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A RESTRICTION-EXTENSION PROPERTY FOR OPERATORS ON BANACH SPACES

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ABSTRACT. We study a natural property of operators between Banach spaces which is shared by the class of operators factoring through a Hilbert space. This leads in particular to an operator theoretic version of the Lindenstrauss-Tzafriri characterization of Hilbertian spaces. Also, we point out connections to a classical result of Johnson-König-Maurey-Retherford.

Lindenstrauss and Tzafriri [**6**] proved the celebrated characterization: *a Banach space is isomorphic to a Hilbert space if and only if there exists a constant* $C > 0$ *such that, whenever* E *is a finite dimensional subspace of* X*, we can find an operator* R *on* X *such that* $R|_E \equiv \mathrm{id}_E$*,* $RX = E$ *and* $||R|| \leq C$. It is clear that the operator R is a projection. This reformulation of the Lindenstrauss-Tzafriri result suggests what seems to be a natural operator theoretic analogon of a space in which every subspace is complemented:

Definition. Let $T : X \to Y$ be an operator between Banach spaces X and Y. T has property (H_∞) if for every closed subspace $Z \subset X$ the restriction $T|_Z$ admits a *bounded extension* $R : X \to Y$ with range $RX \subset \overline{TZ}$.

With only minor changes, the Davis, Dean, Singer argument of [**1**] applies to show that, given T with (H_∞), the "extensions of the restrictions" R corresponding to finite dimensional subspaces Z of X can be chosen to be uniformly bounded. In terms of the next definition this means that *if* T *has* (H_∞)*, then* T *has* (H):

Definition. We say that $T : X \to Y$ has *property* (H) if there exists a positive constant C such that whenever E is a finite dimensional

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subspace of X, we can find an operator $R : X \to Y$ such that $R|_E = T|_E$, $RX = TE$ and $||R|| \leq C||T||$.

Property (H) was first introduced by Figiel and Pełczyński [**2**, Problem 1227], where the question is whether operators with (H) factor through a Hilbert space. In fact, property (H) can be regarded as an attempt to extend the concept of the strongest possible uniform approximation property to the context of operators and, clearly, if an operator T factors through a Hilbert space, then it has (H_∞) (and *a fortiori* (H))). Under an additional hypothesis, we will prove the converse, which looks as a somewhat intriguing generalization of the Lindenstrauss-Tzafriri result above (see Theorem 5). As a further description of property (H), we will point out its connection to an operator-theoretic version of a classical result of Johnson-König-Maurey-Retherford [**5**, Theorem 3.11, 3.13]: an operator $T : X \to Y$ has (H) if and only if, for every nuclear operator $v : Y \to X$, the eigenvalues of vT are absolutely summable (see Theorem 9).

In the sequel, we will concentrate on property (H): it turns out to be self-dual (i.e., T has (H) if and only if T^* has (H): see Theorem 1) and is equivalent to the following property (H^c), which is defined in terms of finite codimensional spaces (see Theorem 2):

Definition. $T : X \to Y$, an operator between Banach spaces X and Y, has *property* (H^c) if there exists a positive constant C such that, whenever X_0 is a finite codimensional subspace of X, we can find an operator $R : X \to Y$ such that $R|_{X_0} = T|_{X_0}$, $RX \subset \overline{TX_0}$ and $||R|| \leq C||T||$.

It is not clear whether (H) is equivalent to (H_∞). Using ultraproducts, one can easily show that this is true in the case when the range of T is reflexive.

Let us agree to say that an operator T has (H) (or (H^c)) with constant C if C satisfies the property in the definition. In the following, we are going to adopt standard terminology and notation, such as in [**7**]. In particular, an operator $T : X \to Y$ is a continuous linear mapping, T^* is the corresponding adjoint operator. If E (respectively, F) is a subspace of a Banach space X (respectively, of its dual X^*), we have

$E^{\perp} \equiv \{x^* \in X^* : (x^*, e) = 0, \text{ for all } e \in E\}$ and ${}^{\perp}F \equiv \{x \in X : (x, f) = 0, \text{ for all } f \in F\}$. As concerns ultrafilters and ultrapowers of Banach spaces and operators, we refer to [**3, 4**].

