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PRIME IDEALS AND FINITENESS CONDITIONS FOR GABRIEL TOPOLOGIES OVER COMMUTATIVE RINGS

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ABSTRACT. We examine some finiteness conditions on Gabriel topologies, particularly over commutative rings. We study special classes of topologies defined by sets of prime ideals and provide examples to establish that a number of these classes are distinct.

Introduction. The equivalent concepts of Gabriel topologies, hereditary torsion theories, idempotent radicals and idempotent kernel functors, were originally investigated, starting about 1960, with an eye to extending localization techniques, so useful in commutative algebra, to the realm of noncommutative rings. However, relatively early in the study of Gabriel topologies, in 1973, P.J. Cahen noted there were interesting questions about such topologies even in the context of commutative rings. More recently, other investigators have focused on this context, looking at either Gabriel topologies (e.g., [**2, 3, 5, 13**]) or idempotent kernel functors (e.g., [**4, 19, 20**]). This paper is related to work done by these people.

Finiteness conditions of one sort or another, either on Gabriel topologies, or on the underlying ring, play a crucial role in the work cited above. So do prime ideals. One indication of the connection between finiteness conditions and prime ideals is that if R is a *noetherian* commutative ring, then every Gabriel topology on R is determined by a set of prime ideals. It is not surprising that many of the finiteness conditions on Gabriel topologies referred to in the literature imply that the topology is determined by prime ideals. The investigation of topologies determined by prime ideals will be the primary goal of this paper, and the focus of Sections 1 and 2.

In studying Gabriel topologies over commutative but not noetherian rings, it is often necessary to restrict the topologies in order to obtain

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reasonable results. As in the broader context of commutative algebra, competing conditions vie as the natural hypotheses which yield generalizations of results known in the noetherian case. In Section 3 we will examine a number of hypotheses which have appeared in the literature, sorting out logical implications among them. It has not always been clear in the literature whether these conditions are all different (at least two have appeared with the same name in different places), and we will provide examples to show that they are indeed logically distinct.

1. Gabriel topologies and prime ideals. In this section we will first review how a set $\mathcal{P}$ of primes determines a Gabriel topology $\mathcal{L}(\mathcal{P})$. We will provide characterizations for those topologies $\mathcal{L}$ which are of the form $\mathcal{L}(\mathcal{P})$, and discuss conditions analogous to, but weaker than the underlying ring R being noetherian, which imply that $\mathcal{L}$ is an $\mathcal{L}(\mathcal{P})$. We will use [**17**], particularly Chapter 6, as a general reference for Gabriel topologies and related concepts. Many of our basic definitions and comments may also be found in the introductory material of [**2, 5**]. Unless otherwise stated, we will assume we are dealing with Gabriel topologies over a commutative ring R.

Let R be any ring (commutative or not). A nonempty set $\mathcal{L}$ of right ideals of R is a Gabriel topology if it satisfies these conditions [**17**, Chap. 6, Lemma 5.2]:

(T3) If I is in $\mathcal{L}$ and b is in R, then $(I : b)$ is in $\mathcal{L}$.

(T4) If J is in $\mathcal{L}$ and I is an ideal of R such that $(I : b)$ is in $\mathcal{L}$ for all b in J, then I is in $\mathcal{L}$.

For R commutative, (T3) and (T4) can be replaced by (T4) and *either* of the simpler conditions:

(T0) R is in $\mathcal{L}$.

(T1) If I is $\mathcal{L}$ and $I \subseteq J$, then J is in $\mathcal{L}$.

Let $\mathcal{P}$ be a set of prime ideals of R. For p in $\mathcal{P}$ write $\mathcal{U}(p)$ for the set of ideals of R which are not contained in p. Define

$$\mathcal{L}(\mathcal{P}) = \{\text{ideals } I \text{ of } R | I_p = R_p \text{ for all } p \text{ in } \mathcal{P}\} = \bigcap_{p\in\mathcal{P}} \mathcal{U}(p).$$

It is straightforward to verify that $\mathcal{L}(\mathcal{P})$ is a Gabriel topology.

A set $\mathcal{P}$ of primes gives rise to the same topology as its *generic closure*, i.e., as the set of all prime ideals contained in some prime of $\mathcal{P}$, and, moreover, the family of generically closed subsets of $\mathrm{Spec}(R)$ parameterizes the set of Gabriel topologies of the form $\mathcal{L}(\mathcal{P})$ [**2**, Section 3].

