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EXTREMAL PROBLEMS OF MARKOV'S TYPE FOR SOME DIFFERENTIAL OPERATORS

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ABSTRACT. Let $\mathcal{P}_n$ be the class of algebraic polynomials of degree at most n and $||P|| = (\int_{\mathbf{R}} |P(t)|^2\, d\sigma(t))^{1/2}$, where $d\sigma(t)$ is a measure corresponding to the classical orthogonal polynomials. We study extremal problems of Markov's type

$$C_{n,m} = \sup_{P\in\mathcal{P}_n} \frac{||\mathcal{D}_m P||}{||A^{m/2}P||},$$

where A is given by (1.1), and the differential operator $\mathcal{D}_m$ is defined by (1.3). The best constants are found for Legendre, Laguerre, and Hermite measures on $(-1,1)$, $(0,+\infty)$ and $(-\infty,+\infty)$, respectively.

1. Introduction. Let $\mathcal{P}_n$ be the class of all algebraic polynomials of degree at most n and $d\sigma(t)$ the measure of the classical orthogonal polynomials, i.e., $d\sigma(t) = w(t)\,dt$ on (a,b) with the weight $t \mapsto w(t)$ satisfying the differential equation of the first order

$$\frac{d}{dt}(A(t)w(t)) = B(t)w(t),$$

where the function $t \mapsto A(t)$ is given by

(1.1)
$$A(t) = \begin{cases} 1-t^2, & \text{for } a=-1,\, b=1 \text{ (the Jacobi case)},\\ t, & \text{for } a=0,\, b=+\infty \text{ (the generalized Laguerre case)},\\ 1, & \text{for } a=-\infty,\, b=+\infty \text{ (the Hermite case)}, \end{cases}$$

and $t \mapsto B(t)$ is a polynomial of the first degree.

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In this paper we study extremal problems of Markov's type

$$C_{n,m}(d\sigma) = \sup_{P\in\mathcal{P}_n} \frac{||\mathcal{D}_m P||_{d\sigma}}{||A^{m/2}P||_{d\sigma}} \tag{1.2}$$

for the differential operator $\mathcal{D}_m$ defined by

$$\mathcal{D}_m P = \frac{d^m}{dt^m}[A^m P], \qquad P\in\mathcal{P}_n,\ m\geq 1, \tag{1.3}$$

where

$$||P||_{d\sigma} = \left(\int_{\mathbf{R}} |P(t)|^2\, d\sigma(t)\right)^{1/2}.$$

We will find the best constant $C_{n,m}(d\sigma)$ in the three following cases:

1° The Legendre measure $d\sigma(t) = dt$ on $[-1,1]$;

2° The Laguerre measure $d\sigma(t) = e^{-t}\,dt$ on $[0,+\infty)$;

3° The Hermite measure $d\sigma(t) = e^{-t^2}\,dt$ on $(-\infty,+\infty)$.

Some extremal problems for differential operators were investigate by Stein [**6**] and Džafarov [**2**].

2. A general consideration. Let $P\in\mathcal{P}_n$, $d\sigma(t) = w(t)\,dt$ on (a,b), and let $\mathcal{D}_m$ be given by (1.3). An application of integration by parts gives

$$\begin{aligned} ||\mathcal{D}_m P||^2_{d\sigma} &= \int_a^b (\mathcal{D}_m P)^2 w\,dt \\ &= (-1)^m \int_a^b A^m P[w\mathcal{D}_m P]^{(m)}\,dt. \end{aligned}$$

Since

$$\begin{aligned} &(-1)^m \int_a^b A^m P[w\mathcal{D}_m P]^{(m)}\,dt \\ &\qquad = \int_a^b (-1)^m(\sqrt{w}A^{m/2}P)\left(\frac{A^{m/2}[w\mathcal{D}_m P]^{(m)}}{\sqrt{w}}\right) dt, \end{aligned}$$

using Cauchy-Schwarz inequality we obtain

$$||\mathcal{D}_m P||_{d\sigma}^2 \le ||A^{m/2}P||_{d\sigma} \left(\int_a^b \frac{A^m}{w} ([w\mathcal{D}_m P]^{(m)})^2 \, dt \right)^{1/2},$$

with equality if and only if

$$\mathcal{F}_m P = \frac{(-1)^m}{w} [w\mathcal{D}_m P]^{(m)} = \lambda P, \qquad P \in \mathcal{P}_n,$$

where λ is an arbitrary constant.

