ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 1, Winter 1992

# ALGEBRAS WITH THE LOCAL INTERPOLATION PROPERTY

Ma TERESA GASSÓ

ABSTRACT. In this paper a class of Boolean algebras is defined in such a way that the classical Nikodym theorem holds for sequences of bounded additive measures defined on said algebras. It is proved that this class of Boolean algebras contains those known to have the property (N), i.e., the ones satisfying the Vitali-Hahn-Saks theorem [**10**] as well as those introduced by Schachermayer [**9**] and by Graves-Wheeler [**5**].

The second problem raised by Graves and Wheeler in [**5**] is solved because the local interpolation (LI) alone proves the property (N). The condition (LI) gives a new example of Boolean algebras with the Nikodym property.

The Boolean algebras of Seever [**10**] and Faires [**3**] and those studied here are defined by means of "interpolation properties" between disjoint sequences in this algebra.

**1. Introduction.** The book by Diestel and Uhl [**2**] gives us an account of the history of Grothendieck, Nikodym and Vitali-Hahn-Saks properties. We must remember that Diestel, Faires and Huff [**1**] proved that a Boolean algebra has the property (VHS) if and only if it has the property (N) and the space of Banach of real and continuous functions on the Stone space of the algebra is a Grothendieck space, (property (G)) [**6**]. A characterization of the algebras with the property (G) would be interesting in the isomorphic classification of the Banach spaces. Such characterization was conjectured by Lindenstrauss [**7**] but has not yet been satisfactorily solved. Once the equivalence between (VHS) and (G)–(N) and the question of Lindenstrauss are established, the implications (N) $\Rightarrow$ (G)? and (G) $\Rightarrow$ (N)? are naturally raised. In fact, the first of these questions appears in [**10**] and was posed by Seever. This problem was solved by Schachermayer [**9**] who shows that (N) $\not\Rightarrow$ (G) by means of two examples. We want to note that the proof of (N) is different in each case. In $J_1$, the algebra of Jordan measurable subsets of [0,1], the property (N) is inferred from the compactivity in [0,1], whereas in the other example $J_2$, it is proved directly in the algebra.

Received by the editors on March 16, 1989.

In 1983 Graves and Wheeler [**5**] obtained more examples in this way. In particular, they defined 11 algebras which satisfied (N) but failed (G). Again they made use of topological properties of the different spaces in which those algebras are defined. Hence, different techniques are used to provide the property (N). In fact, finding a unified argument in order to prove (N) in two of the examples that Graves and Wheeler studied remains in [**5**] as an open question. However, we do know that (VHS) $\Rightarrow$ (N) [**2**]. Consequently, the Boolean algebras with the property $(f)$ [**8**] also satisfy (N). Particularly, when $S$ is a compact totally disconnected $F$-space, $CO(S)$ has the property (N). In this paper we define a class of Boolean algebras by means of the property (LI) (local interpolation) in such a way that the Nikodym property holds. We prove that this class of Boolean algebras contains those known to have the property (N), i.e., the algebras with the property (VHS) as well as those introduced by Schachermayer [**9**] and by Graves and Wheeler [**5**], which have (N) but not (G). This answers the question raised in [**5**] by Graves and Wheeler mentioned above.

Before concluding the paper, we construct a new Boolean algebra in $\mathcal{P}(N)$ by means of the interproperties which arise in (LI). Such algebras satisfy (N) but fail to satisfy the property (G).

**2. Notation and definitions.** Throughout this paper, $\mathcal{S}$ is a Boolean algebra and $CO(S(\mathcal{S}))$ is the algebra (Boolean isomorphic to $\mathcal{S}$) of all clopen subsets of the Stone space $S(\mathcal{S})$, a totally disconnected compact Hausdorff space [**11**]. Elements of $\mathcal{S}$ will be identified with their clopen counterparts in $CO(S(\mathcal{S}))$. In general, $\mathrm{cl}\,(A)$, $\mathrm{int}\,(A)$, $\mathrm{bd}\,(A)$ represent the closure, interior and boundary of a subset $A$ of $S(\mathcal{S})$. If $A$, $B \subseteq S(\mathcal{S})$, then $A\Delta B = (A - B) \cup (B - A)$. If $\{A_i\}_{i=1}^{\infty}$ is a family of elements of $\mathcal{S}$, we write $\vee_{i\in I} A_i$ for the smallest element in that majorizes all $A_i$, if such an element exists. $C(S)$ denotes the Banach algebra of continuous functions on the Stone space $S$. A subset $B$ of $S$ is a zero set if it has the form $\{t : f(t) = 0\}$ for some $f \in C(S)$. The $\sigma$-algebra of Baire sets is the smallest $\sigma$-algebra $\mathrm{Ba}\,(S)$ of subsets of $S$ containing the zero sets. The $\sigma$-algebra of Borel sets is the least $\sigma$-algebra $\mathrm{Bo}\,(S)$ of subsets of $T$ containing the closed sets. A subset $B$ of $S$ is universally measurable if for each finite nonnegative regular Borel measure $\mu$ on $S$, there exist $B_1, B_2$ (depending on $\mu$) in $\mathrm{Bo}\,(S)$ with $B_1 \subseteq B \subseteq B_2$ and $\mu(B_1) = \mu(B_2)$. The $\sigma$-algebra of universally measurable subsets of $S$ is denoted by $U(S)$. Let $F$ be an algebra of subsets of $S$ containing the clopen sets. Then $J_f(F)$ and $J_{na}(F)$ denote, respectively, the algebras of members of $F$ which have finite, countable or scattered boundaries, while $J_\Delta(F) = \{B \in F : B\Delta C$ is finite for some clopen set $C\}$. Clearly, $CO(S) \subseteq J_\Delta(F) \subseteq J_f(F) \subseteq J_{na}(F)$ [**4**].