Definitions. We will call a Gabriel topology of the form $\mathcal{L}(\mathcal{P})$, for some subset $\mathcal{P}$ of $\mathrm{Spec}(R)$, a *primal* topology. The primal topologies are the *half-centered*, or *HC* topologies of [**5, 13, 15**]. Cahen [**6**] introduced the latter terminology to describe a Gabriel topology $\mathcal{L}$ (with associated class of torsion modules $\mathcal{T}$) such that M is in $\mathcal{T}$ if and only if every weakly associated prime of M is in $\mathcal{L}$. We prefer the work *primal* in our setting because it is more suggestive of the (equivalent) property on which we are focusing.

A topology $\mathcal{L}$ is said to be of *finite type* if it has a basis of finitely generated ideals, i.e., if every I in $\mathcal{L}$ contains a finitely generated ideal which is also in $\mathcal{L}$. For $\mathcal{L}(\sigma)$, the Gabriel topology associated to an idempotent kernel functor σ, Caenepeel [**4**] used the phrase *every ideal of* $\mathcal{L}(\sigma)$ *is* σ*-finitely generated* to describe $\mathcal{L}(\sigma)$ being of finite type, and in this case labelled σ as being *noetherian*. It should be noted that the use of the latter word will be inconsistent with our usage in Section 3.

A topology $\mathcal{L}$ is *principal* [**7**], or a 1-*topology* [**17**], if each ideal I of $\mathcal{L}$ contains a principal ideal that is in $\mathcal{L}$.

The following result gives three intrinsic characterizations of primal topologies, one of which is usually easier to work with than the definition.

Theorem 1.1. *The following conditions on a Gabriel topology* $\mathcal{L}$ *are equivalent*:

(a) $\mathcal{L} = \mathcal{L}(\mathcal{P})$ *for some set* $\mathcal{P}$ *of prime ideals of* R.

(b) *If* I *is an ideal of* R *not in* $\mathcal{L}$, *then there is a prime ideal* p *of* R *not in* $\mathcal{L}$ *and containing* I.

(c) $\mathcal{L}$ *is an intersection of topologies of finite type.*

(d) $\mathcal{L}$ *is an intersection of principal topologies.*

Proof. (a) $\Leftrightarrow$ (b) [**17**, Prop. 6.13].

(a) $\Leftrightarrow$ (c) For p a prime ideal of R we have that

$$\mathcal{L}(\{p\}) = \{\text{ideals } I \text{ of } R | I \text{ is not contained in } p\}.$$

Hence, $\mathcal{L}(\{p\})$ is of finite type. Since for $\mathcal{P}$ a set of primes

$$\mathcal{L}(\mathcal{P}) = \bigcap_{p \in \mathcal{P}} \mathcal{L}(\{p\}),$$

it follows that a primal topology is an intersection of topologies of finite type. The converse follows from the observations that if $\mathcal{L}$ is of finite type, then $\mathcal{L} = \mathcal{L}(\mathcal{P})$ for $\mathcal{P}$ the set of primes not in $\mathcal{L}$ [**17**, Corollary 6.15] and that the intersection of primal topologies is primal.

(a) $\Leftrightarrow$ (d) For p a prime ideal the topology $\mathcal{L}(\{p\})$ is principal. Hence, the proof of the equivalence of (a) and (c) shows the equivalence of (a) and (d). $\square$

Apropos of condition (c) of the last theorem, we now turn to the condition that $\mathcal{L}$ is of finite type, which can be characterized in an interesting way. An arbitrary family $\mathcal{L}$ of ideals of R (commutative or not) will be called *noetherian* if whenever the union of a *countable* ascending chain of ideals of R totally ordered by inclusion is in $\mathcal{L}$, then one of the ideals must be in $\mathcal{L}$. This terminology goes back to [**9**]. We will say $\mathcal{L}$ is *strongly noetherian* if the same union condition holds for all totally ordered chains of ideals, rather than just for countable ones.

Golan [**7**, Chapter 3, Section 14] gave a number of conditions equivalent to $\mathcal{L}$ being noetherian (phrased in the language of torsion theories rather than Gabriel topologies). Our next result adds to these conditions.

