sigma_n' = \frac{1}{\rho}\sigma_n, \quad n = 2, 3, \dots , \\
\alpha_n + \beta_n = \sigma_n, \quad n = 1, 2, \dots ,
\end{gathered}$$

gives

$$(n-1)\alpha_n' - (n-2)\alpha_n = \frac{-a+n-2}{-a-1}\sigma_n, \quad n = 2, 3, \dots . \tag{3.4}$$

Combination of (2.24) and (3.4) yields

$$\frac{\alpha_n'}{\alpha_{n-1}} = \frac{-a+n-2}{-a+n-1}, \quad n = 3, 4, \dots .$$

Since also

$$\frac{\alpha_2'}{\alpha_1} = \frac{\sigma_2'}{\sigma_1} = \frac{\sigma_2'}{\sigma_2}\frac{\sigma_2}{\sigma_1} = \frac{1}{\rho} \cdot \frac{-a-1}{-a+1} = \frac{-a}{-a+1},$$

we have

$$\frac{\alpha_n'}{\alpha_{n-1}} = \frac{-a+n-2}{-a+n-1}, \quad n = 2, 3, \dots . \tag{3.5}$$

From (3.3), (3.4) and (3.5) we get

$$\frac{-a+k-2}{k-2}\alpha_{k-1} - \frac{-a+k-1}{k-1}\alpha_k = \frac{-a\alpha_1}{(k-2)(k-1)}, \quad k = 3, 4, \dots ;$$

hence, for $n \geq 3$,

$$\begin{aligned}(-a+1)\alpha_2 - \frac{-a+n-1}{n-1}\alpha_n &= \sum_{k=3}^{n}\left(\frac{-a+k-2}{k-2}\alpha_{k-1} - \frac{-a+k-1}{k-1}\alpha_k\right) \\ &= \sum_{k=3}^{n} a\alpha_1\left(\frac{1}{k-1} - \frac{1}{k-2}\right) \\ &= a\alpha_1\left(\frac{1}{n-1} - 1\right) \\ &= -a\alpha_1\frac{n-2}{n-1}.\end{aligned}$$

This leads to

$$\begin{aligned}\alpha_n &= \frac{(-a+1)\alpha_2(n-1) + a\alpha_1(n-2)}{-a+n-1} \\ &= -\frac{-[(-a+1)\alpha_2 + a\alpha_1]n + (-a+1)\alpha_2 + 2a\alpha_1}{-a+n-1}, \quad n = 1, 2, \dots .\end{aligned} \tag{3.6}$$

If $n = 1$ or $n = 2$, then (3.6) is trivially true. Now we put $K = -(-a+1)\alpha_2 - a\alpha_1$. Then

$$\alpha_n = -\frac{K(n-1) + a\alpha_1}{-a+n-1}, \quad n = 1, 2, \dots ,$$

and we calculate β_n from (3.3):

$$\begin{aligned}\beta_n &= \sigma_n - \alpha_n \\ &= \frac{a(a+1)\alpha_1}{(-a+n-2)(-a+n-1)} + \frac{K(n-1)+a\alpha_1}{-a+n-1} \\ &= \frac{(n-1)[K(n-1)-(a+1)K+a\alpha_1]}{(-a+n-2)(-a+n-1)}, \quad n=1,2,\dots .\end{aligned}$$

Here we distinguish between $K \neq 0$ and $K = 0$.

Assume $K \neq 0$. Then

$$\frac{\alpha_n}{K} = -\frac{\frac{a\alpha_1}{K}+n-1}{-a+n-1}, \ \frac{\beta_n}{K} = \frac{(n-1)(\frac{a\alpha_1}{K}-a+n-2)}{(-a+n-2)(-a+n-1)}, \quad n=1,2,\dots ,$$

and, with $c = (a\alpha_1)/K$, we get

$$\frac{\alpha_n}{K} = -\frac{c+n-1}{-a+n-1}, \ \frac{\beta_n}{K} = \frac{(n-1)(c-a+n-2)}{(-a+n-2)(-a+n-1)}, \quad n=1,2,\dots ,$$

where

$$\begin{gathered} a \neq -1,0,1,2,\dots , \\ -c \neq 0,1,2,\dots \text{ since } \alpha_n \neq 0, n=1,2,\dots \end{gathered}$$

and

$$a-c \neq 0,1,2,\dots \text{ since } \beta_n \neq 0, n=2,3,\dots .$$

Since, for $\lambda \neq 0$, $V_n(z)$ satisfies

$$\begin{gathered} V_n(z) = (z-\alpha_n)V_{n-1}(z) - \beta_n z V_{n-2}(z), \ n=1,2,\dots , \\ \text{with } V_{-1}=0, V_0=1 \end{gathered}$$

if and only if $W_n(z) = \lambda^{-n}V_n(\lambda z)$ satisfies

$$\begin{gathered} W_n(z) = \left(z - \frac{\alpha_n}{\lambda}\right) W_{n-1}(z) - \frac{\beta_n}{\lambda} z W_{n-2}(z), \ n=1,2,\dots , \\ \text{with } W_{-1}=0, W_0=1, \end{gathered}$$

we see that

$$K^{-n}V_n(Kz) = \frac{(c)_n}{(-a)_n}\,{}_2F_1(-n,-a;-c-n+1;z), \quad n=0,1,\dots,$$

and $\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}$ is, apart from a change of scale, the BOS of Example 1.