Theorem 1. *An operator $T : X \to Y$ has (H) if and only if T^* has (H).*

Proof. Step 1. We first prove the "only if" part in the special case when X is finite dimensional. Suppose T has (H) with constant C. Let $G \subset Y^*$ be finite dimensional and define $E \equiv T^{-1}({}^{\perp}G)$. Let $S : X \to Y$ be such that $S|_E = T|_E$, $SX = TE = {}^{\perp}G$, and $||S|| \leq C||T||$. Define $R \equiv (T - S)^* : Y^* \to X^*$. If $g \in G$ and $x \in X$ we have $(S^*g, x) = (g, Sx) = 0$ (since $Sx \in {}^{\perp}G$) and so $S^*g = 0$, which means that $R|_G = T^*|_G$. Further, since $E = T^{-1}({}^{\perp}G) = {}^{\perp}(T^*G) = \ker(T - S)$, we have

$$\begin{aligned}\operatorname{rank} R &= \operatorname{rank}(T - S)^* = \dim[\ker(T - S)]^{\perp} \\ &= \dim[T^{-1}({}^{\perp}G)]^{\perp} = \dim({}^{\perp}(T^*G))^{\perp} = \dim T^*G,\end{aligned}$$

whence $\operatorname{rank} R = \dim T^*G$ (since $R|_G = T^*|_G$) and thus, $RY^* = T^*G$. Finally, $||R|| \leq (C+1)||T^*||$. All this means that T^* has (H) with constant $C + 1$.

Step 2. If $\dim X = \infty$, fix $G \subset Y^*$, with G finite dimensional. Let $E \subset X$ be finite dimensional and such that E 2-norms T^*G (i.e., the natural restriction mapping $J : T^*G \to E^*$ is invertible with $||J^{-1}|| \leq 2$).

Look at $(T|_E)^* : Y^* \to E^*$. By Step 1, $(T|_E)^*$ has (H) with constant $C + 1$ (we will assume that T has (H) with constant C). So we can find an operator $S : Y^* \to (T|_E)^*G$ such that $S|_G = (T|_E)^*|_G$, $||S|| \leq (C+1)||(T|_E)^*|| \leq (C+1)||T||$. Finally, note that $(T|_E)^*G = JT^*TG$ is 2-isomorphic to T^*G (J was chosen as a 2-isomorphism).

Define $R \equiv J^{-1}S : Y^* \to T^*G$. We see that $R|_G = T^*|_G$, and $||R|| \leq 2(C+1)||T^*||$. This shows that T^* has (H) with constant $2(C+1)$.

Step 3. To prove the "if" part of the Theorem, note that if T^* has (H), the above shows that T^{**} has (H) and so (if J_X, respectively J_Y, denotes the canonical embedding $X \to X^{**}$, respectively $Y \to Y^{**}$) $T^{**}J_X = J_Y T$ has (H). It is now immediate to see that $J_Y T$ having (H) implies that T has (H), too. $\square$

Remark. Using the same ideas, an easier argument gives that T has (H_∞) if and only if T^* does.

Corollary 2. *If $T : X \to Y$ is an operator and $Q : Z \to X$ is a quotient mapping such that TQ has (H),then T has (H).*

Proof. Since TQ has (H), by Theorem 1 Q^*T^* has (H). Q^* being an isomorphic embedding, it is clear that T^* has (H) and so, again by Theorem 1, T has (H). $\square$

Thanks to the Corollary, we may restrict our investigation of operators with property (H) to the injective ones. We will use this trick to prove the following

Theorem 3. *An operator $T : X \to Y$ has (H) if and only if it has (H^c).*

Proof. First we show that if $T : X \to Y$ has (H^c), if $Q : X \to X/\ker T$ is the natural quotient mapping and $T_0 : X/\ker T \to Y$ is such that $T = T_0 Q$, then T_0 has (H^c). In fact, let $W \subset X/\ker T$ be finite codimensional. Then, $Q^{-1}W$ is finite codimensional in X. Let $S : X \to Y$ be such that $S|_{Q^{-1}W} = T|_{Q^{-1}W}$, $SX \subset \overline{T(Q^{-1}W)} = \overline{T_0 W}$, and $||S|| \leq C||T||$ (T is assumed to have (H^c) with constant C). Since $\ker T \subset Q^{-1}W$, we have $\ker T \subset \ker S$ and so $S = S_0 Q$ for some $S_0 : X/\ker T \to Y$. Now $S_0|_W = T_0|_W$, $S_0(X/\ker T) = SX \subset \overline{T_0 W}$, and $||S_0|| \leq C||T_0||$, proving that T_0 has (H^c) with constant C.

By the above, if T has (H^c) with constant C, we may as well assume that T is injective. Let $E \subset X$ be finite dimensional. Let Y_0 be a finite codimensional subspace of Y containing TE and such that there is a projection P from Y_0 onto TE with $||P|| \leq 2$. Since $X_0 \equiv T^{-1}Y_0$ is finite codimensional in X (T is injective), there exists an operator $S : X \to Y$ such that $S|_{X_0} = T|_{X_0}$, $SX_0 = Y_0$, and $||S|| \leq C||T||$. If we look at $R \equiv PS : X \to TE$, we see that T has (H) with constant $2C$.