We denote a Banach space by $X$, which for simplicity is assumed real. A function $\mu : \mathcal{A} \to X$ is called a measure if it is additive. We say that a sequence $\{\mu_n\}_{n=1}$ of measures is $\mathcal{S}$-convergent when $\{\mu_n(A)\})_{n=1}$ is convergent for every $A$ in $\mathcal{S}$, and if $D$ is a family of elements in $\mathcal{A}$, then $\{\mu_n\}_{n=1}$ is said to be a $D$-Cauchy sequence if $\lim \mu_n(B)$ exists for all $B \in D$. A measure is called exhaustive if for every sequence $\{A_n\}_{n=1}$ of mutually disjoint elements of $\{||\mu(A_n)||\}_{n=1}$ tends to zero. A family of measures is called "uniformly exhaustive" if $||\mu(A_n)||$ tends to zero uniformly for $\mu \in M$.

**2.1 Definition.** A Boolean algebra $\mathcal{S}$ has the Grothendieck property (G) if every weak* convergent sequence in $C(S)^*$ is weakly convergent [**6**]. $\mathcal{S}$ verifies (G) if and only if every continuous linear map from $C(S)$ to a separable Banach space is weakly compact [**9**].

**2.2 Definition.** A Boolean algebra $\mathcal{S}$ has the Nikodym property (N) if every $\mathcal{S}$-bounded family $M$ of exhaustive measures on $\mathcal{S}$ is uniformly bounded.

$\mathcal{S}$ has (N) if $(m_0(\mathcal{S}), ||\quad||)$ is barreled [**9**], $m_0(\mathcal{S})$ being the linear subspace of $C(S)$ generated by characteristic functions of elements in $\mathcal{S}$.

**2.3 Definition.** A Boolean algebra $\mathcal{S}$ has the Vitali-Hahn-Saks property (VHS) if every $\mathcal{S}$-convergent sequence of exhaustive measure on $\mathcal{S}$ is uniformly exhaustive.

**2.4 Theorem** (Diestel-Faires-Huff [**1**]). *A Boolean algebra verifies* (VHS) *if and only if it satisfies* (N) *and* (G).

**3. The property (LI) and the Nikodym theorem.** We define

Boolean algebras for which the Nikodym theorem is valid by means of the following

**3.1 Definition.** A Boolean algebra $\mathcal{S}$ has the property (LI) if and only if for every $s \in S(\mathcal{S})$, there is a decreasing sequence $\{T_n(s)\}_{n=1}^{\infty}$ of clopen neighborhoods of $s$ such that if $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ are sequences of elements in $\mathcal{S}$ and,

$$A_n \cap B_m = \varnothing \qquad n, m \in \mathbf{N}; \qquad A_n \subseteq T_n(s); \qquad B_n \subseteq T_n(s) \qquad n \in \mathbf{N}.$$

Then there exists a subsequence $\{B_{nk}\}_{k=1}^{\infty}$, $n_{k+1} > n_k$ satisfying:

(1) there exists $A \in \mathcal{S}$

$$B_{nk} \subseteq A, \qquad A_k \cap A = \varnothing \qquad k \in \mathbf{N}$$

(2) for each $J \subseteq \mathbf{N}$ there exists $V_J \in \mathcal{A}$, with

$$B_{nk} \subseteq V_j \qquad k \in J, \qquad B_{nk} \cap V_J = \varnothing \qquad k \in \mathbf{N} - J.$$

