ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 22, Number 2, Spring 1992

DUALITY IN SOME VECTOR-VALUED FUNCTION SPACES

PATRICK N. DOWLING

ABSTRACT. We prove two results concerning duality in some function spaces. First we show that for $1 \leq p \leq \infty$ and X a complex Banach space, the space $H^p(D, X^*)$ is isometrically isomorphic to a dual space and we use this result to get a characterization of the analytic Radon-Nikodym property in dual spaces. Second, we show that if Λ is an infinite Sidon subset of the dual of a compact abelian metrizable group, if X is a Banach space and $1 \leq p \leq \infty$, then $L^p_\Lambda(G, X^*)$ is a dual space if and only if X^* does not contain a copy of c_0.

1. Introduction. In [**3**] Bochner and Taylor proved that if $1 \leq p < \infty$, $1/p + 1/q = 1$ and X is a Banach space, then $(L^p([0,1]; X))^* = L^q([0,1]; X^*)$ if and only if X^* has the Radon-Nikodym property with respect to Lebesgue measure on $[0,1]$. They also gave a representation of $(L^p([0,1]; X))^*$ when $1 \leq p < \infty$ and X is any Banach space. In this note we make use of this representation in two settings. In Section 2 we will show that $H^p(D, X^*)$ is a dual space where X is a Banach space and $1 \leq p \leq \infty$. As an application we obtain a new characterization of the analytic Radon-Nikodym property in dual spaces. In Section 3, we consider the function space $L^p_\Lambda(G, X^*)$, where G is a compact abelian metrizable group, Λ is a Sidon subset of the dual group of G and X is a Banach space. We show that $L^p_\Lambda(G, X^*)$ is a dual space for $1 \leq p \leq \infty$, if and only if X^* does not contain a copy of c_0.

2. The analytic Radon-Nikodym property. We denote by $(\Pi, \mathcal{B}, m)$ the Lebesgue space on the unit circle Π with $m(\Pi) = 1$ and D will denote the open unit disk in the complex plane.

Let X be a complex Banach space and let $1 \leq p \leq \infty$. The space $H^p(D, X)$ consists of all holomorphic functions $f : D \to X$ satisfying

Received by the editors on August 11, 1989, and in revised form on January 22, 1990.

$||f||_p < \infty$ where

$$||f||_p = \sup_{0 \le r < 1} \left(\int_0^{2\pi} ||f(re^{it})||^p \, dm(t) \right)^{1/p}$$

for $1 \le p < \infty$ and

$$||f||_\infty = \sup_{z \in d} ||f(z)||.$$

If $f : \Pi \to X$ is a Bochner integrable function, then its Fourier coefficients are

$$\hat{f}(n) = \int_0^{2\pi} f(e^{it}) e^{-\mathrm{int}} \, dm(t)$$

for each $n \in \mathbf{Z}$.

Similarly, if F is a vector measure on Π, its Fourier coefficients are

$$\hat{F}(n) = \int_0^{2\pi} e^{-\mathrm{int}} \, dF(t)$$

for each $n \in \mathbf{Z}$.

For $1 \le p \le \infty$ we define the following spaces

$$H^p(\Pi, X) = \{f \in L^p(\Pi, X) : \hat{f}(n) = 0 \text{ for all } n < 0\}$$

and

$$H_0^p(\Pi, X) = \{f \in L^p(\Pi, X) : \hat{f}(n) = 0 \text{ for all } n \le 0\}.$$

For a vector measure $F : \mathcal{B} \to X$ we define

$$\mathbf{E}(F|\pi) = \sum_{E \in \pi} \frac{F(E)}{m(E)} \chi_E,$$

where π is a finite measurable partition of Π, along with the convention $0/0 = 0$. For $1 \le p \le \infty$, the space $V^p(\Pi, X)$ consists of all vector measures $F : \mathcal{B} \to X$ with $||F||_p < \infty$ where

$$||F||_p = \sup_\pi ||\mathbf{E}(F|\pi)||_{L^p(\Pi, X)},$$

and the supremum is over all finite measurable partitions π of Π.

In [**3**] Bochner and Taylor proved that for $1 < p \leq \infty$ and $1/p+1/q = 1$ the space $V^p(\Pi, X^*)$ is isometrically isomorphic to $(L^q(\Pi, X))^*$. It was shown by Singer [**13**] that $V^1(\Pi, X^*)$ is isometrically isomorphic to $(C(\Pi, X))^*$, where $C(\Pi, X)$ is the space of continuous X-valued functions on Π, with the supremum norm.

Finally, let us recall the following result of Blasco [**2**].

