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EXTREMAL DISKS AND COMPOSITION OPERATORS ON CONVEX DOMAINS IN $\mathbf{C}^n$

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ABSTRACT. We obtain results about function spaces on strongly convex domains which are associated with the images of Kobayashi extremal disks. As an application we study composition operators on the Hardy function spaces of such domains.

0. Introduction. Let Ω be a smoothly bounded domain in $\mathbf{C}^n$, and let $M \subset \Omega$ be a complex submanifold which intersects $\partial\Omega$ transversally. It is natural to study function spaces on M in conjunction with function spaces on Ω. If $\Omega = B$, the unit ball, and M is a linear subspace, then a theorem of W. Rudin (Theorem 1.2 below) shows that the Hardy spaces $H^p(B)$ enjoy a simple relationship with the weighted Bergman spaces on M. A. Cumenge studied this situation in much greater generality in [**7**].

We focus on the situation where M is the image of a Kobayashi extremal disk. We generalize Rudin's result and make precise some of Cumenge's results for this special case. This work appears in Section 2 of this paper.

In Section 3 we apply these ideas, along with some work of M. Abate, to initiate the study of composition operators associated with Ω: Let $\Phi : \Omega \to \Omega$ be holomorphic. The composition operator T_Φ induced by Φ is defined by $T_\Phi(f) = f \circ \Phi$ (f is a holomorphic function on Ω). For $\Omega = B$ and $n \geq 2$, T_Φ is not in general a bounded operator on $H^p(B)$ [**5, 6**]. For Ω strongly convex we obtain necessary conditions on Φ for T_Φ to be a compact operator on $H^p(\Omega)$, thus generalizing some results of B. MacCluer for which $\Omega = B$ [**14**].

In Section 1 we fix some notation and isolate some results (mainly due to L. Lempert) which will be of subsequent use.

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1. Preliminaries. Let Ω be a bounded domain in $\mathbf{C}^n$. In everything that follows, Ω has a smooth (i.e., C^∞) boundary $\partial\Omega$ and is strongly convex. We denote by B the unit ball in $\mathbf{C}^n$ centered at the origin and, when $n = 1$, we write $B = \Delta$, the unit disk. For $\Omega_1 \subset \mathbf{C}^n$, $\Omega_1 \subset \mathbf{C}^m$, we denote by $H(\Omega_1, \Omega_2)$ the collection of holomorphic maps from Ω_1 to Ω_2.

We denote by k_Ω the Kobayashi distance on Ω [**8, 9**]. Since Ω is convex, it is given by

$$k_\Omega(z, w) = \inf [k_\Delta(0, \lambda) : \exists f \in H(\Delta, \Omega) \text{ with } f(0) = z,\ f(\lambda) = w],$$

where k_Δ is the hyperbolic distance on Δ [**10**]. k_Ω is nonincreasing under holomorphic maps. Denote by $\varphi \in H(\Delta, \Omega)$ an associated extremal disk [**10, 19**]:

$$k_\Omega(\varphi(\lambda), \varphi(\mu)) = k_\Delta(\lambda, \mu) \qquad \forall\, \lambda, \mu \in \Delta.$$

Each extremal disk $\varphi \in H(\Delta, \Omega)$ extends continuously to $\overline{\Delta}$ and has an associated retraction $p = p_\varphi \in H(\Omega, \Omega)$ such that $p(\Omega) = \varphi(\Delta)$ and $p \circ \varphi(\lambda) = \varphi(\lambda)$ for all $\lambda \in \Delta$ [**10, 11**]. One can say much more:

Theorem 1.1. *There is a domain $\Omega' \subset\subset \mathbf{C}^n$ and a biholomorphism $\Psi \in H(\Omega, \Omega')$ such that*:

(a) $\Psi \circ \varphi(\lambda) = (\lambda, 0, \dots, 0)$ *for all* $\lambda \in \Delta$.

(b) *For* $\xi \in \partial\Delta$ *we have* $(\xi, 0, \dots, 0) \in \partial\Omega'$, *and the unit outward normal there is precisely* $(\xi, 0, \dots, 0)$.