Taking a norm with respect to the measure $d\sigma_m(t) = A^m \, d\sigma(t) = A^m w \, dt$, we have

$$\frac{||\mathcal{D}_m P||_{d\sigma}}{||A^{m/2}P||_{d\sigma}} \le \left(\frac{||\mathcal{F}_m P||_{d\sigma_m}}{||P||_{d\sigma_m}} \right)^{1/2}, \tag{2.1}$$

with equality if and only if $\mathcal{F}_m P = \lambda P$, i.e.,

$$(-1)^m \frac{d^m}{dt^m} \left[w \frac{d^m}{dt^m} (A^m P) \right] = \lambda w P, \qquad P \in \mathcal{P}_n. \tag{2.2}$$

We are interested only in polynomial solutions of this equation. If they exist, then from the eigenvalue problem (2.2) and the inequality (2.1), we can determine the best constant in the extremal problem (1.2). Namely,

$$C_{n,m}(d\sigma) = \sqrt{\max_{0 \le \nu \le n} |\lambda_{\nu,m}|}, \tag{2.3}$$

where $\lambda_{\nu,m}$ are eigenvalues of the operator $\mathcal{F}_m$. Then the extremal polynomial is the eigenfunction corresponding to the maximal eigenvalue.

3. The Legendre case. In the Legendre case when $d\sigma(t) = dt$ on $(-1, 1)$, the differential equation (2.2) reduces to

$$(-1)^m \frac{d^{2m}}{dt^{2m}} [(1-t^2)^m P] = \lambda P, \quad P \in \mathcal{P}_n, \tag{3.1}$$

which has polynomial solutions if

$$\lambda = \lambda_{n,m} = (n+2m)(n+2m-1)\cdots(n+1) = \frac{(n+2m)!}{n!}. \tag{3.2}$$

Theorem 3.1. *Let $d\sigma(t) = dt$ on $(-1,1)$. The best constant in the extremal problem (1.2) is given by*

$$C_{n,m}(d\sigma) = \sqrt{\frac{(n+2m)!}{n!}}. \tag{3.3}$$

The supremum in (1.2) is attained only if $P(t) = \gamma C_n^{m+1/2}(t)$, where C_n^μ is the Gegenbauer polynomial of degree n which is orthogonal to $\mathcal{P}_{n-1}$, with respect to the weight function $t \mapsto (1-t^2)^{\mu-1/2}$ on the interval $(-1,1)$, and $\gamma(\neq 0)$ is an arbitrary real constant.

Proof. Let $\mu = m+1/2$. We start with the Gegenbauer equation in the Sturm-Liouville form (cf. Szegő [**7**])

$$\frac{d}{dt}((1-t^2)^{m+1}y') + n(n+2m+1)(1-t^2)^m y = 0. \tag{3.4}$$

A polynomial solution of this equation is the Gegenbauer polynomial $y = C_n^{m+1/2}$. Using (3.4) for $m=1$ and putting $u = (1-t^2)C_n^{3/2}(t)$, a direct calculation shows that u is a solution of the following differential equation of the second order

$$(1-t^2)u'' + (n+1)(n+2)u = 0.$$

By induction on m we can prove a general case, i.e., that the polynomial $t \mapsto u = (1-t^2)^m C_n^{m+1/2}(t)$ satisfies the $(2m)$-order differential equation

$$(1-t^2)^m \frac{d^{2m}u}{dt^{2m}} + (-1)^{m+1}(n+1)(n+2)\cdots(n+2m)u = 0. \tag{3.5}$$

For this reason we suppose that (3.5) is satisfied by $u = (1-t^2)^m C_n^{m+1/2}(t)$ for some integer m $(m \geq 1)$.

Using the following recurrence relations for Gegenbauer polynomials (cf. Szegő [**7**])

$$(7)\quad (1-t^2)\frac{d}{dt}C_n^{m+3/2}(t) = (n+2m+2)C_{n-1}^{m+3/2}(t) - ntC_n^{m+3/2}(t)$$

and

$$C_{n-1}^{m+3/2}(t) - tC_n^{m+3/2}(t) = -\frac{n+1}{2m+1}C_{n+1}^{m+1/2}(t),$$

we have that

$$\frac{d^{2m+2}}{dt^{2m+2}}((1-t^2)^{m+1}C_n^{m+3/2}(t)) = \frac{d}{dt}\left(\frac{d^{2m}}{dt^{2m}}G_{n,m}(t)\right),$$

where

$$\begin{aligned} G_{n,m}(t) &= \frac{d}{dt}((1-t^2)^{m+1}C_n^{m+3/2}(t)) \\ &= (1-t^2)^m\left[(1-t^2)\frac{d}{dt}C_n^{m+3/2}(t) - (2m+2)tC_n^{m+3/2}(t)\right] \\ &= -(1-t^2)^m(n+1)(n+2m+2)\frac{C_{n+1}^{m+1/2}(t)}{2m+1}. \end{aligned}$$

Now, based on the induction hypothesis (3.5), we find that

$$\frac{d^{2m}}{dt^{2m}}G_{n,m}(t) = (-1)^{m+1}(n+1)(n+2)\cdots(n+2m+2)\frac{C_{n+1}^{m+1/2}(t)}{2m+1}.$$

Finally, using the differentiation formula

$$\frac{d}{dt}C_{n+1}^{m+1/2}(t) = (2m+1)C_n^{m+3/2}(t),$$

we conclude that (3.5) holds for $m+1$.