On the other hand, if T has (H) with constant C, T^* has (H) with constant $2(C+1)$ by Theorem 1. Without loss of generality (by Corollary 2), we may assume that T^* is injective. Fix $X_0 \subset X$ finite codimensional. Then, $F \equiv (T^*)^{-1}(X_0^\perp)$ is finite dimensional in Y^*. Therefore we can find an operator $R : Y^* \to X^*$ such that $R|_F = T^*|_F$, $RY^* = T^*F = X_0^\perp$, and $||R|| \leq 2(C+1)||T||$. Define $S \equiv T - R^*|_X$. If $x_0 \in X_0$ and $y^* \in Y^*$, we have $(R^*x_0, y^*) = (x_0, Ry^*) = 0$ (since $Ry^* \in X_0^\perp$), and so $R^*|_{X_0} = 0$, which implies $S|_{X_0} = T|_{X_0}$. Clearly, $||S|| \leq [2(C+1)+1]||T||$ and, further,

$$(TX_0)^\perp = (T^*)^{-1}(X_0^\perp) \subset \ker(T^* - R) \subset (SX)^\perp.$$

This implies $SX \subset {}^\perp((TX_0)^\perp) = \overline{TX_0}$, concluding the proof. $\square$

Before we formulate the generalization of the Lindenstrauss-Tzafriri result, we need a definition:

Definition. An operator $T : X \to Y$ is said to be *super strictly singular* (sss) if it doesn't have the following property: there exist a constant C and a sequence of finite dimensional subspaces (E_n) of X such that $\lim \dim E_n = \infty$ and $||(T|_{E_n})^{-1}|| \leq C$ for all n.

(sss) operators have been introduced by Mitjagin and Pełczyński [**8**]. If T is not (sss), we may roughly say that T "fixes" the E_n's uniformly (this is just a short form to say that T is a uniformly invertible isomorphism whenever restricted to each E_n). By Dvoretzky's theorem, we actually have that T is (sss) if and only if it doesn't "fix" the l_2^n's uniformly.

The reason for the (sss) terminology lies in the theory of the so-called super-ideals of operators. Recall that T is *strictly singular* (ss) if no restriction of T to any infinite dimensional subspace of X is an isomorphism. Following a standard procedure (see [**3**]), we say that T is super-(ss) if every ultrapower of T is (ss).

Lemma 4. *An operator $T : X \to Y$ is* (sss) *if and only if it is super-*(ss)*.*

Proof. If T is not (sss), then there exist a constant C and subspaces E_n of X as in the above definition. Let $\mathcal{U}$ be a free ultrafilter on $\mathbf{N}$. Then $Z \equiv \prod E_n/\mathcal{U}$ is infinite dimensional and it trivially isometrically embeds into the ultrapower $\prod X/\mathcal{U}$ of X. It is easy to check that the ultrapower $\prod T/\mathcal{U} : \prod X/\mathcal{U} \to \prod Y/\mathcal{U}$ is an isomorphism when restricted to Z and so it cannot be (ss). Consequently, T is not super-(ss).

Conversely, if T is not super-(ss), by definition there exists an ultrafilter $\mathcal{U}$ such that $\prod T/\mathcal{U}$ is not (ss), i.e., it satisfies $||(T|_Z)^{-1}|| \leq C$ for some infinite dimensional subspace Z of $\prod X/\mathcal{U}$ and for some constant C. If F is any finite dimensional subspace of Z, by a classical finite representability theorem for ultrapowers (see [**4**]) we can find $F' \subset X$ with $\dim F' = \dim F$ such that $||(T|_{F'})^{-1}|| \leq C+1$. Since the dimension of F is arbitrary, we see that T cannot be (sss). $\square$

The following theorem reduces to the Lindenstrauss-Tzafriri result if we take the special case of T being the identity on some Banach space. Recall that, if $S : W \to Z$ is an operator, then $\gamma_2(S)$ is defined to be the infimum of $||S_1||\,||S_2||$ over all factorizations $S = S_2 S_1$ through a Hilbert space. Fixing W and Z, γ_2 is a Banach norm on the space of all operators $W \to Z$ which factor through a Hilbert space. It is a standard fact that S factors through a Hilbert space if and only if $\sup\{\gamma_2(S|_F) : F \subset W,\ F \text{ finite dimensional}\}$ is bounded above.