(c) Ω' *is strongly convex near* $\{(\xi, 0, \dots, 0) \in \mathbf{C}^n : |\xi| = 1\} \subset \partial\Omega'$.

(d) *If* $\pi \in H(\mathbf{C}^n, \mathbf{C}^n)$ *is given by* $\pi(z_1, \dots, z_n) = (z_1, 0, \dots, 0)$, *then* $\Psi \circ p = \pi \circ \Psi$.

(e) Ψ *extends to a smooth map on* $\overline{\Omega}$ *and has nonvanishing Jacobian determinant there.*

Proof. Parts (a)–(d) are due to Lempert [**10, 11**]. For part (e), Lempert [**12**] observes that Ψ extends smoothly to $\overline{\Omega}$, but does not explicitly state that the Jacobian determinant $J\Psi$ is nonvanishing on $\overline{\Omega}$. It follows from (c) and [**15**, Lemma 1.3] that $J\Psi \neq 0$ at points on $\partial\Omega'$ near $\{(\xi, 0, \dots, 0) \in \mathbf{C}^n : \xi \in \partial\Delta\}$, and I. Graham has observed (unpublished) that Ψ has a biholomorphic extension to a neighborhood of $\overline{\Omega}\backslash\varphi(\partial\Delta)$. These observations ensure that $J\psi \neq 0$ on $\overline{\Omega}$. $\square$

For $p \in [1,\infty)$ we denote by $H^p(\Omega)$ the pth Hardy space of holomorphic functions on Ω, with norm given by

$$||f||_p = ||f||_{H^p(\Omega)} = \left(\sup_{\varepsilon>0}\int_{\partial\Omega_\varepsilon} |f(z)|^p \, d\sigma_\varepsilon(z)\right)^{1/p},$$

where σ_ε is Lebesgue surface measure on $\partial\Omega_\varepsilon = \{z \in \mathbf{C}^n : \rho(z)+\varepsilon = 0\}$ and ρ is a C^∞ defining function for Ω, i.e., $\Omega = \{z \in \mathbf{C}^n : \rho(z) < 0\}$. Whether $||f||_p$ is finite is independent of the choice of ρ [**18**]. There should be no chance of confusion between the number p and the retraction p appearing above.

Let $d_\Omega(z)$ denote the (Euclidean) distance from $z \in \Omega$ to $\partial\Omega$. Theorem 1.1(e) ensures that Ω' is strongly pseudoconvex and that $d_{\Omega'}(\Psi(z)) \sim d_\Omega(z)$. (The symbol $\sim$ means that the ratio of the two quantities is bounded above and away from zero independently of the argument.) Thus an argument similar to that of [**18**, Section 3, Chapter 1] shows that

$$(1) \qquad ||f||_{H^p(\Omega)} \sim ||f \circ \Psi^{-1}||_{H^p(\Omega')} \qquad \forall\, f \in H^p(\Omega).$$

For $n \geq 2$ we denote by $H^p_{n-2}(\Delta)$ the weighted Bergman space of holomorphic functions on Δ, with norm given by

$$||f||_{p,n-2} = \left(\int_0^{2\pi}\int_0^1 |f(re^{i\theta})|^p (1-r^2)^{n-2} r\, dr\, d\theta\right)^{1/p}.$$

The following appears in [**16**, Section 1.4.4].

Theorem 1.2. *Let π denote the orthogonal projection of $\mathbf{C}^n$ onto the first coordinate. There is a constant $c = c(n)$ such that*

$$||f \circ \pi||_{H^p(B)} = c||f||_{p,n-2} \qquad f \in H^p_{n-2}(\Delta).$$

2. Restrictions and extensions. In B, slices through the origin are images of extremal disks, and orthogonal projections onto these slices are their associated retractions. Thus, the following proposition generalizes Theorem 1.2.

Proposition 2.1. *We have* $||f \circ \varphi^{-1} \circ p||_{H^p(\Omega)} \sim ||f||_{p,n-2}$ *for all* $f \in H^p_{n-2}(\Delta)$.

