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ON BIFURCATION AND EXISTENCE OF POSITIVE SOLUTIONS FOR A CERTAIN p-LAPLACIAN SYSTEM

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1. Introduction. In this paper we study bifurcation of positive solutions for an elliptic system of the form

$$\begin{cases} -\Delta_p u_i + g_i(x, u_1, u_2) = \lambda_i |u_i|^{p-2} u_i & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega \end{cases} \qquad i = 1, 2 \tag{1.1}$$

on a smooth bounded domain Ω in $\mathbf{R}^N$, where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $p > 1$. We will prove that under appropriate conditions on g_i, (1.1) has a continuum of positive solutions bifurcating from the trivial solution. In particular, it follows from our main result (Theorem 3.1) that the following competitive system

$$\begin{aligned} -\Delta_p u_1 &= |u_1|^{p-2} u_1(\lambda_1 - a_{11}u_1 - a_{12}u_2) \quad \text{in } \Omega \\ -\Delta_p u_2 &= |u_2|^{p-2} u_2(\lambda_2 - a_{21}u_1 - a_{22}u_2) \quad \text{in } \Omega \\ u_i &= 0, \qquad i = 1, 2 \quad \text{on } \partial\Omega \end{aligned} \tag{1.2}$$

admits positive solutions (u_1, u_2), with $u_i > 0$, for some positive λ_i and a_{ij}, $i, j = 1, 2$.

When $p = 2$, the p-Laplacian becomes the usual Laplacian and system (1.1) has been studied extensively. We refer to the work of Cantrell [**5**] and the reference therein. In the case when $p \neq 2$, Δ_p appears in numerous situations. For example, in the context of reaction-diffusions, Murray [**16**] suggested using diffusion of the form $\Delta_p u$ in the study of diffusion-kinetic enzymes problems. We mention [**7**] and [**4**] for other references. Recently, systems associated with the p-Laplacian have commanded growing interest. Fleckinger et al. [**11, 12**] studied the

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cooperative system

$$(1.3)\qquad \begin{aligned} -\Delta_p u_i &= \sum_{j=1}^{n} a_{ij}|u_j|^{p-2}u_j + f_i \quad \text{in } \Omega \\ u_i &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

where a_{ij}, $i,j = 1,\dots,n$, are constants and $a_{ij} \geq 0$ for $i \neq j$. They obtained, among other results, the existence of positive solutions. For the system

$$(1.4)\qquad \begin{aligned} -\Delta_p u &= \lambda_1 |u|^{\alpha-1}u|v|^{\beta+1} \quad \text{in } \Omega \\ -\Delta_p v &= \lambda_2 |v|^{\beta-1}v|u|^{\alpha+1} \quad \text{in } \Omega \\ u &= v = 0 \quad \text{on } \partial\Omega \end{aligned}$$

de Thélin [**18**] obtained the existence of a positive eigenvalue ($\lambda_1 = \lambda_2$) associated with positive eigenfunction (u,v), while the existence and non-existence of solutions of (1.4) with $\lambda_1 \neq \lambda_2$ are considered by de Thélin and Vélin [**19**]. Felmer et al. [**10**] investigated the system

$$(1.5)\qquad \begin{aligned} -\Delta_p u - a(\alpha+1)|u|^{\alpha-1}u|v|^{\beta+1} &= \lambda|u|^{p-2}u \quad \text{in } \Omega \\ -\Delta_p v - a(\beta+1)|u|^{\alpha+1}|v|^{\beta-1}v &= \lambda|v|^{p-2}v \quad \text{in } \Omega \\ u = v &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Bifurcation of a p-Laplacian system coupled only by eigenvalues is studied by Binding and Huang [**4**]. Note that our prototype (1.2) is not included in (1.3)–(1.5).

The present work is motivated mainly by the work of Cantrell [**5**] and Huang and So [**14**] for the case $p = 2$. In [**14**], abstract bifurcation results were used to show the existence of positive equilibrium solutions for a gradostat model with n-vessels as well as for a model of a chemostat with diffusion. In the case $p \neq 2$, the main differential operator is no longer linear nor self-adjoint, consequently the usual compact linear operator theory, which is used in [**5**] and [**14**], is not directly applicable. Our bifurcation result for (1.1) is proved via the Alexander-Antman bifurcation theorem [**1**], by calculating the topological degree of certain nonlinear operators. The proof relies on a variational characterization of the first variational eigenvalue, which is given in Section 2, as well as a result in [**3**] which enables us to detect a change in the topological degree as a parameter crosses a certain eigenvalue.

