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# ON SOME INEQUALITIES INVOLVING $(n!)^{1/n}$

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When investigating a conjecture on an upper bound for permanents of $(0,1)$-matrices, H. Minc and L. Sathre [**2**] (see also [**1**]) obtained several inequalities involving $f(n)=(n!)^{1/n}$–the geometric mean of the first $n$ positive integers. One of their results is

**Theorem A.** *If $n\geq 1$ is an integer, then*

$$1<\frac{f(n+1)}{f(n)}<1+\frac{1}{n}. \tag{1}$$

Another one, "probably the most interesting ..., and certainly the hardest to prove" [**2**, p. 41] is

**Theorem B.** *If $n\geq 2$ is an integer, then*

$$1<n\frac{f(n+1)}{f(n)}-(n-1)\frac{f(n)}{f(n-1)}. \tag{2}$$

The aim of this note is to establish sharpenings of inequalities (1) and (2). We present a lower bound for the difference on the right-hand side of (2) which is greater than 1. Furthermore, we give an answer to the question: What is the largest real number $\alpha$ and the smallest real number $\beta$ such that

$$1+\frac{\alpha}{n+1}\leq\frac{f(n+1)}{f(n)}<1+\frac{\beta}{n+1}$$

is valid for all integers $n\geq 1$?

First we provide a monotonicity theorem.

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**Theorem 1.** *The sequence*

$$n \mapsto (n+1)\frac{f(n+1)}{f(n)} - n\frac{f(n)}{f(n-1)}, \qquad n = 2, 3, \dots,$$

*is strictly decreasing and converges to* 1*, if* $n$ *tends to* $\infty$*.*

*Proof.* Our proof is modelled after the one given by Minc and Sathre [**2**] to establish inequality (2). Let $f(x) = \Gamma(x+1)^{1/x}$, $0 < x \in \mathbf{R}$; first we show that the function

$$P(x) = (x+1)\frac{f(x+1)}{f(x)}$$

is strictly concave on $[21, \infty)$. We set

$$g(x) = \frac{f(x+1)}{f(x)} \quad \text{and} \quad h(x) = xg(x).$$

Then we have

$$P(x) = h(x) + g(x).$$

Differentiation leads to

$$xP''(x) = (x+1)h''(x) - 2g'(x). \tag{3}$$

Our aim is to show that $P''(x) < 0$ for $x \geq 21$. A simple calculation yields

$$g'(x) = g(x)F(x)$$

and

$$h''(x) = g(x)[F(x) + xF^2(x) + H(x)]$$

with

$$F(x) = \frac{2x+1}{x^2(x+1)^2}\log\Gamma(x+1) - \frac{\psi(x+1)}{x(x+1)} + \frac{1}{(x+1)^2} - \frac{\log(x+1)}{(x+1)^2}$$

and

$$H(x) = \frac{3x+1}{x(x+1)^2}\psi(x+1) - \frac{4x^2+3x+1}{x^2(x+1)^3}\log\Gamma(x+1) - \frac{\psi'(x+1)}{x+1} - \frac{2x-1}{(x+1)^3} + \frac{x-1}{(x+1)^3}\log(x+1).$$

From (3) we conclude that we have to show that the function

$$Q(x) = F(x) - \frac{x+1}{2}[F(x) + xF^2(x) + H(x)] \tag{4}$$

attains only positive values for $x \geq 21$. In [**2**] it is proved that the inequality

$$F(x) + xF^2(x) + H(x) < \frac{2 - \log(2\pi) - \log(x+1)}{(x+1)^3} + \frac{1}{x^3} \tag{5}$$

is valid for $x \geq 3$. From (4) and (5) we obtain for $x \geq 3$:

$$\begin{aligned} 2(x+1)^2 Q(x) > -\log(x+1) + \frac{2(2x+1)}{x^2} \log\Gamma(x+1) \\ - \frac{2(x+1)}{x}\psi(x+1) - \frac{(x+1)^3}{x^3} + \log(2\pi). \end{aligned}$$

An application of

$$\psi(y) < \log(y) - \frac{1}{2y}, \qquad y > 1,$$

and

$$\log\Gamma(y) > (y - 1/2)\log(y) - y + \log(2\pi)/2, \qquad y > 1,$$

(see [**2**]) yields for $x \geq 3$:

$$2x^2 Q(x) > \log(2\pi(x+1)) - 5 + \frac{2x^2 - 1}{x(x+1)^2}.$$

Let

$$G(x) = \log(2\pi(x+1)) - 5 + \frac{2x^2 - 1}{x(x+1)^2}.$$

From

$$x^2(x+1)^3 G'(x) = x^4 + 3x^2 + 3x + 1 > 0$$

and

$$G(21) = 0.015\ldots$$

we get $G(x) > 0$ and $Q(x) > 0$ for $x \geq 21$. Hence, $P$ is strictly concave on $[21, \infty)$, so that the inequality

$$P\left(\frac{x+y}{2}\right) > \frac{P(x)+P(y)}{2} \tag{6}$$

holds for all real $x$ and $y$ with $x, y \geq 21$ and $x \neq y$. Setting $x = n-1$ and $y = n+1$ ($22 \leq n \in \mathbf{Z}$) we obtain from (6):