We remark that [**7**, Theorem 0.1] can be used to show that an extension from a function g, where $g \circ \varphi \in H^p_{n-2}(\Delta)$, to a function $G \in H^p(\Omega)$ exists; our result gives such an extension explicitly.

We prove this result by a sequence of lemmas. First, we assume that Ω has the special form that Ω' enjoys in Theorem 1.1 (a)–(d). We assume further that Ω is strongly convex, so that Ω is given by a smooth strongly convex defining function ρ.

It is convenient to use real cylindrical coordinates on $\mathbf{R}^N \simeq \mathbf{C}^n$, so that the point $(re^{i\theta}, x_3, \dots, x_N) \in \mathbf{R}^N$ is the point $(z_1, \dots, z_n) \in \mathbf{C}^n$, where $z_1 = re^{i\theta}$ and $N = 2n$. For r and θ fixed we identify the point $x = (re^{i\theta}, x_3, \dots, x_N)$ with the point $\hat{x} = (x_3, \dots, x_N) \in \mathbf{R}^{N-2}$, and we define the function $\hat{\rho}$ on $\mathbf{R}^{N-2}$ by $\hat{\rho}(\hat{x}) = \rho(re^{i\theta}, \hat{x})$.

The domain $\Omega_{r\theta} = \{\hat{x} \in \mathbf{R}^{N-2} : \hat{\rho}(\hat{x}) < 0\}$ is smoothly bounded and strongly convex. It therefore has a positive definite (symmetric) $(N-2) \times (N-2)$ Hessian matrix $\widehat{H}(\hat{x}) = [\hat{\rho}_{jk}(\hat{x})]$, where subscripts denote partial derivatives. $\widehat{H}(\hat{x})$ is obtained by deleting the first two rows and columns of the positive definite (symmetric) $N \times N$ matrix $H(x) = [\rho_{jk}(x)]$.

Let $\lambda_0(x)$, $\lambda^0(x) > 0$ be the smallest and largest eigenvalues of $H(x)$. We have then for each unit vector $v \in \mathbf{R}^{N-2}$:

$$\lambda_0(x) \le v^T \widehat{H}(\hat{x}) v \le \lambda^0(x) \qquad \forall\, x \in \partial\Omega.$$

Now the eigenvalues of $H(x)$ are a continuous function of x, so by compactness we may assume that λ_0 and λ^0 are independent of x.

The normal radius of curvature $r_v(\hat{x})$ at $\hat{x} \in \partial\Omega_{r\theta}$ in the direction of the unit vector $v \in T_{\hat{x}}(\partial\Omega_{r\theta})$ is given by

$$r_v(\hat{x}) = \frac{||\nabla \hat{\rho}(\hat{x})||}{v^T \widehat{H}(\hat{x}) v}.$$

Thus, we have the following:

Lemma A. *We have* $r_v(\hat{x}) \sim ||\nabla\hat{\rho}(\hat{x})||$ *for all* $(re^{i\theta}, \hat{x}) \in \partial\Omega$.

Lemma B. *Once magnified by a factor of* $(1-r^2)^{-1/2}$ *in each variable, then as* $r \to 1$, $\partial\Omega_{r\theta}$ *tends to a nondegenerate ellipsoid in* $\mathbf{R}^{N-2}$ *(which depends on* θ*).*

Proof. Let $\hat{0} = (0, \dots, 0) \in \mathbf{R}^{N-2}$ and expand ρ about $(re^{i\theta}, 0, \dots, 0)$:

$$\begin{aligned}\rho(re^{i\theta}, x_3, \dots, x_N) &= \hat{\rho}(\hat{0}) + \hat{x}\cdot\nabla\hat{\rho}(\hat{0}) + \frac{1}{2}\hat{x}^T\widehat{H}(\hat{0})\hat{x}\\ &\quad + \mathcal{O}(|\hat{x}|^3) \quad \text{as } |\hat{x}| \to 0.\end{aligned}$$