The rest of the paper is organized as follows. In Section 2, the necessary notations and facts concerning the p-Laplacian are given as well as a proof of the aforementioned variational characterization of the first eigenvalue. The statement of our main result on the bifurcation and existence of positive solutions for (1.1) and its proof are given in Section 3.

2. Preliminaries. Let $p > 1$ and Ω be a smooth bounded domain in $\mathbf{R}^N$. In this paper, we work in the Sobolev space $W_0^{1,p}(\Omega)$ and consider only weak solutions. More precisely, $u \in W_0^{1,p}(\Omega)$ is a (weak) solution of the "general" problem

$$\begin{aligned} -\Delta_p u &= g(x,u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

if the following holds for all "test functions" $\phi \in C_0^\infty(\Omega)$:

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi = \int_\Omega g(x,u)\phi. \tag{2.2}$$

Since all our integrals will be taken on the whole of Ω, for simplicity, we will suppress the notation Ω from the integrals from now on.

We define the operator $K_p = (-\Delta_p)^{-1} : L^{p'}(\Omega) \to W_0^{1,p}(\Omega)$ with $1/p + 1/p' = 1$ as follows: $K_p w = u$ if and only if $u \in W_0^{1,p}(\Omega)$ and $-\Delta_p u = w$ in Ω. It is well known that K_p is well defined and is strictly positive, i.e., $0 \not\equiv w \geq 0$ implies $u > 0$ in Ω. Moreover, K_p is compact (cf. [**13**] and [**15**]). We further denote $\phi_p(u) := |u|^{p-2}u$.

Next we will recall a bifurcation result whose variants and proof can be found in [**3**, Theorem 5.1, **8**, Theorem 1.1] and [**9**, Theorem 4].

Proposition 2.1. *Assume $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is Carathéodory and satisfies*

$$\lim_{s\to 0} f(x,s) = 0 \tag{2.3}$$

and

$$\lim_{|s|\to\infty} f(x,s)|s|^{p-q} = 0 \tag{2.4}$$

uniformly for $x \in \Omega$, *for some* $q \in (p, \bar{p})$, *where* $\bar{p} = p + p^2/N$. *Assume that* $a(x) \in L^{\tilde{p}}(\Omega)$, *where* $\tilde{p} = N/p$ *if* $p < N$ *and* $\tilde{p} = 1$ *if* $p \geq N$. *Then for any sufficiently small* $\alpha > 0$, *there exist positive* $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *with* $\int |u|^p = \alpha^p$ *and* $\lambda \in \mathbf{R}$ *such that* (λ, u) *satisfies*

$$\begin{aligned} -\Delta_p u + \phi_p(u)[a(x) + f(x,u)] &= \lambda\phi_p(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.5}$$

Note that $\bar{p} < p^*$, where $p^* = Np/(N-p)$ if $p < N$ and $p^* = \infty$ if $p \geq N$. Our next theorem provides the variational characterization we need of a certain eigenvalue.

Theorem 2.2. *Assume that* $f : \Omega \times \mathbf{R}^2 \to \mathbf{R}$ *is continuous and there exists* $c > 0$ *such that*

$$|f(x,u,v)| \leq c(|u|^{q-p} + |v|^{q_1}), \qquad x \in \Omega,\ u \in \mathbf{R},\ v \in \mathbf{R} \tag{2.6}$$

where $q \in (p, \bar{p})$ *and* $q_1 \in [0, p^*p/N)$. *Then*

(i) *For* $\alpha > 0$ *sufficiently small, the solution* $(\lambda(v), u(v))$ *of*

$$\begin{aligned} -\Delta_p u + f(x,u,v)\phi_p(u) &= \lambda\phi_p(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.7}$$

given by Proposition 2.1 *with* $\int |u(v)|^p = \alpha^p$ *and* $u(v) > 0$ *is a continuous function from* $L^p(\Omega)$ *to* $\mathbf{R} \times W_0^{1,p}(\Omega)$, *i.e., if* $v \to \tilde{v}$ *in* $L^p(\Omega)$, *then* $\lambda(v) \to \lambda(\tilde{v})$ *in* $\mathbf{R}$ *and* $u(v) \to u(\tilde{v})$ *in* $W_0^{1,p}(\Omega)$, *and*