Theorem 1.2. *Let R be any ring, possibly noncommutative, $\mathcal{L}$ a family of right ideals of R, satisfying property* (T1). *The following conditions are equivalent*:

(a) *$\mathcal{L}$ is of finite type.*

(b) *$\mathcal{L}$ is strongly noetherian.*

Proof. (a) $\Rightarrow$ (b). Suppose $\mathcal{L}$ is of finite type and that $\{I_i\}$ is a family of right ideals of R, totally ordered by inclusion, such that $I = \cup I_i$ is in $\mathcal{L}$. Then I contains a finitely generated right ideal J which is in $\mathcal{L}$. Being finitely generated, J is contained in some I_i. Because $\mathcal{L}$ satisfies (T1), I_i is in $\mathcal{L}$.

(b) $\Rightarrow$ (a). Suppose $\mathcal{L}$ is strongly noetherian. Let A be a right ideal of $\mathcal{L}$. Let T be a set of generators of A. We may assume T is infinite and well ordered. Hence, we may identify T with a limit ordinal p. For $z < p$, define B_z to be the right ideal of R generated by all y with $y < z$. Then $\{B_z | z < p\}$ is a family of right ideals totally ordered by inclusion and whose union is A, hence, in $\mathcal{L}$. Because $\mathcal{L}$ is strongly noetherian, some B_z is in $\mathcal{L}$. Let

$$Z = \{z | z < p,\ B_z + J \text{ is in } \mathcal{L},\ J \text{ some finitely generated right ideal in } A\}.$$

Z is nonempty, so it has a least element t. Let J be a finitely generated right ideal contained in A such that $B_t + J$ is in $\mathcal{L}$. If t is a finite ordinal, then B_t is finitely generated, and it follows from the definition of Z that A contains a finitely generated right ideal of $\mathcal{L}$. We shall show that a contradiction arises from assuming t is not finite.

Suppose t is not finite. Then $t = q + n$ with q a limit ordinal, $q < p$ and n finite. Let K be the right ideal generated by $q, q+1, \ldots, q+n$. Arguing as above, there exists $s < t$ with $B_s + K + J$ in $\mathcal{L}$. This contradicts the minimality of t in Z and completes the proof of the theorem. $\square$

Theorem 1.2 is related to an interesting topological fact that does not appear to be widely known, namely that a topological space X is compact if and only if

(∗) For every covering $X = \cup_{i \in I} X_i$, with $\{X_i\}_{i \in I}$ a family of open sets totally ordered by inclusion, $X = X_j$ for some j in I

[**12**, Chap. 5, Exercise H]. The method used to show that (b) implies (a) can be easily adapted to prove this characterization of compactness. On the other hand, the topological fact provides an alternative proof that (b) implies (a) in the special case where $\mathcal{L} = \mathcal{L}(\mathcal{P})$ is a primal topology (keeping in mind that $\mathcal{L}(\mathcal{P})$ is of finite type if and only if $\mathcal{P}$ is compact.)

As we indicated in the introduction, if R is noetherian, then every Gabriel topology $\mathcal{L}$ is an $\mathcal{L}(\mathcal{P})$, since $\mathcal{P}$ may be taken as the set of prime ideals not in $\mathcal{L}$. This is a consequence of Theorem 1.1. The next result shows that the same conclusion holds if every *prime* ideal in $\mathcal{L}$ contains a finitely generated ideal which is in $\mathcal{L}$. This result also suggests a variant theorem which generalizes I.S. Cohen's theorem that if every prime ideal of R is finitely generated, then every ideal of R is finitely generated [**1**, Chap. 7, Exercise 1] or [**11**, Section 1.1, Theorem 7]). Our proof uses the ideas in the proof of Cohen's theorem.

Theorem 1.3. *Let R be a commutative ring. Let $\mathcal{L}$ be a family of ideals of R satisfying property* (T1)*. Let $\mathcal{S}$ be a family of ideals of R having the following properties*:

(i) *If $\{I_i\}$ is a family of ideals totally ordered by inclusion and $\cup I_i$ is in $\mathcal{S}$, then some I_i is in $\mathcal{S}$.*

(ii) *If I is an ideal of R such that $I+(b)$ and $(I:b)$ are in $\mathcal{S}$ for some b in R, then I is in $\mathcal{S}$*;

Suppose every prime ideal of $\mathcal{L}$ is in $\mathcal{S}$. Then $\mathcal{L} \subseteq \mathcal{S}$.