**3.2 Lemma.** *Let $\mathcal{S}$ be a Boolean algebra with the property* (LI)*. Let $s \in S(\mathcal{S})$ and suppose $\{T_n(s)\}_{n \geq 1}$ be as in the definition of* (LI)*. Assume $\{E_n\}_{n \geq 1}$ is a sequence of pairwise disjoint elements so that for each $n$, $E_n \subseteq T_n(s)$. Then if $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of exhaustive measures on $\mathcal{S}$, there is a subsequence $\{Ep_k\}_{k=1}^{\infty}, p_{k+1} > p_k$ such that if $D$ is the $\sigma$-algebra of sets generated by $\{Ep_k\}_{k=1}^{\infty}$, for each $\mu_n$, a function $\lambda_n : D \to \mathbf{R}$ can be associated such that the following are satisfied*:

(I) *$\lambda_n$ is a $\sigma$-additive measure*

(II) *$\lambda_n(A) = \mu_n(A)$ if $A \in \mathcal{S} \cap D$*

(III) *given $B \in D$, we can find $A_B \in \mathcal{S}$ such that $\mu_n(A_B) = \lambda_n(B)$ $\forall n$.*

In particular, if $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of exhaustive measures in $\mathcal{S}$, then $\{\lambda_n\}_{n=1}^{\infty}$ will be exhaustive measures.

*Proof.* Since $\mathcal{S}$ has the property (LI), there exists a subsequence $\{E_{n_k}\}_{k=1}^{\infty}$ such that given $J \subseteq N$ there is $A_J \in \mathcal{S}$ satisfying

(1) $\qquad En_P \cap A_J \qquad p \in J; \qquad En_P \cap A_J = \varnothing \qquad p \in \mathbf{N} - J.$

In order to simplify notation, we set $En_P = E_P$. From (1) we obtain a sequence $\{W_n\}_{n=1}^{\infty}$ of pairwise disjoint elements in satisfying

(2) $$E_P \subseteq W_i \qquad p \in \mathbf{N}_i \qquad i \in \mathbf{N}.$$

Let $\mu$ be

$$\mu = \sum_{n=1}^{\infty} \frac{||\mu_n||}{2^n(||\mu_n|| + 1)}.$$

$\mu$ is an exhaustive measure because each $\mu_n$ is exhaustive. Now we perform an inductive process to construct a sequence $\{D_k\}_{k=1}$ of elements in $\mathcal{S}$ and a subsequence $\{Ep_k\}_{k=1}$ such that

(3) $$p_{k+1} > p_k \qquad k \in N$$

(4) $$D_{k+1} \subseteq D_k \qquad k \in N$$

(5) $$Ep_k \cap D_{k0} = \varnothing \qquad k < k_0$$

(6) $$Ep_k \subseteq D_{k0} \qquad k > k_0$$

(7) $$\mu(D_k) \leq 1/2^k \qquad k \in N$$

(8) $$\text{the set } \{p \in N : E_P \subseteq D_k\} \text{ is infinite } k \in N.$$

(9) $$D_k \subseteq Tp_k \qquad k \in N.$$

The sets $V_k = D_k - D_{k+1} = Ep_k$, $k \in N$ form a sequence such that

$$V_k \in \mathcal{S} : V_k \subseteq Tp_k; \qquad Ep_k \subseteq Tp_k; \qquad Ep_k \cap V_k = \varnothing \ \forall k.$$

The property LI yields a subsequence $\{Ep_{ki}\}_{i=1}, k_{i+1} > k_i$ for each $i \in N$ and an element $A$ in $\mathcal{S}$ with

$$Ep_{ki} \subseteq A \ \forall i \quad \text{and} \quad A \cap V_k = \varnothing \ \forall k.$$

Let $V = T_1 - A$ be

(10) $$V_k \subseteq T_1 - A = V \qquad \forall k.$$

(11) $$V \cap Ep_{ki} = \varnothing \qquad \forall i.$$

We will prove that the sequence $\{Ep_{ki}\}_{i=1}$ verifies the statement of the lemma. The $\sigma$-algebra $D$ generated by $\{Ep_{ki}\}_{i=1}$ is

$$D=\left\{\bigcup_{i\in J} Ep_{ki},\ J\subset N\right\}.$$

In order to define the functions $\{\lambda_n\}_{n=1}$ we note that the series $\sum\{\mu_i(Ep_k), k\in N\}$ is unconditionally convergent because if $F$ is a finite subset of $N$,

$$\left|\sum\{\mu_i(Ep_k), k\in F\}\right|\leq 2^i(1+||\mu_i||)1/2^\alpha,\qquad \alpha=\min(\alpha\in F). \tag{12}$$

Then we can set $\lambda_n: D\to R$ with

$$\lambda_n\left(\bigcup_{j\in J} Ep_{kj}\right)=\sum\{\mu_n(Ep_{kj}), j\in J\}.$$