(ii) (λ, u) *is also continuous in* α, *and* $\lambda \to \tilde{\lambda}$ *as* $\alpha \to 0^+$, *where* $\tilde{\lambda}$ *is the first eigenvalue of*

$$\begin{aligned} -\Delta_p u + f(x,0,v)\phi_p(u) &= \lambda\phi_p(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.8}$$

(iii)

$$\lambda(v) = \inf_{\int |u|^p = \alpha^p} \alpha^{-p} \left(\int |\nabla u|^p + \int |u|^p f(x, u(v), v) \right) \tag{2.9}$$

and $(\lambda(v), u(v))$ *is unique provided* $f(x, u, v)$ *is increasing in* u *for* $u > 0$.

Proof. The first two parts of the theorem follow from a standard procedure, see, e.g., the proof of [**3**, Theorem 4.1]. To prove (iii), fix $v \in L^p(\Omega)$. Let (λ_1, u_1) be the corresponding eigenpair given by part (i), i.e., $u_1 > 0$, $\int |u_1|^p = \alpha^p$ and

$$\begin{aligned} -\Delta_p u_1 + f(x, u_1, v)\phi_p(u_1) &= \lambda_1 \phi_p(u_1) \quad \text{in } \Omega \\ u_1 &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.10}$$

Let (λ_2, u_2) be the eigenpair of the "homogeneous" problem

$$\begin{aligned} -\Delta_p u + f(x, u_1, v)\phi_p(u) &= \lambda \phi_p(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.11}$$

with $\int |u_2|^p = \alpha^p$, $u_2 > 0$. Then (cf. [**17**])

$$\begin{aligned} \lambda_2 &= \alpha^{-p} \left(\int |\nabla u_2|^p + \int |u_2|^p f(x, u_1(v), v) \right) \\ &= \inf_{\int |u|^p = \alpha^p} \alpha^{-p} \left(\int |\nabla u|^p + \int |u|^p f(x, u_1(v), v) \right). \end{aligned}$$

Multiplying (2.10) by $(u_1^p - u_2^p)/\phi_p(u_1) = (u_1^p - u_2^p)/u_1^{p-1}$ and (2.11) (with (λ, u) replaced by (λ_2, u_2)) by $(u_1^p - u_2^p)/\phi_p(u_2)$ and integrating the difference of the two resulting equations, we obtain

$$\begin{aligned} I(u_1, u_2) &:= \int \left(-\Delta_p u_1, \frac{u_1^p - u_2^p}{u_1^{p-1}} \right) - \int \left(-\Delta_p u_2, \frac{u_1^p - u_2^p}{u_2^{p-1}} \right) \\ &= (\lambda_1 - \lambda_2) \int (|u_1|^p - |u_2|^p) = 0. \end{aligned}$$

By Proposition 2 of [**2**], u_1 is a constant multiple of u_2. Thus, $u_1 = u_2$. Consequently $\lambda_1 = \lambda_2$ and (2.9) follows. Furthermore, Theorem 2.2 of [**3**] implies the uniqueness of $(\lambda(v), u(v))$ and the proof is complete. □

Remark 2.3. In general, the minimizer of the quotient

$$\tilde{\lambda} = \inf_{\int |u|^p = \alpha^p} \frac{\int |\nabla u|^p + \int |u|^p f(x,u)}{\int |u|^p}$$

is not necessarily a solution of

$$-\Delta_p u + f(x,u)\phi_p(u) = \lambda \phi_p(u)$$

unless $D_u(uf(x,u)) = f(x,u)$, i.e., $f(x,u) \equiv a(x)$ is independent of u.