Proof. Suppose $\mathcal{B}$, the family of ideals in $\mathcal{L}$ which are not in $\mathcal{S}$, is not empty. Then $\mathcal{B}$ has a maximal element, call it I, by (i). It suffices to show that I is a *prime* ideal.

If I is not prime, then there exists b in R such that $I+(b)$, $(I:b)$ contain I properly. These ideals are in $\mathcal{L}$ by (T1), and I is a maximal element of $\mathcal{B}$. It follows that $I+(b)$, $(I:b)$ are in $\mathcal{S}$. Then I is in $\mathcal{S}$, by (ii).

Corollary 1. *Suppose $\mathcal{L}$ is a Gabriel topology such that every prime ideal of $\mathcal{L}$ is finitely generated. Then every ideal of $\mathcal{L}$ is finitely generated.*

Proof. Take $\mathcal{S}$ the family of finitely generated ideals of R. It is easy to see that (i) holds for this $\mathcal{S}$. Condition (ii) holds by an argument used in the proof of Cohen's theorem: if $I+(b)=(a_1+r_1b,\dots,a_n+r_nb)$ and $(I:b)=(c_1,\dots,c_m)$, then $I=(a_1,\dots,a_n,c_1b,\dots,c_mb)$. $\square$

Corollary 2. *Suppose $\mathcal{L}$ is a Gabriel topology such that every prime ideal of $\mathcal{L}$ contains a finitely generated ideal of $\mathcal{L}$. Then every ideal of $\mathcal{L}$ contains a finitely generated ideal of $\mathcal{L}$, i.e., $\mathcal{L}$ is of finite type.*

Proof. Let $\mathcal{S}$ be the family of ideals of $\mathcal{L}$ which contain a finitely generated ideal of $\mathcal{L}$. It is easy to see that $\mathcal{S}$ satisfies (i) and (ii). $\square$

Pakala shows [**13**, Prop. 7.1] that the hypotheses of our Corollary 2 imply that $\mathcal{L}$ is WC (Cahen's *well-centered* criterion, that a module M is torsion free with respect to $\mathcal{L}$ if and only if each of its weakly associated primes is outside $\mathcal{L}$). This result follows from our Corollary 2 since if $\mathcal{L}$ is of finite type, it is WC [**13**, Corollary 7.3].

2. Special classes of primal topologies. In this section we will give some natural constructions which give rise to primal Gabriel topologies and characterize which primal topologies arise in this way.

The first construction is related to a class of examples given in [**17**, Chap. 6, Prop. 6.10]. The examples there are defined for a family $\mathcal{B}$ of finitely generated ideals of R. We will restrict ourselves to $\mathcal{B} = \{B\}$, but we will not need to assume B is finitely generated.

For B any ideal of R define

$$\mathcal{R}(B) = \{I | I \text{ is an ideal of } R \text{ with } B \subseteq \sqrt{I}\},$$

where

$$\sqrt{I} = \{x | x^n \in I \text{ for some integer } n \geq 0\} = \bigcap_{p \in V(I)} p$$

and $V(I)$ is the set of prime ideals of R containing I.

Let $\mathcal{P} = D(B)$, the set of prime ideals not containing B. It is straightforward to verify that an ideal I of R is in $\mathcal{R}(B)$ if and only if I is not contained in any prime of $D(B)$. Hence, $\mathcal{R}(B) = \mathcal{L}(\mathcal{P})$, showing that $\mathcal{R}(B)$ is not only a Gabriel topology, but a primal one as well.

Theorem 2.1. *Let $\mathcal{L}$ be a Gabriel topology. $\mathcal{L} = \mathcal{R}(B)$ for some ideal B of R if and only if $\mathcal{L}$ satisfies these conditions*:

(1) *If* $\sqrt{I}$ *is in* $\mathcal{L}$*, then* I *is in* $\mathcal{L}$*.*

(2) $\cap_{I\in\mathcal{L}}\sqrt{I}$ *is an ideal of* $\mathcal{L}$*.*

Proof. It is easy to see that if (1) and (2) hold, then $\mathcal{L} = \mathcal{R}(B)$ for $B = \cap_{I\in\mathcal{L}}\sqrt{I}$. We omit the other details. □

It is clear that $\mathcal{R}(B) = \mathcal{R}(\sqrt{B})$ for B any ideal of R, and that

$$\sqrt{B} = \bigcap_{I\in\mathcal{R}(B)} \sqrt{I}.$$

Hence, there is a bijective correspondence between Gabriel topologies $\mathcal{R}(B)$ and *radical ideals*, i.e., ideals B such that $B = \sqrt{B}$. We will therefore refer to the topologies $\mathcal{R}(B)$ as *radical topologies*.