Theorem 1. *Let X be a complex Banach space and let $1 \leq p \leq \infty$. Then $H^p(D, X)$ is isometrically isomorphic to $V_a^p(\Pi, X)$, where*

$$V_a^p(\Pi, X) = \{F \in V^p(\Pi, X) : \hat{F}(n) = 0 \textit{ for all } n < 0\}.$$

Combining Blasco's result with those of Bochner and Taylor [**3**] and Singer [**13**] gives us

Corollary 2. *Let X be a complex Banach space.*

(a) *For $1 < p \leq \infty$, $H^p(D, X^*)$ is isometrically isomorphic to $(L^q(\Pi, X)/H_0^q(\Pi, X))^*$, where $1/p + 1/q = 1$ and*

(b) *$H^1(D, X^*)$ is isometrically isomorphic to $(C(\Pi, X)/A_0(\Pi, X))^*$ where $A_0(\Pi, X) = \{f \in C(\Pi, X) : \hat{f}(n) = 0$ for all $n \leq 0\}$.*

Proof. (a) By Theorem 1, $H^p(D, X^*)$ is isometrically isomorphic to $V_a^p(\Pi, X^*)$. By [**3**], $V^p(\Pi, X^*)$ is isometrically isomorphic to $(L^q(\Pi, X))^*$, for $1/p + 1/q = 1$, under the obvious correspondence. It is easy to show that $(H_0^q(\Pi, X))^\perp$ is isometrically isomorphic to $V_a^p(\Pi, X^*)$ and so (a) clearly follows. The proof of (b) is similar to (a) and can also be found in [**10**].

Definition. A complex Banach space X is said to have the analytic Radon-Nikodym property if $H^1(\Pi, X)$ is isometrically isomorphic to $H^1(D, X)$ under the correspondence

$$F(re^{i\theta}) = \int_0^{2\pi} P_r(\theta - t) f(e^{it})\, dm(t),$$

where $F \in H^1(D, X)$, $f \in H^1(\Pi, X)$ and $P_r(\theta - t)$ is the Poisson kernel ($0 \leq r < 1$, $0 \leq \theta \leq 2\pi$).

This property was introduced in [**5**] by Bukhvalov and Danilevich who showed that if 1 is replaced by p for any $p \in [1, \infty]$, then the property remains the same.

Theorem 3. *Let X be a complex Banach space.*

(a) *If $1 < p \le \infty$ and $1/p + 1/q = 1$, then X^* has the analytic Radon-Nikodym property if and only if the natural inclusion of $H^p(\Pi, X^*)$ into $(L^q(\Pi, X)/H_0^q(\Pi, X))^*$ is surjective.*

(b) *X^* has the analytic Radon-Nikodym property if and only if the natural inclusion of $H^1(\Pi, X^*)$ into $(C(\Pi, X)/A_0(\Pi, X))^*$ is surjective.*

The proof of Theorem 3 is an easy application of Theorem 1 and Corollary 2.

Remark. $H^p(\Pi, X)$ is always isometrically isomorphic to a subspace of $H^p(D, X)$, but, in general, the two spaces may be quite different. For example, by Corollary 2, $H^p(D, l_\infty)$ is a dual space when $1 \le p \le \infty$. However, by a slight modification of Bourgain's proof in [**4**], we can show that $H^p(\Pi, l_\infty)$ is not a dual space when $1 \le p \le \infty$. In fact, if $1 \le p < \infty$, then $H^p(\Pi, l_\infty)$ contains a complemented copy of c_0 (see [**8**]) and so it is not a dual space [**1**].

3. The Λ-Radon-Nikodym property. Let G be a compact abelian metrizable group, let $\mathcal{B}(G)$ denote the σ-algebra of Borel subsets of G and let λ be normalized Haar measure on $\mathcal{B}(G)$. We can define $L^p(G, X)$ and $V^p(G, X)$ for a Banach space X in the obvious manner. Let Γ denote the dual group of G. If $\mu \in V^p(G, X)$ and $\gamma \in \Gamma$, then the Fourier coefficient $\hat{\mu}(\gamma)$ is defined by

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma(g)}\, d\mu(g).$$

We can similarly define $\hat{f}(\gamma)$ for $f \in L^p(G, X)$. If $\Lambda \subseteq \Gamma$, we let

$$L^p_\Lambda(G, X) = \{f \in L^p(G, X) : \hat{f}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}$$

and

$$V^p_\Lambda(G, X) = \{\mu \in V^p(G, X) : \hat{\mu}(\gamma) = 0 \text{ for all } \gamma \notin \Lambda\}$$

for $1 \leq p \leq \infty$.

Definition. [**9**]. A Banach space X is said to have the Λ-Radon-Nikodym property if and only if $V_\Lambda^\infty(G,X) = L_\Lambda^\infty(G,X)$.