Putting $P(t) = C_n^{m+1/2}(t)$ and comparing (3.5) and (3.1), where $\lambda = \lambda_{n,m}$ is given by (3.2), we see that $\lambda_{n,m}$ and $C_n^{m+1/2}(t)$ are eigenvalues and eigenfunction of the operator $\mathcal{F}_m$, respectively. Then, from (2.3) we obtain the best constant (3.3). The supremum in (1.2) is attained only if $P(t) = \gamma C_n^{m+1/2}(t)$, where γ ($\neq 0$) is an arbitrary constant. □

Remark 3.1. In the Jacobi case with the weight function $t \mapsto (1-t)^\alpha(1+t)^\beta$ $(\alpha, \beta > -1)$ the equation (2.2) has no polynomial solution for $|\alpha| + |\beta| > 0$.

4. The Laguerre and the Hermite cases. In the Laguerre case when $d\sigma(t) = e^{-t}\,dt$ on $(0, +\infty)$ the differential equation (2.2) becomes

$$(-1)^m e^t \frac{d^m}{dt^m}\left[e^{-t}\frac{d^m}{dt^m}(t^m P)\right] = \lambda P, \qquad P \in \mathcal{P}_n,$$

with polynomial solutions for

$$\lambda = \lambda_{n,m} = (n+m)(n+m-1)\cdots(n+1) = \frac{(n+m)!}{n!}.$$

Let $t \mapsto L_n^m(t)$ be the generalized Laguerre polynomial of degree n which is orthogonal to all polynomials of degree at most $n-1$ with respect to the weight function $t \mapsto t^m e^{-t}$ on $(0, +\infty)$. Similarly as in the previous section, using the differential equation for generalized Laguerre polynomials (cf. Szegő [**7**])

$$\frac{d}{dt}(t^{m+1}e^{-t}y') + nt^m e^{-t} y = 0,$$

the recurrence relations

$$\begin{aligned} t\frac{d}{dt}L_n^{m+1}(t) &= nL_n^{m+1}(t) - (n+m+1)L_{n-1}^{m+1}(t), \\ L_n^m(t) &= L_n^{m+1}(t) - L_{n-1}^{m+1}(t), \end{aligned}$$

and the differentiation formula $(e^{-t}L_n^m(t))' = -e^{-t}L_n^{m+1}(t)$, we can prove that the polynomial $t \mapsto v = t^m L_n^m(t)$ satisfies the following linear differential equation

$$t^m \frac{d^m}{dt^m}(e^{-t}v^{(m)}) + (-1)^{m+1}(n+1)(n+2)\cdots(n+m)v = 0.$$

According to this result and the general consideration in Section 2, we have:

Theorem 4.1. *Let $d\sigma(t) = e^{-t}\,dt$ on $(0, +\infty)$. The best constant in the extremal problem* (1.2) *is given by*

$$C_{n,m}(d\sigma) = \sqrt{\frac{(n+m)!}{n!}}.$$

The supremum in (1.2) *is attained only if $P(t) = \gamma L_n^m(t)$, where L_n^m is the generalized Laguerre polynomial of degree n, and γ $(\neq 0)$ is an arbitrary real constant.*

Remark 4.1. In the generalized Laguerre case with the weight function $t \mapsto t^s e^{-t}$ $(s > -1)$ the equation (2.2) has no polynomial solution for $s \neq 0$.

Finally, in the Hermite case when $d\sigma(t) = e^{-t^2}\,dt$ on the real line $\mathbf{R}$, the differential equation (2.2) reduces to

$$(-1)^m e^{t^2} \frac{d^m}{dt^m}[e^{-t^2} P^{(m)}] = \lambda P, \qquad P \in \mathcal{P}_n,$$

with polynomial solutions for

$$\lambda = \lambda_{n,m} = 2^m (n-m+1)\cdots(n-1)n.$$

The Hermite case is the simplest. The best constant is

$$C_{n,m}(d\sigma) = 2^{m/2}\sqrt{n!/(n-m)!}.$$

This result can be found in the Ph.D. thesis of Shampine [**5**] (see also, Dörfler [**1**] and Milovanović [**3**]). The case $m = 1$ was investigated by Schmidt [**4**] and Turán [**8**].

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