The proofs of (I), (II) and (III) can be obtained from the following facts. According to (12), if $\{A_n\}_{n=1}$ is a sequence of pairwise disjoint subsets of $N$, the series

$$\sum_{1\leq n\leq\infty}\left(\sum\{\mu_i(E_{p_k}):\ k\in A_n\}\right)$$

converges for every $i$ and

$$\sum_{1\leq n\leq\infty}\left(\sum\{\mu_i(Ep_k,\ k\in A_n\}\right)=\sum\{\mu_i(E_{p_k}):\ k\in\cup_{n=1}A_n\}. \tag{13}$$

Moreover, if $M$ is a subset of $N$ such that $\cup_{k\in M}E_{p_k}\in\mathcal{S}$, then by (6) and (7),

$$\mu_i\left(\bigcup_{k\in M} Ep_k\right)=\sum\{\mu_i(Ep_k), k\in M\}\qquad\forall\ i. \tag{14}$$

From (1), given a subset $M$ of $N$ there exists an element of $H_M$ such that

$$Ep_{ki}\subseteq H_M\qquad i\in M;\qquad E_{p_k}\cap H_M=\varnothing\qquad k\neq k_i\qquad i\in M.$$

It follows that

$$(15) \qquad \sum\{\mu_q(Ep_{ki}), i \in M\} = \mu_q((D_1 - V) \cap H_M), \qquad q \in N,$$

is $V$, the set defined in (10) (because if $m, q \in N$, then (5), (6) and (7) are satisfied).

$$\Big|\mu_q[(D_1 - V) \cap H_M] - \sum\{\mu_q(E_{p_{ki}}), 1 \leq i \leq m, i \in M\}\Big| \leq 2^q(1+||\mu_q||)/2^{m+1}.$$

From (13), (14) and (15), conditions (I), (II) and (III) are proved. □

**3.3 Theorem.** *Let $\mathcal{S}$ be a Boolean algebra satisfying the property* (LI)*. Then $\mathcal{S}$ has the property* (N)*.*

*Proof.* Proceeding by contradiction, let us suppose that there exists a seqeunce $\{\mu_{n_k}\}_{n=1}$ of exhaustive measures which are pointwise bounded on $\mathcal{S}$ but such that

$$\lim_k ||\mu_{n_k}|| = \infty \qquad n_{k+1} > n_k.$$

To simplify notation, we set $\mu_{np} = \mu_p$, $p \in N$. Since $S(\mathcal{S})$ is compact, we can choose an element $x_0$ of $S(\mathcal{S})$ such that for each clopen neighborhood $U_0$ of $x_0$,

$$\lim_n |\mu_n|(U_0) = \infty.$$

In particular the seqeunce $\{|\mu_n|(T_k^0)\}_{n=1}$ is unbounded for each $k$. We may find a partition $(L_1, F_1)$ of $T_1$ into disjoint members of $\mathcal{S}$ and an integer $n_1$ such that

$$|\mu_{n_1}(L_1)|, \qquad |\mu_{n_1}(F_1)| > 2.$$

And at least one of

$$\inf{}_k \sup_n \inf{}_{E\in\mathcal{S}} |\mu_n(E \cap L_1 \cap T_k)|$$
$$\inf{}_k \sup_n \inf{}_{E\in\mathcal{S}} |\mu_n(E \cap F_1 \cap T_k)|$$

is infinite. If the former is infinite, set $E_1 = F_1$, otherwise set $E_1 = L_1$. In any case, there is an $n_2 > n_1$ and $(L_2, F_2)$, a disjoint partition of $(T \cap E_1 \cap T_2)$, such that

$$|\mu_{n_2}(L_2), \qquad |\mu_{n_2}(F_2)| > 3 + |\mu_{n_2}(E_1)|.$$

Now at least one of

$$\inf{}_k \sup_n \sup_{E \in \mathcal{S}} |\mu_n(E \cap L_2 \cap T_k)|$$

and

$$\inf{}_k \sup_n \sup_{E \in \mathcal{S}} |\mu_n(E \cap F_2 \cap T_k)|$$

is infinite. In the case of the former, set $E_2 = F_2$; otherwise set $E_2 = L_2$. By continuing in this way, we obtain a sequence $\{E_n\}_{n=1}$ of pairwise disjoint elements of $\mathcal{S}$, such that $E_n \subseteq T_n$ and a strictly increasing sequence of positive integers $\{n_k\}_{k=1}$ such that, for each $k \geq 1$,

$$(16) \qquad |\mu_{n_k}(E_k)| > \sum \{|\mu_{n_k}(E_j)|, \qquad 1 \leq j \leq k-1\} + k + 1.$$

Now, given that $\mathcal{S}$ has the property (LI), by using lemma 3.2 there exists a subsequence $\{E_{p_k}\}_{k=1}$ and a sequence $\{\lambda_n\}_{n=1}$ of $\sigma$-additive measures on $D$ such that

$$|\lambda_k(E)| = |\mu_{nk}(E)| \qquad E \in \mathcal{A} \cap D.$$

Consequently, by (16)

$$|\lambda_{p_k}(E_{p_k})| \geq \sum_{1 < j \leq p_k - 1} |\lambda_{p_k}(E_j)| + p_k + 1, \qquad p_k \geq 1.$$