Set $t = (1-r^2)^{1/2}$, $x_j = tu_j$, $3 \le j \le N$, and $\hat{u} = (u_3, \dots, u_N)$:

$$\begin{aligned}\rho(re^{i\theta}, tu_3, \dots, tu_N) &= \hat{\rho}(\hat{0}) + t\hat{u}\cdot\nabla\hat{\rho}(\hat{0})\\ &\quad + \frac{1}{2}t^2\hat{u}^T\widehat{H}(\hat{0})\hat{u} + \mathcal{O}(t^3) \quad \text{as } t \to 0.\end{aligned} \tag{2}$$

Expand $\hat{\rho}(\hat{0}) = \rho(re^{i\theta}, 0, \dots, 0)$ about $r = 1$:

$$\begin{aligned}\hat{\rho}(\hat{0}) &= \rho(e^{i\theta}, 0, \dots, 0) + (r-1)\frac{\partial}{\partial r}\rho(re^{i\theta}, 0, \dots, 0)|_{r=1}\\ &\quad + \mathcal{O}((r-1)^2) \quad \text{as } r \to 1\\ &= 0 + \mathcal{O}(r-1) \quad \text{as } r \to 1 \quad \text{(by Theorem 1.1 (b))}\\ &= \mathcal{O}(t^2) \quad \text{as } t \to 0.\end{aligned}$$

Expand $\hat{u}\cdot\nabla\hat{\rho}(\hat{0}) = (0,0,\hat{u})\cdot\nabla\rho(re^{i\theta}, 0, \dots, 0)$ about $r = 1$:

$$\begin{aligned}\hat{u}\cdot\nabla\hat{\rho}(\hat{0}) &= (0,0,\hat{u})\cdot\nabla\hat{\rho}(e^{i\theta}, 0, \dots, 0)\\ &\quad + (r-1)\frac{\partial}{\partial r}[(0,0,\hat{u})\cdot\nabla\rho(re^{i\theta}, 0, \dots, 0)]|_{r=1}\\ &\quad + \mathcal{O}((r-1)^2) \quad \text{as } r \to 1\\ &= 0 + \mathcal{O}(r-1) \quad \text{as } r \to 1 \quad \text{(by Theorem 1.1(b))}\\ &= \mathcal{O}(t^2) \quad \text{as } t \to 0.\end{aligned}$$

On the surface $\rho(re^{i\theta}, tu_3, \dots, tu_N) = 0$ then, (2) reads

$$-\mathcal{O}(t^2) = t\mathcal{O}(t^2) + \frac{1}{2}t^2\hat{u}^T\widehat{H}(\hat{0})\hat{u} + \mathcal{O}(t^3) \quad \text{as } t \to 0.$$

Divide by t^2 and let $t \to 0$ to obtain

$$\frac{1}{2}\hat{u}^T\widehat{H}(x_0)\hat{u} = K_\theta,$$

where $x_0 = (e^{i\theta}, 0, \dots, 0)$ and $K_\theta = -\lim_{r\to 1}\rho(re^{i\theta}, 0, \dots, 0)/(1 - r^2) > 0$. As $\widehat{H}$ is (symmetric) positive definite, the proof is complete. □

Since $\nabla\hat{\rho} \neq 0$ on $\partial\Omega_{r\theta}$, we may assume that $\hat{\rho}_N = \rho_N \neq 0$ locally on $\partial\Omega_{r\theta}$. Thus (Lebesgue) surface measure on $\partial\Omega$ and $\partial\Omega_{r\theta}$ are given respectively by

$$d\sigma = \frac{||\nabla\rho(x)||}{\rho_N(x)}\, dx_1 \wedge \dots \wedge dx_{N-1},$$

and

$$d\sigma_{r\theta} = \frac{||\nabla\hat{\rho}(x)||}{\hat{\rho}_N(\hat{x})}\, dx_3 \wedge \dots \wedge dx_{N-1}.$$

Substituting the latter expression into the former and observing Theorem 1.1 (a), (d), we obtain the following:

Lemma C. *We have*

$$d\sigma = \frac{||\nabla\rho(re^{i\theta}, x_3, \dots, x_N)||}{||\nabla\hat{\rho}(x_3, \dots, x_N)||}\, d\sigma_{r\theta} r\, dr\, d\theta.$$

Proof of Proposition 2.1. By (1) we may assume that Ω has the special form that Ω' enjoys in Theorem 1.1, so it suffices to show that

$$||F||_p \sim ||f||_{p,n-2} \qquad \forall\, f \in H^p_{n-2}(\Delta),$$

where $F \in H(\Omega)$ is given by $F(z_1, \dots, z_n) = f(z_1)$.