We will refer to Gabriel topologies satisfying condition (1) above as *radical saturated* topologies, modifying somewhat the use of *saturated* alone by Shores [**15**]. Shores showed that these topologies can be characterized as arising from certain topologically defined families of closed subsets of Spec(R), generalizing the construction of primal topologies $\mathcal{L}(\mathcal{P})$. The next result gives conditions which, together with (1), suffice to ensure that $\mathcal{L}$ is a primal topology.

Theorem 2.2. *Let* $\mathcal{L}$ *be a radical saturated Gabriel topology. If* $\mathcal{L}$ *satisfies either of the following conditions, then* $\mathcal{L}$ *is primal.*

(i) *Each radical ideal not in* $\mathcal{L}$ *is a finite intersection of prime ideals.*

(ii) *The intersection of any family of prime ideals of* $\mathcal{L}$ *is in* $\mathcal{L}$ (*or equivalently, the intersection of any family of radical ideals of* $\mathcal{L}$ *is in* $\mathcal{L}$)*.*

The proof is straightforward, and we may omit it.

It would be interesting to have a way of generalizing Stenström's construction of the bounded topologies to a family $\mathcal{B}$ of ideals which are not necessarily finitely generated.

Another way of constructing Gabriel topologies is by taking a multiplicatively closed set S and forming

$$\mathcal{L}_S = \{I | I \text{ is an ideal of } R \text{ with } I\cap S \neq \varnothing\}.$$

The topologies $\mathcal{L}_S$ are parameterized by *saturated* multiplicative sets, i.e., by those sets S such that ab in S implies a, b are in S. Moreover, they are precisely the principal topologies [**17**, Prop. 6.1], hence primal. Although the following result is known [**13**, Lemma 7.7], it seems natural to mention it in this discussion of special primal topologies. The equivalence of conditions (b) and (c) was treated by Smith [**16**] and by Reis and Viswanathan [**14**]. For the proof of (b) $\Rightarrow$ (a) one can take for S the complement in R of the union of the primes in $\mathcal{P}$.

Theorem 2.3. *Let $\mathcal{P}$ be a set of prime ideals of R. The following statements are equivalent*

(a) *$\mathcal{L}(\mathcal{P})$ is a principal topology.*

(b) *$\mathcal{L}(\mathcal{P}) = \mathcal{L}_S$ for some multiplicative set S of R.*

(c) *For p a prime of R, $p \subseteq \cup \mathcal{P}$ implies $p \subseteq q_0$ for some q_0 in $\mathcal{P}$.*

(d) *For I an ideal of R, $I \subseteq \cup \mathcal{P}$ implies $I \subseteq q_0$ for some q_0 in $\mathcal{P}$.*

3. A comparison of some finiteness and chain conditions. In this section we will consider conditions on a Gabriel topology which have occurred in the literature in lieu of the assumption that R is noetherian. We will indicate which logical implications hold among these conditions and provide examples to show that they are not equivalent. We begin by listing the various hypotheses for a Gabriel topology $\mathcal{L}$. We assume our rings are commutative, although when the conditions involved make sense for general rings R, the assertions involving these conditions remain valid.

(a) Every ideal in $\mathcal{L}$ is finitely presented.

(b) Every ideal in $\mathcal{L}$ is finitely generated.

(c) Every ideal I of R contains a finitely generated ideal J such that $(J : x)$ is in $\mathcal{L}$ for every x in I. This condition was dubbed "*R is $\sigma_{\mathcal{L}}$-noetherian*" by Caenepeel [**4**, 119–120], where $\sigma_{\mathcal{L}}$ is the idempotent kernel functor associated to $\mathcal{L}$.

(d) Every ascending chain of ideals of $\mathcal{L}$ stabilizes.

(e) $\mathcal{L}$ is of finite type.

(f) Any ideal of R not in $\mathcal{L}$ is contained in an ideal of R maximal with respect to exclusion from $\mathcal{L}$.

(g) $\mathcal{L}$ is primal.

(h) $\mathcal{L}$ is the intersection of a countable set of topologies of finite type.

(i) $\mathcal{L}$ is noetherian.