3. Main theorem. Consider the following p-Laplacian system

$$\begin{cases} -\Delta_p u_i + f_i(x, u_1, u_2)\phi_p(u_i) = \lambda_i \phi_p(u_i) & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega \end{cases}, \quad i = 1, 2. \tag{3.1}$$

Assume that $f_i : \Omega \times \mathbf{R}^2 \to \mathbf{R}$ $(i = 1, 2)$ are continuous functions and they satisfy the following hypotheses:

(H1) There exists $c > 0$ such that

$$|f_1(x, u_1, u_2)| \leq c(|u_1|^{q-p} + |u_2|^{q_1}) \tag{3.2}$$

and

$$|f_2(x, u_1, u_2)| \leq c(|u_2|^{q-p} + |u_1|^{q_1}) \tag{3.3}$$

for all $x \in \Omega$ and $u_1, u_2 \in \mathbf{R}$, where $q \in (p, \overline{p})$ and $q_1 \in [0, pp^*/N)$, and

(H2) f_1 is strictly increasing in u_1 for $u_1 > 0$, f_2 is increasing in u_2 for $u_2 > 0$, $f_1(x, 0, 0) \geq 0$, and

$$|f_1(x, u, 0) - f_1(x, v, 0)| \leq c_1 \cdot |u - v|^{q-p}, \tag{3.4}$$

for some constant $c_1 > 0$.

Note that by modifying f_1 and f_2 as follows: $f_i(x, u_1, u_2) = f_i(x, 0, u_2)$ if $u_1 \le 0$ and $f_i(x, u_1, u_2) = f_i(x, u_1, 0)$ if $u_2 \le 0$ $(i = 1, 2)$, one can apply the maximum principle in [**17**] to each equation in (3.1) to ensure that any non-trivial solution (u_1, u_2) must be positive in Ω.

We remark that, for $a_{ij} > 0$, $i, j = 1, 2$, functions of the forms

$$f_1 = a_{11}u_1 + a_{12}u_2, \qquad f_1 = \frac{a_{11}u_1}{1 + a_{12}u_2},$$

and

$$f_2 = \pm a_{21}u_1 + a_{22}u_2, \qquad f_2 = \frac{a_{22}u_2}{1 + a_{21}u_1},$$

satisfy hypotheses (H1) and (H2). These functions appear in various competitive and predator-prey models.

For $\beta > 0$ fixed and sufficiently small, we denote by (λ_1^0, u_1^0) the solution of

$$\begin{aligned} -\Delta_p u_1 + f_1(x, u_1, 0)\phi_p(u_1) &= \lambda_1 \phi_p(u_1) \quad \text{in } \Omega \\ u_1 &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.5}$$

with $u_1^0 > 0$ and $\int (u_1^0)^p = \beta^p$. The existence of the pair (λ_1^0, u_1^0) is guaranteed by Theorem 2.2. Then $(\lambda_1^0, u_1^0, \lambda_2, 0)$ is a solution of (3.1) for any λ_2. We will refer to this solution as the trivial branch and consider it as parametrized by $\beta > 0$ and λ_2. In addition, we will denote by (λ_2^0, u_2^0) the solution of the homogeneous problem

$$\begin{aligned} -\Delta_p u_2 + f_2(x, u_1^0, 0)\phi_p(u_2) &= \lambda_2 \phi_p(u_2) \quad \text{in } \Omega \\ u_2 &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{3.6}$$

with $u_2^0 > 0$ and $\int (u_2^0)^p = \beta^p$. The existence of (λ_2^0, u_2^0) again follows from Theorem 2.2. Note that, under the above assumptions, $\lambda_1^0 > \lambda_0$, where λ_0 is the first eigenvalue of

$$\begin{aligned} -\Delta_p u &= \lambda \phi_p(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

However, λ_2^0 might be negative if f_2 is sufficiently negative.

We are now ready to state our main theorem.

Theorem 3.1. *Assume f_1 and f_2 satisfy* (H1) *and* (H2). *Then there exists a continuum $\Lambda \subset \mathbf{R}^2 \times [W_0^{1,p}(\Omega) \cap L^\infty(\Omega)]^2$ of solutions to* (3.1) (*in the sense of* [**1**] *and* [**5**]). *Moreover, Λ contains the "trivial branch" $(\lambda_1^0, \lambda_2, u_1^0, 0)$, given as above, and a positive branch $(\lambda_1, \lambda_2, u_1, u_2)$ with $u_i > 0$, $i = 1, 2$, bifurcating out from $(\lambda_1^0, \lambda_2^0, u_1^0, 0)$.*

As a corollary of Theorem 3.1, we have the following existence result.