$$(n+1)\frac{f(n+1)}{f(n)} - n\frac{f(n)}{f(n-1)} > (n+2)\frac{f(n+2)}{f(n+1)} - (n+1)\frac{f(n+1)}{f(n)}, \tag{7}$$

that is, $a(n) = (n+1)f(n+1)/f(n) - nf(n)/f(n-1)$ is strictly decreasing for $n \geq 22$. For $2 \leq n \leq 21$ we get (7) by direct computation. The approximate values of $a(n)$, $n = 2, 3, \ldots, 22$, are given in the following table:

| $n$ | $a(n)$ | $n$ | $a(n)$ |
|---|---|---|---|
| 2 | 1.0262 | 12 | 1.0032 |
| 3 | 1.0175 | 13 | 1.0029 |
| 4 | 1.0128 | 14 | 1.0026 |
| 5 | 1.0099 | 15 | 1.0024 |
| 6 | 1.0080 | 16 | 1.0021 |
| 7 | 1.0066 | 17 | 1.0020 |
| 8 | 1.0055 | 18 | 1.0018 |
| 9 | 1.0048 | 19 | 1.0017 |
| 10 | 1.0041 | 20 | 1.0015 |
| 11 | 1.0036 | 21 | 1.0014 |
| | | 22 | 1.0013 |

This implies that $a(n)$ is strictly decreasing for all $n \geq 2$.

Next we prove $\lim_{n\to\infty} a(n) = 1$. Let

$$b(n) = \left(\frac{f(n+1)}{f(n)} - 1\right)(n+1).$$

From the second inequality of (1) we get

$$b(n) < 1 + 1/n. \tag{8}$$

The inequality

$$x-1>\log(x)$$

holds for $x>1$. If we set $x=f(n+1)/f(n)$, then we have

$$b(n)>(n+1)\log\frac{f(n+1)}{f(n)}=\log\frac{n+1}{f(n)}. \tag{9}$$

Since $\lim_{n\to\infty}(n+1)/f(n)=e$, we conclude from (8) and (9) that $b(n)$ tends to 1, if $n\to\infty$. Thus,

$$a(n)-1=b(n)-b(n-1)\to 0, \qquad \text{if } n\to\infty.$$

This completes the proof of Theorem 1. □

An application of Theorem 1 leads to a refinement of inequality (2).

**Theorem 2.** *Let $n\ge 2$ be an integer. Then we have*

$$\begin{aligned} 1&<1+\frac{f(n)}{f(n-1)}-\frac{f(n+1)}{f(n)}\\ &<n\frac{f(n+1)}{f(n)}-(n-1)\frac{f(n)}{f(n-1)}. \end{aligned} \tag{10}$$

*Proof.* The second inequality of (10) is an immediate consequence of Theorem 1. Using the arithmetic mean-geometric mean inequality we obtain

$$\begin{aligned} \frac{1}{n-1}\sum_{i=1}^{n-1}\log(i)&=\log\prod_{i=1}^{n-1}i^{1/(n-1)}\\ &\le\log\left(\frac{1}{n-1}\sum_{i=1}^{n-1}i\right)=\log\frac{n}{2}. \end{aligned} \tag{11}$$

From $(1+1/n)^n<4$ we conclude

$$\log\frac{n}{2}<\frac{n+2}{2}\log(n)-\frac{n}{2}\log(n+1), \tag{12}$$

so that (11) and (12) yield

$$\frac{2}{n(n^2-1)}\sum_{i=1}^{n-1}\log(i) < \frac{n+2}{n(n+1)}\log(n) - \frac{1}{n+1}\log(n+1),$$

which is equivalent to the first inequality of (10). □

*Remark.* The left-hand inequality of (10) states that $f(n)$, $n = 1, 2, \dots$, is strictly logarithmically concave.

A second application of Theorem 1 provides sharp upper and lower bounds for the ratio $f(n+1)/f(n)$. The following refinement of double-inequality (1) is valid.

**Theorem 3.** *The inequalities*

$$1 + \frac{\alpha}{n+1} \le \frac{f(n+1)}{f(n)} < 1 + \frac{\beta}{n+1} \tag{13}$$

*hold for all integers* $n \ge 1$ *if and only if* $\alpha \le 2(\sqrt{2}-1) = 0.828\dots$ *and* $\beta \ge 1$.

*Proof.* At the end of the proof of Theorem 1 we have shown that

$$b(n) = \left(\frac{f(n+1)}{f(n)} - 1\right)(n+1)$$

tends to 1, if $n \to \infty$. From

$$0 < a(n) - 1 = b(n) - b(n-1), \qquad n = 2, 3, \dots,$$

we conclude that $b(n)$ is strictly increasing. Hence we get

$$2(\sqrt{2}-1) = b(1) \le b(n) < 1, \qquad n = 1, 2, \dots,$$

which is equivalent to (13) with $\alpha = 2(\sqrt{2}-1)$ and $\beta = 1$. □

## REFERENCES

**1.** H. Minc, *Upper bounds for permanents of* $(0,1)$*-matrices*, Bull. Amer. Math. Soc. **69** (1963), 789–791.

**2.** H. Minc and L. Sathre, *Some inequalities involving* $(r!)^{1/r}$, Edinburgh Math. Soc. **14** (1964/65), 41–46.

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