The implications among conditions (a) to (i) are summarized by Figure 1, provided as a visual aid to Theorem 3.1.

FIGURE 1.

Theorem 3.1. *The following implications hold among conditions* (a)–(i) *listed above*: (a) $\Rightarrow$ (b); (b) $\Rightarrow$ (d); (b) $\Rightarrow$ (e); (c) $\Rightarrow$ (e); (e) $\Rightarrow$ (f), (h), (i); (f) $\Rightarrow$ (g); (h) $\Rightarrow$ (g); (h) + (i) $\Rightarrow$ (e); (d) + (e) $\Rightarrow$ (b).

Proof. Most of the implications are immediate consequences of the definitions. (c) $\Rightarrow$ (e) uses property (T4) of a Gabriel topology. (e) $\Rightarrow$ (f) follows from Zorn's Lemma. (f) $\Rightarrow$ (g) uses the observation that if $\mathcal{L}$ is a Gabriel topology, then any ideal maximal with respect to exclusion from $\mathcal{L}$ is a prime ideal. (h) $\Rightarrow$ (g) holds by Theorem 1.1.

Now assume (h) and (i) hold. We wish to show (e) holds. Let $\mathcal{L} = (\mathcal{L}_1 \cap \mathcal{L}_2 \cap \dots)$, with $\mathcal{L}_i$ of finite type and let I be an ideal in $\mathcal{L}$. For each $i = 1, 2, \dots$, I contains a finitely generated ideal J_i in $\mathcal{L}_i$. Letting $I_n = J_1 + \cdots + J_n$ we have that I_n is in $\mathcal{L}_n$ and is contained in I. The sum of all the I_n is in all $\mathcal{L}_i$, hence in $\mathcal{L}$. Because (i) holds, some I_n is in $\mathcal{L}$. Thus, (e) holds.

The implication (d) + (e) $\Rightarrow$ (b) is mentioned [**8**, p. 251]. □

We now turn to examples we will call upon to show that conditions (a) to (i) are pairwise logically inequivalent. The main thrust of the first example is to show that a family $\mathcal{L}$ we construct is a Gabriel topology.

Example 3.2. Let F be a field and B the polynomial ring over F in uncountably many commuting indeterminates, say Y_t, t in $[0, 1]$. Let A be the ideal generated by the Y_t^2, t in $[0, 1]$. Let $R = B/A$ and let X_t be the image of Y_t in R. The ideal M generated by all the X_t is a maximal, hence prime, ideal of R. Since any prime ideal of R contains each X_t, M is the unique prime ideal of R.

By a monomial in R we shall mean a nonzero element $aZ_1 \cdot Z_2 \cdots Z_m$, where each Z_i is one of the X_t, $a \in F$ and $a \neq 0$. A set S of monomials will be called tame if none of the X_t, t in $[0, 1]$, occurs as a factor in infinitely many elements of S. Let $\mathcal{L}$ be the set of all ideals I of R which contain an uncountable tame set $S(I)$ of monomials. $\mathcal{L}$ is not $\{R\}$, as it contains the ideal generated by all the X_t. We claim that $\mathcal{L}$ satisfies the conditions (T0) and (T4) characterizing a Gabriel topology over a commutative ring.

Condition (T0) clearly holds, hence only (T4) is left. Let J, I be as in (T4). Let $S(J)$ be an uncountable tame set of monomials in J. Well-order $S(J)$ and view it as an uncountable ordinal. For each monomial m in $S(J)$, $(I : m)$ is in $\mathcal{L}$ by hypothesis. Thus $(I : m)$ contains an uncountable tame set of monomials, call it T_m.

Let f be the first element in the ordering on $S(J)$. Let f' be any element in T_f having no factor X_t in common with f. Then ff' is a monomial in I since f' is in $(I : f)$ and ff' is not zero. To make the construction below more transparent let g be the successor of f. Because T_g is uncountable, it contains a monomial g' which has no factor X_t in common with ff' or with g. As before, gg' is a monomial in I, and if ff' and gg' have any X_t as a common factor, X_t must be a factor of g.

Now let m be a countable ordinal in $S(J)$. Suppose that for each l in $S(J)$ satisfying $l < m$, we have defined l' satisfying:

(i) l' is in T_l and ll' is not zero.

(ii) If k is in $S(J)$ and $k < l$, then kk' and l' have no X_t as a common factor.