Suppose now that $K=0$. Then

$$\frac{\alpha_n}{a\alpha_1} = -\frac{1}{-a+n-1}, \; \frac{\beta_n}{a\alpha_1} = \frac{n-1}{(-a+n-2)(-a+n-1)}, \quad n=1,2,\dots,$$

where $a \neq -1,0,1,2,\dots$. So, as above, we see that

$$(a\alpha_1)^{-n}V_n(a\alpha_1 z) = \frac{1}{(-a)_n}\,{}_2F_0(-n,-a;-z), \quad n=0,1,\dots,$$

and $\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}$ is, apart from a change of scale, the BOS of Example 3.

The case $\rho=1$. In this case we have

$$\sigma_n' = \sigma_n, \quad n=2,3,\dots, \quad \text{by (3.1)}$$

and

$$\sigma_n = \sigma_1 =: \sigma, \quad n=1,2,\dots, \quad \text{by (3.2)}.$$

From $\alpha_n + \beta_n = \sigma$ and $\alpha_n' + ((n-2)/(n-1))\beta_n = \sigma$, $n=2,3,\dots$, we get

$$(n-1)\alpha_n' - (n-2)\alpha_n = \sigma, \quad n=1,2,\dots. \tag{3.7}$$

(It is obvious that (3.7) holds for $n=1$ since $\alpha_1 = \sigma_1 = \sigma$.) Combination of (2.24) with $\sigma_n = \sigma$ and (3.7) now gives $\alpha_n'/\alpha_{n-1} = 1$, $n=3,4,\dots$, while also $a_2'/\alpha_1 = \sigma_2'/\sigma_1 = (\sigma_2'/\sigma_2)\cdot(\sigma_2/\sigma_1) = 1$, so $\alpha_n' = \alpha_{n-1}$, $n=2,3,\dots$. Together with (3.7), this yields

$$\frac{\alpha_{k-1}}{k-2} - \frac{\alpha_k}{k-1} = \frac{\sigma}{(k-2)(k-1)}, \quad k=3,4,\dots,$$

and

$$\begin{aligned}\alpha_2 - \frac{\alpha_n}{n-1} &= \sum_{k=3}^{n}\left(\frac{\alpha_{k-1}}{k-2} - \frac{\alpha_k}{k-1}\right)\\ &= \sum_{k=3}^{n}\left(\frac{1}{k-2} - \frac{1}{k-1}\right)\sigma\\ &= \left(1 - \frac{1}{n-1}\right)\sigma\\ &= \frac{n-2}{n-1}\sigma, \quad n = 3, 4, \ldots ,\end{aligned}$$

hence,

$$\alpha_n = \alpha_2(n-1) - \sigma(n-2) = -\beta_2 n + \sigma + \beta_2, \quad n = 3, 4, \ldots .$$

Since this relation is obvious for $n = 1$ and for $n = 2$, we have

$$\alpha_n = \alpha_2(n-1) - \sigma(n-2) = -\beta_2 n + \sigma + \beta_2, \quad n = 1, 2, \ldots . \tag{3.8}$$

From $\alpha_n + \beta_n = \sigma$ and (3.8) we obtain

$$\beta_n = \beta_2(n-1), \quad n = 1, 2, \ldots . \tag{3.9}$$

With $c = -\sigma/\beta_2$, formulae (3.8) and (3.9) can be written as

$$\frac{\alpha_n}{-\beta_2} = c + n - 1, \quad \frac{\beta_n}{-\beta_2} = -(n-1), \quad n = 1, 2, \ldots ,$$

where $-c \neq 0, 1, 2, \ldots$ since $\alpha_n \neq 0$, $n = 1, 2, \ldots$. Hence, in this case

$$(-\beta_2)^{-n} V_n(-\beta_2 z) = (-1)^n (c)_n \, {}_1F_1(-n; -c-n+1; -z), \quad n = 0, 1, 2, \ldots ,$$

and $\{V_n, z^{-n-1} Y_n\}_{n=0}^{\infty}$ is essentially the BOS of Example 2.

We summarize the results of this section in the following theorem. Recall that the class $\mathcal{D}$ is the class of all the regular BOSs $\{V_n, z^{-n-1} Y_n\}_{n=0}^{\infty}$ which have a regular BOS $\{\frac{1}{n+1} V'_{n+1}, z^{-n-1} \frac{1}{n+1} Y'_{n+1}\}_{n=0}^{\infty}$ as their derivative.

Theorem 3.1. *If* $\{V_n, z^{-n-1}Y_n\}_{n=0}^{\infty}$ *belongs to class* $\mathcal{D}$*, then there is a nonzero* $\lambda \in \mathbf{C}$ *such that* $\{\lambda^{-n}V_n(\lambda z), z^{-n-1}\lambda^{-n}Y_n(\lambda z)\}_{n=0}^{\infty}$ *is a classical biorthogonal system of Laurent polynomials, i.e.,* $\{\lambda^{-n}V_n(\lambda z), z^{-n-1}\lambda^{-n}Y_n(\lambda z)\}_{n=0}^{\infty}$ *is one of the systems of the Examples* 1–3.

Remark 3.1. In [**3**] it is essential that the polynomials considered, being orthogonal with respect to a nonnegative weight function on a real interval, have simple zeros. As in [**6**], Hahn's method is followed in the case of generalized orthogonality, it is not clear whether, in [**6**], an assumption about simple zeros is needed or that the polynomials have simple zeros as a consequence of the orthogonality of these polynomials and the orthogonality of their derivatives. See also [**1**]. In the approach of the present paper assumptions about simple zeros are not needed.

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