Since the sequence $\{\lambda_{p_k}\}_{k=1}$ is $D$-convergent, this contradicts the Vitali-Hahn-Saks theorem [**5**]. □

We now show that the property (LI) alone is sufficient to prove (N). Let us note at first that Molto in [**8**] defines a class of Boolean algebras verifying the Vitali-Hahn-Saks theorems and containing a class discussed in papers by Seever [**10**] and Faires [**3**]. According to the definition of the property $(f)$ [**8**] and (LI), we obtain the following

**3.4 Proposition.** *A Boolean algebra having the property* $(f)$ *has the property* (LI)*, i.e.,* $(f) \to$ (LI).

*Remark* . With this result we have proved that the algebras verifying the Vitali-Hahn-Saks theorem also satisfy (LI). In particular,

**3.5 Corollary.** *If* $F$ *is a compact* $f$*-space totally disconnected Hausdorff, then* $CO(S)$ *has the property* (LI).

In [**9**] Schachermayer has shown that $(N) \not\to (VHS)$ by means of two examples $J_1$ and $J - 2$ which have the property $(N)$ but fail the Grothendieck and Vitali-Hahn-Saks properties. We will prove that these Boolean algebras also satisfy (LI). In order to do that, the following definition will be useful.

**3.6 Definition.** A Boolean algebra $\mathcal{S}$ has the ($\sigma$CL) if given $t \in S(\mathcal{S})$ we can find a decreasing sequence $\{T_n\}_{n\geq 1}$ of clopen neighborhoods of $t$ such that whenever $\{T_{k_n}\}_{n\geq 1}$ is a subsequence of $\{T_n\}$ and $\{A_n\}$ is a sequence of clopen sets in $S(\mathcal{S})$ with $A_n \subseteq T_{k_n}$ we have $\sup_n A_n$ exists.

It is easily proved that

**3.7 Proposition.** *If* $\mathcal{S}$ *has the property* ($\sigma$CL) *then* $\mathcal{S}$ *satisfies the property* (LI)*, i.e.,* ($\sigma$CL) $\to$ (LI).

This implication allows us to prove that the Boolean algebras quoted above have the property (LI). The converse is not true, for example $CO(\beta N - N)$ satisfies (LI) but fails ($\sigma$CL).

**3.8.** Let $J_1$ denote the family of Jordan measurable sets in [0.1].

**3.9 Proposition.** $J_1$ *has the property* (LI).

*Proof.* By using the compactness of [0.1] for each $A \in J_1$ we can consider it as a clopen subset of $S(J_1)$ the elements of [0,1] as elements of $S(J_1)$. Let $t$ be an element of [0,1], we choose

$$T_n = ]t - 1/n, t + 1/n[ \cap [0,1]$$

for every $n \in \mathbf{N}$. Since the sequence $\{T_n\}_{n=1}$ is a neighborhood base for $t$ which satisfies the hypothesis of ($\sigma$CL). Given that for each sequence $\{A_n\}_{n=1}$ in $J_1$ with $A_n \subseteq T_{kn}, k_{n+1} > k_n$ for all $n$, as the Lebesgue measure $\mu$ is purely nonatomic. Then,

$$\left(\mathrm{cl}\left(\bigcup_{n=1} A_n\right) - \mathrm{int}\left(\bigcup_{n=1} A_n\right)\right) \leq \sum_{n=1} (\mathrm{cl}\,(A_n) - \mathrm{int}\,(A_n)) + \mu(t) = 0$$

and from the definition of $J_1$, it follows that $\cup_{n=1} A_n \in J_1$. □

**3.10.** Let $X = \oplus[0,1]^i$ and let $J_2$ be the next Boolean algebra, $\{A_i\}_{i=1} \in J_2$ if

i) $A_i$ is a set of Borel in $[0.1]^i$

ii) $\varepsilon > 0\ \exists n_0/m,\ n \geq n_0 \Rightarrow \mu(A_m \Delta A_n) \leq \varepsilon$

where $\mu$ is the Lebesgue measure.

**3.11 Proposition.** *$J_2$ has the property* (LI).

*Proof.* The following convention will be helpful: Let $A$ be an element $J_2$. We write $A$ if we consider it a member of the field of $X$ and write $\psi(A)$ if we consider it as a clopen subset of $S(J_2)$. Let $t$ be an element of $S(J_2)$ and

$$U = \{A \in \mathrm{CO}(S(J_2))/t \in A\}.$$

Then $U$ is an ultrafilter in $S(J_2)$; consequently, we can consider the following $J_2$ ultrafilter in $X$, $V = \{B \in J_2/\psi(B) \in U\}$. By constructing a sequence $\{T_n\}_{n=1}$ of clopen neighborhoods for $t$, we will prove that $J_2$ satisfies ($\sigma$CL).