To this end, we begin by showing that

$$\int_{\partial\Omega_{r\theta}} \frac{||\nabla\rho||}{||\nabla\hat{\rho}||}\, d\sigma_{r\theta} \sim (1-r^2)^{n-2} \qquad \text{for } r \text{ near } 1. \tag{3}$$

By Theorem 1.1 (c) there is a $\delta > 0$ such that

$$\{(z_1, \dots , z_n) \in \mathbf{C}^n : |z_1| > 1-\delta\} \cap \Omega$$

is strongly convex near $\partial\Omega$. Thus, Lemma A–C are applicable in establishing (3).

Since $\nabla\rho \neq 0$ on the compact set $\partial\Omega$ and ρ is smooth, by Lemma A we have

$$\frac{||\nabla\rho||}{||\nabla\hat{\rho}(\hat{x})||} \sim \frac{1}{r_v(\hat{x})} \qquad \forall\, (re^{i\theta}, \hat{x}) \in \partial\Omega.$$

In the integral

$$\int_{\partial\Omega_{r\theta}} \frac{||\nabla\rho||}{||\nabla\hat{\rho}||}\, d\sigma_{r\theta} \sim \int_{\rho(re^{i\theta},\hat{x})=0} \frac{d\sigma_{r\theta}(\hat{x})}{r_v(\hat{x})}$$

we make the change of variables $x_j = tu_j$, $3 \leq j \leq N$, where $t = (1-r^2)^{1/2}$ to obtain

$$\int_{\rho(re^{i\theta},t\hat{u})=0} \frac{t^{N-3}\, d\sigma_{r\theta}(\hat{u})}{tr_v(\hat{u})} = t^{N-4} \int_{\rho(re^{i\theta},t\hat{u})=0} \frac{d\sigma_{r\theta}(\hat{u})}{r_v(\hat{u})}.$$

Now for r near 1 (i.e., t near 0), Lemma B shows that this last integral is bounded above and away from zero independently of θ. Finally, $t^{N-4} = (1-r^2)^{n-2}$ and we have established (3).

If $f \in H^p_{n-2}(\Delta)$ is continuous on $\overline{\Delta}$, then F is continuous on $\overline{\Omega}$, and by Lemma C we have

$$\begin{aligned}\int_{\partial\Omega} |F|^p\, d\sigma &= \int_0^{2\pi}\int_0^1 \int_{\partial\Omega_{r\theta}} |F|^p \frac{||\nabla\rho||}{||\nabla\hat{\rho}||}\; d\sigma_{r\theta} r\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^1 |f(re^{i\theta})|^p \int_{\partial\Omega_{r\theta}} \frac{||\nabla\rho||}{||\nabla\hat{\rho}||}\; d\sigma_{r\theta} r\, dr\, d\theta.\end{aligned}$$

So in this case the result follows from (3). For a general $f \in H^p_{n-2}(\Delta)$ the result follows from a limiting argument arising from the definition of $||\cdot||_p$; for $\varepsilon > 0$ small enough, the level sets $\rho + \varepsilon = 0$ retain all the relevant properties of $\rho = 0$. □

The next proposition can be proved directly with the help of Theorem 1.1 or by applying [**7**, Theorem 0.1]. We omit the proof. The subsequent corollary is immediate.

Proposition 2.2. *We have* $||f \circ \varphi||_{p,n-2} \lesssim ||f||_p$ *for all* $f \in H^p(\Omega)$.