Theorem 5. *If an operator $T : X \to Y$ has (H) and is not* (sss), *then it factors through a Hilbert space.*

Proof. To avoid confusion, note that we can assume that T is injective, by Corollary 2. Since T is not (sss) there exist a constant $C > 1$ and finite dimensional subspaces E_n of X such that $\lim \dim E_n = \infty$ and $||(T|_{E_n})^{-1}|| \leq C$ for all n. We may assume that $\dim E_n = n$ and (by Dvoretzky's theorem) that $d(E_n, l_2^n) \leq C$ for all n; this means that, for each n, we can find $R : l_2^n \to E_n$ an isomorphism such that $||R|| \leq 1$ and $||R^{-1}|| \leq C$.

Fix $n \in N$ and an arbitrary n-dimensional subspace E of X. We may assume that $E \cap E_n = TE \cap TE_n = \varnothing$. Let $T_1 : E \to l_2^n$, $T_2 : l_2^n \to TE$ be such that $T|_E = T_2 T_1$ and $||T_1|| = ||T_2|| = \gamma_2(T|_E)^{1/2}$.

Look at $E \oplus E_n$. Define the subspace $D \equiv \{e \oplus RT_1 e : e \in E\}$. Since $T|_{E \oplus E_n}$ has (H) (say, with constant C'), there exists an operator

$$T_D : E \oplus E_n \to TD = \{Te \oplus TRT_1 e : e \in E\}$$

such that T_D extends $T|_D$ and $||T_D|| \leq C'||T||$. Define the operators $\alpha : E \to E$ and $\beta : E_n \to E$ implicitly by

$$T_D(x \oplus y) = (T\alpha x + T\beta y) \oplus (TRT_1\alpha x + TRT_1\beta y).$$

Since T_D extends $T|_D$, we have

$$T_D(e \oplus RT_1 e) = T(e \oplus RT_1 e) = (T\alpha e + T\beta RT_1 e) \oplus \cdots = Te \oplus \cdots$$

for all $e \in E$, where we put "$\dots$" instead of the component in TE_n, which is not important for us. It follows that

$$T|_E = T\alpha + T\beta RT_1.$$

Finally,

$$\begin{aligned} \gamma_2(T|_E) &\leq \gamma_2(T_2T_1\alpha) + \gamma_2(T\beta RT_1) \\ &\leq \gamma_2(T|_E)^{1/2}(||T_1\alpha|| + ||T_D||) \\ &\leq \gamma_2(T|_E)^{1/2}(||R^{-1}(T|_{E_n})^{-1}TRT_1\alpha|| + C'||T||). \end{aligned}$$

Now $||R^{-1}(T|_{E_n})^{-1}TRT_1\alpha|| \leq ||R^{-1}||\,||(T|_{E_n})^{-1}||\,||T_D|| \leq C^2C'||T||$, and so we get

$$\gamma_2(T|_E) \leq (1 + C^2)C'||T||\gamma_2(T|_E)^{1/2}$$

showing that $\sup\{\gamma_2(T|_E) : E \subset X,\ E \text{ finite dimensional}\}$ is finite, but this means that T factors through a Hilbert space. $\square$

Using Theorem 1 we immediately deduce

Corollary 6. *If $T : X \to Y$ has (H) and doesn't factor through a Hilbert space, then $T, T^*, T^{**}, \dots$ are* (sss) *and TX is not closed in Y.*

Proof. Let T have (H). If any of the adjoints of T is not (sss), the same adjoint has (H) by Theorem 1, and so it factors through a

Hilbert space by Theorem 5. On the other hand, if TX is closed in Y, $T = T_0 Q$ where $Q : X \to X/\ker T$ is the canonical quotient map and $T_0 : X/\ker T \to Y$ is an isometric embedding. By Corollary 2, T_0 has (H) and so, by Lindenstrauss-Tzafriri's result, $X/\ker T$ is isomorphic to a Hilbert space. □

As pointed out above, it is open whether operators with (H) always factor through a Hilbert space (this was the original question of Figiel and Pełczyński). Anyway, after this paper had been submitted for publication, Pełczyński [**10**] showed me a very nice unpublished argument of his to prove that an operator with (H) *always* is weakly compact. This obviously makes the following corollary ridiculously obsolete. The proof presented here, however, clearly is much simpler than Pełczyński's more powerful approach.

Corollary 7. (a) *If $T : X \to Y$ has (H) and X is an $\mathcal{L}_\infty$-space, then T is weakly compact.*

(b) *If $T : X \to Y$ has (H) and Y is an $\mathcal{L}_1$-space, then T is weakly compact. In particular, if $Y = l_1$ then T is compact.*

Proof. (a) If T has (H) and is not (sss), then, by the above, T "fixes" the l_2^n's and, moreover, these l_2^n's are uniformly complemented in X. Since this cannot happen when X is an $\mathcal{L}_\infty$-space, T must by (sss) and so,in particular, T is strictly singular. By [**9**], T is weakly compact.