*Case* 1. Let us suppose that there exists an element $A = \{A_i\}_{i=1}$ in $V$ such that $\Pi_n(A) = A_n = \{q\}$ for some $q \in X$ for each $n \in N$.

Then let $T_n = \psi(A)$ for each $n \in N$. If $\{\psi(F_n)\}_{n=1}$ is a sequence in $\mathrm{CO}(S(J_2)$ with $\psi(F_n) \subseteq T_{k_n}$, $k_{n+1} > k_n$ for all $n$, it follows that

$$\bigvee_{k=1} \psi(F_k) \in \psi(J_2) \equiv \mathrm{CO}(S(J_2)).$$

*Case* 2. Let us consider

$$\begin{aligned} A_1^1 &= \{\pi_i(A_1^1)\}_{i=1}, & \pi_i(A_1^1) &= [0,1/2], & i &\in N \\ A_1^2 &= \{\pi_i(A_1^2)\}_{i=1}, & \pi_i(A_2^2) &= [1/2,1], & i &\in N. \end{aligned}$$

It is clear that $A_1^1 \in J_2$, $A_1^2 \in J_2$, $A_1^1 \cup A_1^2 = \oplus[0,1]^i$ and $\mu(\pi_i(A_1^j)) = 1/2$, $j = 1,2$. Since $V$ is a $J_2$-ultrafilter, we can suppose that $A_1^1 \in V$. Note that if $A_1^1 \cap A_1^2 \in V$, we return to Case 1. Similarly, now we consider

$$\begin{aligned} A_2^1 &= \{\pi_i(A_2^1)\}_{i=1}, & \pi_i(A_2^1) &= [0,1/4], & i &\in N \\ A_2^2 &= \{\pi_i(A_2^2)\}_{i=1}, & \pi(A_2^2) &= [1/4,1/2], & i &\in N. \end{aligned}$$

From their construction,

$$A_2^i \in J_2, \qquad \mu(\pi_j(A_2^i)) = 1/2^2 \qquad i = 1,2, \qquad j \in N$$

and

$$A_2^1 \cup A_2^2 = A_1^1.$$

So, $A_2^1$ or $A_2^2$ are elements of the ultrafilter $V$. By working in this way, we construct a sequence $\{A_n\}_{n=1}$ satisfying the following properties:

$$A_n \in V \tag{17}$$

$$A_m \subseteq A_n, \qquad m \geq n \tag{18}$$

$$\mu(\pi_i(A_n)) = 1/2^n, \qquad i,n \in N. \tag{19}$$

Let $T_n = \psi(A_n)$ hold for each $n \in N$. We can prove that $J_2$ satisfies ($\sigma$CL) with this sequence using the above facts. Let us suppose $\{\psi(F_n)\}_{n=1}$ to be a sequence in $\mathrm{CO}(S(J_2))$ with

$$\psi(F_n) \subseteq T_{k_n}, \qquad k_{n+1} > k_n \qquad \forall n. \tag{20}$$

By virtue of (18), (19) and (20), it follows that $j = 1$, $F_j \in J_2$ and hence $\vee_{j=1}\psi(F_j) = \psi(\cup_{j=1}F_j) \in \mathrm{CO}(S(J_2))$.

Thus, we have demonstrated that $J_2$ hs the property ($\sigma$CL) and by (3.7), $J_2$ satisfies (LI). □

Recently, Graves and Wheeler [**5**] have shown the fulfillment of the property (N) for various algebras $J$ of subsets in a compact, totally disconnected space $T$. The methods used by Graves and Wheeler are based upon the compactivity of $T$, but not upon $S(J)$, which in general does not coincide with $T$. By means of the following result, we prove that the quoted examples also satisfy (LI).

**3.12 Theorem.** *Let us assume that $\mu$ is a purely nonatomic measure which satisfies the following. For each $t \in T$ there exists a decreasing sequence $\{A_i\}_{i=1}$ of clopen neighborhoods of $t$ such that $\mu(F) = 0$ for every $F \in U(T)$ with $F \subset \cap_i A_i$.*

*Then the Boolean algebra $J_\mu(\mathcal{S})$ has the property* (LI), *for $\mathcal{S} = Ba(T), Bo(T)$ or $U(T)$.*

*Proof.* We will prove that $J_\mu(\mathcal{A})$ has the property (LI) by means of the implication ($\sigma$CL) → (LI).