For each $l < m$, let V_l be the set of monomials in T_m which have a factor X_t in common with ll'. V_l is a finite set, and the union of the V_l for $l < m$, together with the set of monomials in T_m which have a factor in common with m, is a countable set. Since T_m is uncountable, there is a monomial m' in T_m but not in this union.

We have thus constructed m' in T_m for m a countable ordinal in $S(J)$. Let

$$S(I) = \{mm' | m \text{ is a countable ordinal in } S(J)\}.$$

$S(I)$ is an uncountable set of monomials in I. We claim $S(I)$ is tame. For suppose X_t divides mm'. Then X_t divides m or m'. There are only finitely many m divisible by X_t, since $S(J)$ is tame. As for X_t dividing m', once it divides one m', X_t does not divide n' for n different from m. Hence, X_t divides only finitely many mm' and $S(I)$ is tame.

The Gabriel topology $\mathcal{L}$ is noetherian since if the union of a countable sequence of ideals I_n contains an uncountable and tame set of monomials, some I_n must contain such a set. We note that $\mathcal{L}$ is not primal, since the only prime ideal of R is in $\mathcal{L}$, and there are ideals of R not in $\mathcal{L}$.

It follows from Theorem 3.1 that (e) implies (g). Since the topology $\mathcal{L}$ of Example 3.2 does not satisfy (g), it does not satisfy (e) either. This gives us an example of a Gabriel topology $\mathcal{L}$ which is noetherian but not of finite type. Bergman has also provided an example of a Gabriel topology which is noetherian but not of finite type [**7**, p. 319], but we have not been able to decide whether the topology in that example is primal.

Example 3.3. Let F be a field, S a polynomial ring over F in the commuting indeterminates Y_t, t in $[0,1]$. Let A be the ideal of S generated by the $Y_t^2 - Y_t$, t in $[0,1]$. Let $R = \mathcal{S}/A$ and let X_t denote the image of Y_t in R, M the maximal ideal of R generated by the X_t. Then $M^2 = M$ and $\mathcal{L} = \{R, M\}$ is a Gabriel topology on R [**17**, Chap. 6, Prop. 6.11].

We shall now show that condition (f) holds for this topology $\mathcal{L}$. Suppose I is an ideal of R not in $\mathcal{L}$. Then X_s is not in I for some s in $[0,1]$. Let $J = I + R(1 - X_s)$. Since $1 - X_s$ is in J, J is not contained in M. Also, $J \neq R$ since otherwise $a + r(1 - X_s) = 1$, with a in I; multiplying by X_s we get that $X_s a = X_s$, hence that X_s is in I, a contradiction. Thus, J is a proper ideal not contained in M, hence is contained in a maximal ideal of R which is not in $\mathcal{L}$. Thus (f) holds.

Condition (h) does not hold for this $\mathcal{L}$. To show this, we first note that if (h) holds for a Gabriel topology $\mathcal{L}$, then every ideal in $\mathcal{L}$ contains a countably generated ideal which is also in $\mathcal{L}$ (see the proof that (h) and (i) together imply (e), in Theorem 3.1). But M is in $\mathcal{L}$ yet does not contain any countably generated ideal in $\mathcal{L}$.

Example 3.4. Let F be a field, S the polynomial ring over F in the commuting indeterminates $X_1, X_2, \dots$. Let N be the ideal generated by all the X_i, R the localization of S at the complement of N, and let M be the unique maximal ideal of R. For each $n = 1, 2, \dots$, let P_n be the (prime) ideal of R generated by $X_1, \dots, X_n$. Let $\mathcal{L}_i$ be the topology $\mathcal{L}(P_i)$ consisting of all ideals not contained in P_i and let $\mathcal{L}$ be the intersection of the $\mathcal{L}_i$. I is in $\mathcal{L}$ if and only if we have that for each n, I is not contained in P_n.

Suppose there were an ideal I maximal with respect to exclusion from $\mathcal{L}$. Then I is contained in some P_n, and since P_n is not in $\mathcal{L}$ and I has the indicated maximality property, $I = P_n$. But P_{n+1} properly contains P_n and is not in $\mathcal{L}$, contradicting the maximality of I. There are no ideals I as described. $\mathcal{L}$ does not satisfy (f), but it does satisfy (h).