Remarks. 1) When we write "$V_\Lambda^\infty(G,X) = L_\Lambda^\infty(G,X)$" we mean that the natural inclusion of $L_\Lambda^\infty(G,X)$ into $V_\Lambda^\infty(G,X)$ is surjective.

2) If $G = \Pi$, then $\Gamma = \mathbf{Z}$. In this case $\mathbf{Z}$-Radon-Nikodym property is equivalent to the Radon-Nikodym property and $\mathbf{N}$-Radon-Nikodym property is equivalent to the analytic Radon-Nikodym property.

3) If Λ is finite then every Banach space has the Λ-Radon-Nikodym property and if Λ is infinite then c_0 fails the Λ-Radon-Nikodym property.

Proposition 4. [**9**]. *Let G be a compact abelian metrizable group, $\Lambda \subseteq \Gamma$, $\Lambda' = \{\gamma \in \Gamma : \bar{\gamma} \notin \Lambda\}$ and let X be a Banach space. Then X has the Λ-Radon-Nikodym property if and only if for every bounded linear operator $T : L^1(G)/L^1_{\Lambda'}(G) \to X$, the operator Tq is representable where $q : L^1(G) \to L^1(G)/L^1_{\Lambda'}(G)$ is the natural quotient.*

This result will now be used to characterize the Λ-Radon-Nikodym property when Λ is a Sidon subset of Γ. Recall that Λ is a Sidon set if and only if $C_\Lambda(G)$ is isomorphic to $l^1(\Lambda)$.

Proposition 5. *If Λ is a Sidon subset of Γ, then every Banach space not containing a copy of c_0 has the Λ-Radon-Nikodym property.*

Proof. If Λ is a finite subset of Γ, then we have the result trivially.

If Λ is an infinite Sidon set, then $L^1(G)/L^1_{\Lambda'}(G)$ is isomorphic to c_0 [**11**, p. 121]. Therefore, if X is a Banach space not containing a copy of c_0, then every bounded linear operator $T : L^1(G)/L^1_{\Lambda'}(G) \to X$ is compact [**6**; p. 113, exercise 2]. Consequently, Tq is a compact operator and so it is representable. By Proposition 4, X has the Λ-Radon-Nikodym property.

In [**9**] Edgar asked the following question: If Λ is a Riesz subset of Γ

and X has the Λ-Radon-Nikodym property is $V^1_\Lambda(G,X) = L^1_\Lambda(G,X)$? (A subset Λ of Γ is a Riesz set if $V^1_\Lambda(G) = L^1_\Lambda(G)$).

We will now give a sufficient condition for $V^1_\Lambda(G,X) = L^1_\Lambda(G,X)$, which, in particular, applies to Sidon sets.

Proposition 6. *Let Λ be a Riesz subset of Γ and let X be a Banach space. If $L^1(G,X)$ has the Λ-Radon-Nikodym property then $V^1_\Lambda(G,X) = L^1_\Lambda(G,X)$.*

Proof. Let $\mu \in V^1_\Lambda(G,X)$ and define an operator

$$T : L^1(G) \to L^1(G,X)$$

by $T(f) = f * \mu$ for all $f \in L^1(G)$. For $\gamma \in \Gamma$, $T(\gamma) = \gamma * \mu = \hat{\mu}(\gamma)\gamma$. Therefore, $T(\gamma) = 0$ for all $\gamma \notin \Lambda$. Let us note that in the notation of [**9**], $\bar{\Lambda}' = \{\gamma \notin \Lambda\}$. Hence, $T|_{L^1_{\bar{\Lambda}'(G)}} \equiv 0$. Thus, there exists a bounded linear operator $S : L^1(G)/L^1_{\bar{\Lambda}'}(G) \to X$ such that $T = Sq$ where $q : L^1(G) \to L^1(G)/L^1_{\bar{\Lambda}'}(G)$ is the natural quotient. It is easily seen that if a Banach space has the Λ-Radon-Nikodym property then it has the $\bar{\Lambda}$-Radon-Nikodym property. Consequently, if $L^1(G,X)$ has the Λ-Radon-Nikodym property, it has the $\bar{\Lambda}$-Radon-Nikodym property and so T is a representable operator by Proposition 4. Hence, there exists a function $g \in L^\infty(G, L^1(G,X))$ such that

$$T(f) = \int_G f(t)g(t)\, d\lambda(t)$$

for all $f \in L^1(G)$; that is,

$$f * \mu = \int_G f(t)g(t)\, d\lambda(t).$$

To complete the proof, apply the same methods as in Coste's Theorem [**7**, pages 90–92].