Given that $\mathrm{CO}(T) \subseteq J_\mu(\mathcal{S})$, $\{A_i\}_{i=1}$ will be a sequence of elements of $J_\mu(\mathcal{S})$ containing $t$. Let us suppose that there is a sequence $\{F_n\}_{n=1}$ of elements in $J_\mu(\mathcal{S})$ such that $F_n \subseteq A_{k_n}$, $k_{n+1} \geq k_n$. We will prove that

$$\bigvee_{n=1} F_n = \mathrm{cl}\left(\bigcup_{n=1} F_n\right) \in J_\mu(\mathcal{S}).$$

Since the sequence of clopens $\{A_{kn}\}_{n=1}$ is decreasing, we can deduce

$$\mathrm{cl}\Bigg(\mathrm{cl}\left(\bigcup_{n=1} F_n\right)\Bigg) - \mathrm{int}\left(\mathrm{cl}\left(\bigcup_{n=1} F_n\right) - \bigcup_{n=1}(\mathrm{cl}\,(F_n)\right) - \mathrm{int}\ \ (F_n)) \cup \bigcap_{n=1} A_{kn}.$$

Given that $F_n \in J_\mu(\mathcal{S})$ and the sequence $\{A_n\}_{n=1}$ satisfies the hypothesis of this theorem, it is clear that

$$\mathrm{cl}\left(\bigcup_{n=1} F_n\right) = \bigvee_{n=1} F_n \in J_\mu(\mathcal{S}). \qquad \square$$

Hence, we obtain the following

**3.13 Corollary.** *If $X$ is a Frechet space and $\mu : \mathcal{S} \to X$ is a purely nonatomic exhaustive measure, then $J_\mu(\mathcal{S})$ has the property* (LI).

**3.14 Corollary.** *If $T$ is measurable and $\mu$ is a nonatomic probability measure on $T$, then $J_\mu(\mathcal{S})$ has* (LI).

**3.15 Corollary.** *If $T$ is measurable, then $J_{na}(\mathcal{S})$ has* (LI).

**3.16 Corollary.** *If $T$ is extremally disconnected and has no isolated points, then $J_{na}(Ba)$ satisfies* (LI).

Consequently, because of that, the implication (LI) $\to$ (N) (3.3) answers the second question raised by Graves and Wheeler [**5**] because the local interpolation is a single method useful in showing the property (N) on $J_{na}(Ba)$ when $T$ is a first countable (3.15) as well as when $T$ is an $F$-space (3.15).

We will prove that the property (LI) is a means of constructing Boolean algebras with the property (N). In particular, we will construct a new algebra of subsets of $N$ which has the property (N) and fails the property (G).

**3.17 Example.** Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint subsets of $\mathbf{N}$ so that $\mathbf{N} = \cup_{n=1} A_n$ and $\operatorname{card}(A_n) = 2^n$. We set for each $n \in N$,

$$\lambda_n : 2^N \to R$$
$$\lambda_n(E) = \operatorname{card}(E \cap A_n)/2^n.$$

We can find for each $n \in N$ a finite sequence $\{W_i^n 1 \leq i \leq 2^n\}$ of pairwise disjoint subsets of $N$ such that

$$\bigcup_{i=1}^{2^n} W_i^n = \bigcup_{p=n}^{\infty} A_p \tag{21}$$

$$\text{if } m > n \quad \text{and} \quad 1 \leq i \leq 2^m, \qquad 1 \leq j \leq 2^n \quad \text{then} \tag{22}$$
$$W_i^m \subseteq W_j^m \quad \text{or} \quad W_i^m \cap W_j^m = \varnothing$$

$$(23) \qquad n \in N \qquad \lambda_p(W_i^n) = 1/2^n \qquad 1 \leq i \leq 2^n \qquad p \geq n.$$

The elements of $\mathcal{S}_0$ (which is the algebra of sets generated by $\{W_p^n, 1 \leq p \leq 2^n, n \in N\}$) can be expressed as

$$(24) \qquad \left( \bigcup_{j=1}^{p} \overset{n_j}{W_{ij}} \right) F$$

$W_{ij}^{nj}$ being pairwise disjoint sets and $F$ a finite set. Then, according to (23), for each $E \in \mathcal{S}_0$ there exists $\lim \lambda_n(E)$. Now with Zorn's axiom, we assume the existence of a maximal algebra verifying

$$(25) \qquad \mathcal{S} \subseteq P(N)$$

$$(26) \qquad \mathcal{S}_0 \subseteq \mathcal{S}$$

$$(27) \qquad \text{there exists} \quad \lim \lambda_n(E) \qquad \forall E \in \mathcal{S}.$$

**3.18 Lemma.** *If $N$ is a subset of $N$ such that $\lim \lambda_n(N \cap E)$ exists for each $E \in \mathcal{S}$, then $N \in \mathcal{S}$.*