(b) If $T : X \to Y$ has (H) and Y is an $\mathcal{L}_1$-space, then T^* has (H) and part (a) applies. Hence, T is weakly compact. If $Y = l_1$, then T is actually compact because l_1 has the Schur property. □

In the positive direction, the following is evident after Theorem 5:

Corollary 8. *$X \to Y$ factors through a Hilbert space if and only if $T \oplus \mathrm{id}_{l_2} : X \oplus l_2 \to Y \oplus l_2$ has (H).*

We conclude with an application of this Corollary to the proof of the following characterization of property (H), which generalizes [**5**, Theorem 3.11, 3.13]:

Theorem 9. *An operator $T : X \to Y$ has (H) if and only if, for every nuclear operator $v : Y \to X$, the eigenvalues of vT are absolutely summable. Equivalently, if and only if there exists a constant c such that T satisfies*

$$\sum_{i=1}^{\infty} |\lambda_i(vT)| \leq c||v||_\wedge \tag{1}$$

for all $v : Y \to X$ of finite rank (where $|| \cdot ||_\wedge$ is the nuclear norm).

Proof. That condition (1) is equivalent to saying that, for every nuclear operator $v : Y \to X$, the eigenvalues of vT are absolutely summable, can be seen as in [**5**, Theorem 3.11, 3.13].

Now let T satisfy (1). We proceed in the spirit of [**5**, Lemma 3.12]: fix a finite dimensional subspace E of X, $\varepsilon > 0$, and call $T_0 : E \to TE$ the restriction and astriction of T. The operator norms

$$||S||_1 \equiv \inf\{||\widetilde{S}|| : \widetilde{S} \in \mathcal{L}(X, TE), \widetilde{S}|_E = S\} \quad \text{on } \mathcal{L}(E, TE)$$

and

$$||R||_{\wedge 1} \equiv ||i_E R||_\wedge \quad \text{on } \mathcal{L}(TE, E)$$

(where $i_E : E \to X$ is the natural inclusion) are easily seen to be in duality with respect to the trace functional (see [**5**, Lemma 3.13]). So there exists $R \in \mathcal{L}(TE, E)$, $||R||_{\wedge 1} = 1$, such that $||T_0||_1 = \operatorname{tr}(RT_0)$. Let $\widetilde{R} \in \mathcal{L}(Y, X)$ be a "Hahn-Banach extension" of R such that $||\widetilde{R}||_\wedge \leq (1 + \varepsilon)||i_E R||_\wedge = 1 + \varepsilon$. Then,

$$\begin{aligned} ||T_0||_1 = \operatorname{tr}(RT_0) &\leq \sum_{i=1}^{\infty} |\lambda_i(RT_0)| \\ &\leq \sum_{i=1}^{\infty} |\lambda_i(\widetilde{R}T)| \\ &\leq c||\widetilde{R}||_\wedge \leq c(1 + \varepsilon). \end{aligned}$$

Since ε was arbitrary, it is easy to see that the above inequality means that there exists an extension $\widetilde{T} : X \to TE$ of $T|_E$ with $||\widetilde{T}|| \leq c$, i.e., that T has property (H) with constant c.

Conversely, let T have (H) (say, with constant C) and let $v : Y \to X$ be nuclear. Let $(\lambda_i(vT))$ be the sequence of the eigenvalues of vT, with repetitions according to multiplicities. To prove that $(\lambda_i(vT))$ is absolutely summable, it is enough to show that

$$\left| \sum_{i=1}^{n} \lambda_{k_i}(vT) \right| \leq CN(v)$$

for any finite subsequence $(\lambda_{k_i}(vT))_{i=1}^{n}$. Pick such a subsequence and let x_{k_i}, $i = 1, \ldots, n$ be corresponding (linearly independent) eigenvectors. Let $E \equiv \operatorname{span}[x_{k_i}] \subset X$. Let $T_0 : E \to TE$ be the restriction and astriction of T and, similarly, let $v_0 : TE \to E$ be the restriction and astriction of v. Let $T_E : X \to TE$ be an extension of T_0 with $||T_E|| \leq C$ (T has (H) with constant C). We then have

$$\left| \sum_{i=1}^{n} \lambda_{k_i}(vT) \right| = |\operatorname{tr}(v_0 T_0)| = |\operatorname{tr}(vT_E)| \leq CN(v)$$

(see [**11**, Lemma 0.4]) which is what we needed. $\square$

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