Corollary 7. *If Λ is a Sidon subset of Γ and X does not contain a copy of c_0, then $V^1_\Lambda(G,X) = L^1_\Lambda(G,X)$.*

Proof. If X does not contain a copy of c_0, then $L^1(G, X)$ does not contain a copy of c_0 [**11**]. Thus, $L^1(G, X)$ has the Λ-Radon-Nikodym property when Λ is a Sidon set by Proposition 5. Apply Proposition 6 to complete the proof.

Remark. It is easily seen that if $V^1_\Lambda(G, X) = L^1_\Lambda(G, X)$, then $V^p_\Lambda(G, X) = L^p_\Lambda(G, X)$ for all $1 \leq p < \infty$.

Theorem 8. *Let Λ be an infinite Sidon subset of Γ, let X be a Banach space and let $1 \leq p \leq \infty$. Then $L^p_\Lambda(G, X^*)$ is a dual space if and only if X^* does not contain a copy of c_0.*

Proof. If X^* does not contain a copy of c_0, then X^* has the Λ-Radon-Nikodym property by Proposition 6. By Corollary 7 and the remark, $L^p_\Lambda(G, X^*) = V^p_\Lambda(G, X^*)$ for $1 \leq p \leq \infty$.

By [**2, 7**] and methods similar to those in Section 2,

$$V^p_\Lambda(G, X^*) = \left(\frac{L^q(G, X)}{L^q_{\bar{\Lambda}'}(G, X)}\right)^*$$

for $1 < p \leq \infty$ and $1/p + 1/q = 1$, and

$$V^1_\Lambda(G, X^*) = \left(\frac{C(G, X)}{C_{\bar{\Lambda}'}(G, X)}\right)^*$$

where $C(G, X)$ is the space of continuous X-valued functions on G, with the supremum norm, and $C_{\bar{\Lambda}'}(G, X)$ is the space of $C(G, X)$-functions whose Fourier coefficients vanish off Λ'. Therefore, $L^p_\Lambda(G, X^*)$ is a dual space for $1 \leq p \leq \infty$.

Conversely, suppose X^* contains a copy of c_0. Then an application of Bourgain's result [**4**] shows that $L^p_\Lambda(G, X^*)$ is not a dual space for $1 \leq p \leq \infty$.

Remark. If Λ is infinite, $1 \leq p \leq \infty$ and X is a Banach space such that X^* contains a copy of c_0, then $L^p_\Lambda(G, X^*)$ contains a complemented copy of c_0 (see [**8**]), and so $L^p_\Lambda(G, X^*)$ is not a dual space [**1**].

Acknowledgments. This paper was written while the author was at Ohio State University. The author wishes to thank Professor G.A. Edgar for many helpful discussions relating to this paper. Finally, the author wishes to thank the referee for some very helpful comments.

REFERENCES

1. C. Bessaga and A. Pelczynski, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–164.

2. O. Blasco, *Les valeurs sur la frontiere des functions de* $H^p_B(D)$ $(1 \leq p \leq \infty)$, VII Congresso do Grupo de Mathematicos de Expressão Latino, Vol. **II** (1985), 53–56.

3. S. Bochner and A.E. Taylor, *Linear functionals on certain spaces of abstractly-valued functions*, Ann. Math. **39** (1938), 913–944.

4. J. Bourgain, *A note on the Lebesgue spaces of vector-valued functions*, Bull. Soc. Math. Belg. **31** (1978), 45–47.

5. A.V. Bukhvalov and A.A. Danilevich, *Boundary properties of analytic functions with values in Banach space* (Russian), Mat. Zametki **31** (1982), 103–114; English translation, Math. Notes Acad. Sci. USSR **31** (1982), 104–110.

6. J. Diestel, *Sequences and series in Banach spaces*, Springer-Verlag, Grad. Texts Math. **92** (1984).

7. J. Diestel and J.J. Uhl, Jr., *Vector measures*, Math. Surveys **15**, Amer. Math. Soc., Providence, RI, 1977.

8. P.N. Dowling, *Complemented copies of* c_0 *in vector-valued Hardy spaces*, Proc. Amer. Math. Soc. **107** (1989), 251–254.

9. G.A. Edgar, *Banach spaces with analytic Radon-Nikodym property and compact abelian groups*, Proc. of the International Conference on Almost Everywhere Convergence in Probability and Ergodic Theory, Academic Press, Inc., Boston (1989), 195–213.

10. W. Hensgen, *Hardy-Räume vektorwertiger funktionen*, Dissertation, Munich, 1986.

11. S. Kwapien, *On Banach spaces containing* c_0, Studia Math. **52** (1974), 187–188.

12. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure Appl. Math. **12**, Interscience, New York, 1962.

13. I. Singer, *Les duals des certains espaces de Banach de champ de vecteurs* I, Bull. Sci. Math. **82** (1958), 29–40.

Department of Mathematics and Statistics, Miami University, Oxford, OH 45056