Example 3.5. We would like to thank M. Roitman for the following example. Let F be a field. Let X, Y_i, $i = 1, 2, \ldots$, be a set of commuting indeterminates over F, and let $S = F[\mathcal{U}]$ where

$$\mathcal{U} = \{X, Y_i/X^s : i = 1, 2, \ldots, s = 0, 1, 2, \ldots\}.$$

Let A be the ideal of S generated by the elements XY_i, $i = 1, 2, \ldots$, and let $R = S/A$. Write x, etc., for the image of X in R. We claim that the ideal M of R generated by x is a maximal ideal that is finitely generated but not finitely presented.

The element X of S generates an ideal containing all Y_i/X^s since $Y_i/X^s = (Y_i/X^{s+1})X$. The maximality of M follows easily from this. To show that M is not finitely presented, it suffices to show that the annihilator of x in R is not finitely generated, since there is an exact sequence

$$0 \to \operatorname{ann}(x) \to R \to M \to 0.$$

Ann (x) contains all the y_i so it suffices to show that the y_i are not all contained in any finitely generated ideal of R.

The generators of R as an F-algebra satisfy these relations for $i, j = 1, 2, \ldots$:

$$(y_i/x^s)(y_j/x^t) = (xy_i)(y_j/x^{s+t+1}) = 0 \qquad \text{for } s, t = 0, 1, 2, \ldots ;$$
$$x(y_i/x^s) = y_i/x^{s-1} \qquad \text{for } i = 1, 2, \ldots \quad \text{and} \quad s = 1, 2, \ldots .$$

It follows that R is spanned as a vector space over F by the elements x^k and y_i/x^s, $k = 0, 1, 2, \ldots$, $i = 1, 2, \ldots$, $s = 0, 1, 2, \ldots$. If the y_i were all contained in a finitely generated ideal of R, then they would all be contained in the F-subspace spanned by a finite set

$$\mathcal{B} = \{1, x, \ldots, x^n, y_i/x^s : i = 1, 2, \ldots, m-1, \ s = 0, 1, 2, \ldots, N\}.$$

In particular, y_m would be a linear combination of the elements of $\mathcal{B}$. Lifting this relation to S and multiplying by X^N leads to an impossible relation involving polynomials. We have thus established that M is a maximal ideal that is finitely generated but not finitely presented.

Let $\mathcal{L}$ be the set of ideals of R which contain M^k for some k. R/M^k is a noetherian ring since M/M^k is its only prime ideal, and the theorem of I.S. Cohen, cited before Theorem 1.3, states that if every prime ideal of a commutative ring is finitely generated, then the ring is noetherian. It follows that $\mathcal{L}$ is a Gabriel topology whose every ideal is finitely generated. But $\mathcal{L}$ contains M, an ideal that is not finitely presented. Thus $\mathcal{L}$ satisfies (b) but not (a).

In [**8**] Golan and Teply describe a group of conditions, including (a) and (b), as being nonequivalent, but we have not been able to find a specific example in the literature.

Remark 3.6. We can now make a list of various nonimplications among conditions (a) to (i) and provide an example relevant to each.

(b) does not imply (a)–noted in Example 3.5.

(c) does not imply (b) or (d)–Let $\mathcal{L}$ be the set of all ideals in a non-noetherian ring.

(d) does not imply (b)–Let $\mathcal{L}$ be as in Example 3.3.

(a) does not imply (c)–Let $\mathcal{L} = \{R\}$, R not noetherian.

(h) does not imply (f)–noted in Example 3.4.

(d) does not imply (g)–Let R be a one-dimensional but not noetherian valuation domain with maximal ideal M [**10**, Chapter 2, Section A and Example 34, p. 68]. We then have $M^2 = M$. Hence, $\mathcal{L} = \{R, M\}$ is a Gabriel topology which satisfies (d). Let I be any ideal other than (0) or R (take I to be all r in R with valuation $v(r)$ at least d, with $d > 0$ in the value group of v). I is not in $\mathcal{L}$ but the only prime ideal in which it is contained is M, since (0) and M are the only prime ideals of R.

(i) does not imply (g)–noted in Example 3.2.

(f) does not imply (h)–noted in Example 3.3.

(d) + (f) does not imply (i)–Let R, $\mathcal{L}$, etc., be as in Example 3.3, but with countably many indeterminates.

Note added in proof. We would like to thank M. Teply for pointing out [**20**] to us. In particular, [**20**, Example 3.19] gives a non-commutative example illustrating the nonequivalence of (a) and (b).

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