Corollary 2.3. *We have* $||f \circ p||_p \lesssim ||f||_p$ *for all* $f \in H^p(\Omega)$.

($A(x) \lesssim B(x)$ means that the quotient of A by B is bounded above by a constant independent of x.)

By [**14**, Lemma 1.5] we have that

$$\int_0^1 \int_{-r}^r |f \circ \varphi(t)|^p (1-r)^{n-2}\, dt\, dr \lesssim ||f \circ \varphi||_{p,n-2}^p \qquad \forall\, f \in H^p(\Omega).$$

Changing the order of integration we obtain

(4)
$$(n-1)^{-1} \int_{-1}^1 |f \circ \varphi(t)|^p (1-|t|)^{n-1}\, dt \lesssim ||f \circ \varphi||_{p,n-2}^p \qquad \forall\, f \in H^p(\Omega),$$

which will be of subsequent use.

3. An application. Let $\Phi \in H(\Omega, \Omega)$ and define the composition operator $T = T_\Phi$ on $H(\Omega, \mathbf{C})$ by $T(f) = f \circ \Phi$. Corollary 2.3 says that T_p is a bounded operator on $H^p(\Omega)$.

Example 3.1. The composition operator $T = T_p$ is not compact on $H^p(\Omega)$.

Proof. There is a sequence $\{f_j\} \subset H^p_{n-2}(\Delta)$ with $||f_j||_{p,n-2} = 1$ such that $||f_j - f_k||_{p,n-2} \gg 0$ for all $j \neq k$. Set $F_j = f_j \circ \varphi^{-1} \circ p$, then by Proposition 2.1, $\{F_j\} \subset H^p(\Omega)$ is a bounded set, and T is the identity there. Again, by Proposition 2.1, we have

$$0 \ll ||f_j - f_k||_{p,n-2} \lesssim ||TF_j - TF_k||_p \qquad \forall\, j \neq k$$

and the proof is complete. $\square$

Assume now (with no loss of generality) that $0 \in \Omega$. Let $\Phi \in H(\Omega, \Omega)$ be without fixed points. Let $r_j \in (0,1)$ with $r_j \uparrow 1$ and set $\Phi_j = r_j \Phi$. Then $\Phi_j(\Omega) \Subset \Omega$ and so there is a $w_j \in \Omega$ such that $\Phi_j(w_j) = w_j$. Up to a subsequence we may assume that $w_j \to x \in \partial\Omega$. The point x (which is unique) is called the Denjoy-Wolff point for Φ; it is the point to which the iterates of Φ converge [**2**]. Obtaining it in this fashion is by now standard [**13, 2**].

Boundary estimates for k_Ω ([**1**, Propositions 1.2, 1.3]) provide constants C_1, C_2 and $C_3 > 0$ such that

$$\begin{aligned} k_\Omega(0, w_j) - k_\Omega(0, \Phi(w_j)) &= k_\Omega(0, w_j) - k_\Omega(0, w_j/r_j) \\ &\le C_1 - \frac{1}{2}\log\, d_\Omega(w_j) + C_2 + \frac{1}{2}\log\, d_\Omega(w_j/r_j) \\ &\le C_3 + \frac{1}{2}\log \frac{d_\Omega(w_j/r_j)}{d_\Omega(w_j)} \le C_3 \qquad \forall\, j. \end{aligned}$$

On the other hand, the quantity $k_\Omega(0,w) - k_\Omega(0,\Phi(w))$ is bounded below independently of w by the triangle inequality and the nonincreasing property of k_Ω. Thus there is an $A > 0$ such that

$$\liminf_{w \to x} [k_\Omega(0,w) - k_\Omega(0,\Phi(w))] = \frac{1}{2}\log A. \tag{5}$$

We fix some more notation. For $\xi \in \partial\Omega$, denote by $\varphi_\xi \in H(\Delta, \Omega)$ the unique continuously extended extremal disk such that $\varphi_\xi(0) = 0$ and $\varphi_\xi(1) = \xi$ [**3**, Proposition 1.7]. Denote by p_ξ the associated retraction onto $\varphi_\xi(\Delta) \subset \Omega$.