*Proof.* Let $V$ be a subalgebra generated by $\mathcal{S}$ and $N$. It suffices to prove that $V = \mathcal{S}$. We demonstrate that $V$ satisfies (27). Note that the elements of the algebra $V$ are in the form

$$\{(A \cap N)\Delta B / A, B \in \mathcal{S}),$$

then

$$\lambda_p[(A \cap N)\Delta B] = \lambda_p([(A - B) \cap N] \cup (B - A) \cup [(A \cap B) - N]).$$

Since $(A - B) \cap N$, $B - A$ and $[(A \cap B) - N]$ are disjoint

$$\lambda_p[(A \cap N)\Delta B - N] = \lambda_p[(A - B) \cap N] + \lambda_p(B - A) + \lambda_p[(A \cap B) - N].$$

The first two limits exist by hypothesis and

$$\lambda_p[(A \cap B) - N] = \lambda_p(A \cap B) - \lambda_p[(A \cap B) \cap N].$$

Consequently, $V = \mathcal{S}$.

**3.19 Proposition.** *$\mathcal{S}$ has the property* (N) *and fails the property* (G).

*Proof.* The sequence $\{\lambda_p\}_{p=1}^{\infty}$ shows that $\mathcal{S}$ does not have the property (VHS), (note that $\lambda_p(A_p) = 1$). We will see that it verifies the property (LI). Let $U$ be a free $\mathcal{S}$-ultrafilter as in (2.1). For each $n$ we can choose an $i_n \in N$ such that $W_{i_n}^n \in U$. We will prove that the sequence $\{W_{i_n}^n\}_{n=1}$ has the following property:

$$\text{(28)} \qquad \text{If } B_n \in \mathcal{S}, \qquad B_n \subseteq W_{i_n}^n \quad \text{then} \quad \bigcup_{n=1}^{\infty} B_n \in \mathcal{S}.$$

Since $B_n \cap E \subseteq W_{i_n}^n$ it suffices to prove that the sequence

$\{\lambda_p(\cup_{n=1} B_n)\}_{n=1}$ is convergent. Let $\varepsilon > 0$. We can choose $r \in N$ such that

$$1/2^{r-1} < \varepsilon/2.$$

Given that the sequence $\{\lambda_m(B_n)\}_{m=1}$ is convergent, we can take $r_0 \geq r$ such that if $p, q \geq r_0$, then

$$|\lambda_p(B_n) - \lambda_q(B_n)| < \varepsilon/2^r \qquad 1 \leq n \leq r.$$

Therefore, if $p, q \geq r_0$,

$$\left| \lambda_p\left( \bigcup_{n=1} B_n \right) - \lambda_q\left( \bigcup_{n=1} B_n \right) \right| < \varepsilon$$

which proves that $\cup_{n=1} B_n \in \mathcal{S}$. Consequently, the algebra $\mathcal{S}$ has the property (LI) (3.7). □

**Open question.** The property (LI) implies (N), but is the other implication also true?

**Acknowledgment.** I would like to express my appreciation to Dr. V. Del Olmo for his helpful comments and supervising the development of this paper.

## REFERENCES

**1.** J. Diestel, B. Faires, and R. Huff, *Convergence and boundedness of measures non sigma complete algebras*, preprint, 1976.

**2.** J. Diestel and J.J. Uhl, Jr., *Vector measure*, Amer. Math. Soc., Providence, RI, 1977.

**3.** B. Faires, *On the Vitali-Hahn-Saks-Nikodym type theorems*, Ann. Inst. Fourier (genoble) **26** (1976), 99–114.

**4.** L. Gillman and M. Jerison, *Rings of continuous functions*, Springer, Berlin, 1975.

**5.** W.H. Graves and R.F. Wheeler, *The Grothendieck and Nikodym properties*, Rocky Mountain J. Math. **13** (1983), 333–353.

**6.** A Grothendieck, *Sur les applications linearies faiblement compactes d'spaces du type $C(K)$*, Canad. J. Math. Soc. **5** (1953), 129–173.

**7.** J. Lindenstrauss, *On the extension of operators with range in a $C(K)$-space*, Proc. Amer. Math. Soc. **15** (1964), 218–225.

**8.** A. Molto, *On the Vitali-Hahn-Saks theorem*, Proc. Royal Soc. Edinburgh **90** (1981), 163–173.

**9.** W. Schachermayer, *On some classical measure-theoric theorems for non $\sigma$-complete Boolean algebras*, Linz University Technical Report, 1978.

**10.** G. Seever, *Measures on $F$ spaces*, Trans. Amer. Math. Soc. **133** (1968), 267–280.

**11.** R. Sikorski, *Boolean algebras*, 3rd edition, Springer, Berlin, 1964.

Universidad Politécnica Valencia, Departamento de Matemática Aplicada, Escuela Universitaria de Informática, Camino de Vera s/n, Valencia, Spain