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A NOTE ON ORTHOGONAL POLYNOMIALS

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Let $d\mu$ be a finite positive Borel measure on the interval $[0, 2\pi]$ such that its support is an infinite set. Then there is a unique system $\{s_n\}_{n=0}^{\infty}$ of polynomials orthonormal with respect to $d\mu$ on the unit circle, i.e., polynomials

$$s_n(z) := s_n(d\mu, z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \qquad a_n := a_n(d\mu) > 0$$

satisfying

$$(1) \qquad \frac{1}{2\pi}\int_0^{2\pi} s_m(z)\overline{s_n(z)}\, d\mu(\theta) = \delta_{mn}, \qquad z = e^{i\theta};\ m, n > 0,$$

where $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ otherwise. The purpose of this note is to give a simple and elementary proof of the following identity (without using any recurrence relations).

$$(2) \qquad \begin{gathered} \int_0^{2\pi} z^k |s_n(z)|^{-2}\, d\theta = \int_0^{2\pi} z^k\, d\mu(\theta), \\ z = e^{i\theta}, \quad |k| \leq n,\ n = 0, 1, 2, \dots . \end{gathered}$$

This identity plays a very important role in the study of the asymptotics of orthonormal polynomials (cf., e.g., [**4, 5**]). Other proofs of (2) can be found in [**2**, Theorem 5.2.2, p. 198, **1**, Lemma 2 or **3**, formula (1.20), p. 7].

Proof of (2). For simplicity, we write $d\mu$ for $d\mu(\theta)$ and z for $e^{i\theta}$.

By (1), we have

$$\int_0^{2\pi} s_n(z) z^{-k}\, d\mu = \frac{2\pi}{a_n}\delta_{nk}, \qquad k = 0, 1, 2, \dots , n,$$

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i.e.,

$$\sum_{l=0}^{n} a_l \int_0^{2\pi} z^{l-k}\, d\mu = \frac{2\pi}{a_n}\delta_{nk}, \qquad k = 0, 1, 2, \dots, n. \tag{3}$$

Taking conjugate of (3), we get

$$\sum_{l=0}^{n} \bar{a}_l \int_0^{2\pi} z^{-l+k} d\mu = \frac{2\pi}{a_n}\delta_{nk}, \qquad k = 0, 1, 2, \dots, n. \tag{4}$$

From (3) and (4), it is easy to see that

$$\underline{\mu} := \left(\int_0^{2\pi} z^n\, d\mu, \dots, \int_0^{2\pi} d\mu, \int_0^{2\pi} z^{-1}\, d\mu, \dots, \int_0^{2\pi} z^{-n}\, d\mu \right)^{\tau} \in \mathbf{C}^{2n+1}$$

is a solution of $\underline{x}$ in $\mathbf{C}^{2n+1}$ satisfying

$$\mathbf{A}\underline{x} = \underline{\alpha}, \tag{5}$$

where

$$\mathbf{A} := \begin{pmatrix} a_n & a_{n-1} & \cdots & a_0 & & \bigcirc & \\ 0 & a_n & \cdots & a_1 & a_0 & & \\ & & \ddots & \vdots & & \ddots & \\ \bigcirc & & & a_n & a_{n-1} & \cdots & a_0 \\ & & & \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_n \\ & & \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_n & 0 \\ & & \vdots & \vdots & & \bigcirc & \\ 0 & \bar{a}_0 & \cdots & \bar{a}_{n-1} & \bar{a}_n & & \end{pmatrix}_{(2n+1)\times(2n+1)}$$

and

$$\underline{\alpha} := \Big(\overbrace{0\cdots 0}^{\text{n}}\ \frac{2\pi}{a_n}\ \overbrace{0\cdots 0}^{\text{n}} \Big)^{\tau}.$$

Note that we can write

$$\det \mathbf{A} = a_n \cdot (-1)^{\frac{n(n-1)}{2}} R_n,$$

where

$$R_n := \begin{vmatrix} a_n & a_{n-1} & \cdots & a_0 & & & \\ & a_n & \cdots & a_1 & a_0 & & \\ & & \cdots & & & & \\ & & a_n & a_{n-1} & \cdots & a_0 \\ \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_n & & \\ & \bar{a}_0 & \cdots & & \bar{a}_n & \\ & & \cdots & & & \\ & & \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_n \end{vmatrix},$$

which is the resultant of $s_n(z)$ and $s_n^*(z)$. Now since all the zeros of $s_n(z)$ lie in $|z| < 1$ (cf. [**6**, p. 292]), so all the zeros of $s_n^*(z)$ lie in $|z| > 1$, and hence $s_n(z)$ and $s_n^*(z)$ have no common zeros. Furthermore, note that $a_n \neq 0$. Now, by the property of the resultant (cf. [**7**, p. 84]) we know that $R_n \neq 0$. Therefore, $\det \mathbf{A} \neq 0$. Thus, equation (5) has a unique solution. It follows from this that, in order to prove (2), we only need to show that

$$\underline{s} := \left(\int_0^{2\pi} z^n |s_n(z)|^{-2}\, d\theta, \dots, \int_0^{2\pi} |s_n(z)|^{-2}\, d\theta, \dots, \int_0^{2\pi} z^{-n} |s_n(z)|^{-2}\, d\theta \right)^\tau$$

is also a solution of (5).

In fact, if we write $s_n^*(z) := z^n \overline{s_n(1/\bar{z})}$, then the k-th element of $\mathbf{A}\underline{s}$ is

(6)
$$\int_0^{2\pi} \frac{s_n(z) z^{-k+1}}{|s_n(z)|^2}\, d\theta = \int_0^{2\pi} \frac{z^{-k+1}}{\overline{s_n(z)}}\, d\theta = \int_{|z|=1} \frac{z^{n-k+1}}{s_n^*(z)} \cdot \frac{dz}{iz}, \quad \text{for } k = 1, 2, \dots, n+1,$$

and

(7)
$$\int_0^{2\pi} \frac{\overline{s_n(z)} z^{k-n-2}}{|s_n(z)|^2}\, d\theta = \overline{\int_0^{2\pi} \frac{s_n(z) z^{-k+n+2}}{|s_n(z)|^2}\, d\theta} = \overline{\int_{|z|=1} \frac{z^{2n-k+2}}{s_n^*(z)} \cdot \frac{dz}{iz}}, \quad \text{for } k = n+2, \dots, 2n+1.$$

Now note that all the zeros of $s_n^*(z)$ lie in $|z| > 1$, so by the residue theorem,

$$\int_{|z|=1} \frac{z^l}{s_n^*(z)} \cdot \frac{dz}{iz} = 0, \quad \text{for } l = 1, 2, \dots, n, \tag{8}$$

and for $l = 0$,

$$\int_{|z|=1} \frac{z^l}{s_n^*(z)} \cdot \frac{dz}{iz} = \int_{|z|=1} \frac{1}{is_n^*(z)z}\, dz = \frac{2\pi}{s_n^*(0)} = \frac{2\pi}{a_n}. \tag{9}$$

By (6), (7), (8) and (9), we can see that $\underline{s}$ is a solution of (5). This completes the proof. □

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