Lemma 3.2. [**4**, Corollary 3.3]. *Let $\Omega \Subset \mathbf{C}^n$ be smoothly bounded and strongly convex, and let $\Phi \in H(\Omega, \Omega)$ be such that* (5) *holds for some $x \in \partial\Omega$. There is a $y \in \partial\Omega$ such that*

$$\lim_{t \to 1} \frac{1 - \varphi_y^{-1} \circ p_y \circ \Phi \circ \varphi_x(t)}{1 - t} \quad \textit{exists.}$$

We come to the main result of this section, the proof of which is modeled after the analogous result for $B = \Omega$ appearing in [**14**]. We omit some of the details.

Proposition 3.3. *Let $\Omega \Subset \mathbf{C}^n$, $n \geq 2$, be smoothly bounded and strongly convex, and let $\Phi \in H(\Omega, \Omega)$. If $T = T_\Phi$ is a compact operator on $H^p(\Omega)$, then Φ has a unique fixed point in Ω.*

Proof. Assume that Φ has no fixed points; let $x, y \in \partial\Omega$ be given by Lemma 3.2. The functions

$$f_\alpha(\lambda) = \left(\frac{\alpha}{(1-\lambda)^{n-\alpha}}\right)^{1/p} \qquad 0 < \alpha < 1$$

are a bounded family in $H^p_{n-2}(\Delta)$ and so by Proposition 2.1, $\{f_\alpha \circ \varphi_y^{-1} \circ p_y\}$ is a bounded family in $H^p(\Omega)$. It is clear that $F_\alpha = f_\alpha \circ \varphi_y^{-1} \circ p_y \to 0$ uniformly on compact subsets of Ω as $\alpha \to 0$. If T is compact, it must be the case that $T(F_\alpha) = F_\alpha \circ \Phi \to 0$ in $H^p(\Omega)$ as $\alpha \to 0$. (This (elementary) compactness criterion is provided by [**17**, Theorem 2.5], which is easily seen to hold in our more general situation.)

By Proposition 2.2, (4) and Lemma 3.2, we have, however,

$$\begin{aligned}
||F_\alpha \circ \Phi||_p^p &\gtrsim ||F_\alpha \circ \Phi \circ \varphi_x||_{p,n-2}^p \\
&\gtrsim (n-1)^{-1} \int_{-1}^{1} |F_\alpha \circ \Phi \circ \varphi_x(t)|^p (1-|t|)^{n-1}\, dt \\
&\gtrsim (n-1)^{-1} \int_0^1 \frac{\alpha(1-t)^{n-1}\, dt}{|1 - \varphi_y^{-1} \circ p_y \circ \Phi \circ \varphi_x(t)|^{n-\alpha}} \\
&\gtrsim (n-1)^{-1} \int_0^1 \alpha(1-t)^{\alpha-1}\, dt \gg 0,
\end{aligned}$$

which is the desired contradiction.

If Φ has two fixed points $z, w \in \Omega$, then there is an extremal disk $\varphi \in H(\Delta, \Omega)$ such that [**20**, Theorem 4.1]

$$\{z, w\} \subset \varphi(\Delta) \subset \{\xi \in \Omega : \Phi(\xi) = \xi\}.$$

If p is the associated retraction onto $\varphi(\Delta)$ we have for each $f \in H^p(\Omega)$

$$T_p T_\Phi(f) = F \circ \Phi \circ p = F \circ p = T_p(f).$$

Now T_p is bounded by Corollary 2.3, so if T_Φ is compact, then $T_p T_\Phi = T_p$ must be compact. This contradicts Example 3.1, and we are done. □

We point out that the angular derivative of a map $\Phi \in H(\Omega, \Omega)$ at a point $x \in \partial\Omega$ is defined to be the limit appearing in Lemma 3.2, when it exists (see [**4**]). Thus the above proof contains the following:

Corollary 3.4. *If T_Φ is a compact operator on $H^p(\Omega)$, then the angular derivative of Φ exists at no point of $\partial\Omega$.*

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