Theorem 3.2. *Consider the system*

$$
\begin{aligned}
-\Delta_p u_1 &= \phi_p(u_1)(\lambda_1 - a_1(x)u_1 - b_1(x)u_2) \quad \text{in } \Omega \\
-\Delta_p u_2 &= \phi_p(u_2)(\lambda_2 \pm a_2(x)u_1 - b_2(x)u_2) \quad \text{in } \Omega \\
u_i &= 0, \quad i = 1, 2, \quad \text{on } \partial\Omega,
\end{aligned} \tag{3.7}
$$

with $0 \leq a_i, b_i \in L^\infty(\Omega)$, $i = 1, 2$. Then (3.7) *has a solution $(\lambda_1, \lambda_2, u_1, u_2)$ satisfying $\lambda_i > 0$ and $u_i > 0$, $i = 1, 2$.*

Proof of Theorem 3.1. Define the operator $T : \mathbf{R}^2 \times [W_0^{1,p}(\Omega) \cap L^\infty(\Omega)]^2 \to [W_0^{1,p}(\Omega) \cap L^\infty(\Omega)]^2$ as follows:

$$
T(\lambda_1, \lambda_2, u_1, u_2) = \begin{pmatrix} K_p[\phi_p(u_1)(\lambda_1 - F_1(u_1, u_2))] \\ K_p[\phi_p(u_2)(\lambda_2 - F_2(u_1, u_2))] \end{pmatrix}, \tag{3.8}
$$

where F_i is the Nemytskii operator induced by f_i, $i = 1, 2$. Evidently, T is well defined and for fixed (λ_1, λ_2), $T(\lambda_1, \lambda_2) : [W_0^{1,p}(\Omega) \cap L^\infty(\Omega)]^2 \to [W_0^{1,p}(\Omega) \cap L^\infty(\Omega)]^2$ is compact, since $\phi_p(\cdot)$ and F_i are continuous and K_p is compact. We also note that, $T(\lambda_1, \lambda_2, u_1, u_2) = \binom{u_1}{u_2}$ if and only if $(\lambda_1, \lambda_2, u_1, u_2)$ is a solution of (3.1). According to [**1**], it suffices to show that there is a change in the fixed point index $\operatorname{ind}(T(\lambda_1^0, \mu), (u_1^0, 0))$ of the operator $T(\lambda_1^0, \mu)$ at the fixed point $(u_1^0, 0)$ as μ crosses λ_2^0.

We introduce two auxiliary operators $\widetilde{T}$ and T^0 as follows:

$$
\widetilde{T}(\mu, u_1, u_2) = \begin{pmatrix} K_p[\phi_p(u_1)(\lambda_1^0 - F_1(u_1^0, u_2))] \\ K_p[\phi_p(u_2)(\mu - F_2(u_1, u_2))] \end{pmatrix}
$$

and

$$
T^0(\mu, u_1, u_2) = \begin{pmatrix} T_1^0 u_1 \\ T_2^0(\mu) u_2 \end{pmatrix} = \begin{pmatrix} K_p[\phi_p(u_1)(\lambda_1^0 - F_1(u_1^0, 0))] \\ K_p[\phi_p(u_2)(\mu - F_2(u_1^0, u_2))] \end{pmatrix}.
$$

We further introduce the sets

$$B_1^\alpha = \left\{ u \in W_0^{1,p}(\Omega) \ : \ \int |u - u_1^0|^p < \alpha^p \right\},$$
$$B_2^\alpha = \left\{ u \in W_0^{1,p}(\Omega) \ : \ \int |u|^p < \alpha^p \right\},$$
$$B^\alpha = \{(u,v) \in (W_0^{1,p}(\Omega))^2 \ : \ (u,v) \in B_1^\alpha \times B_2^\alpha\},$$

and

$$S_1^\alpha = \partial B_1^\alpha, \qquad S_2^\alpha = \partial B_2^\alpha, \qquad S^\alpha = \partial B^\alpha.$$

Fix any $\mu \neq \lambda_2^0$ and $|\mu - \lambda_2^0|$ small so that there exists $\alpha > 0$ sufficiently small (in fact also assume $\alpha < \beta$) such that

$$\begin{aligned} -\Delta_p u_2 &= \phi_p(u_2)(\mu - f_2(x, u_1, u_2)) \quad \text{in } \Omega \\ u_2 &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.9}$$

has no positive solution $u_2 \in \overline{B_2^\alpha}$ as long as $u_1 \in \overline{B_1^\alpha}$. This is possible due to the continuity of eigenpair on parameters (cf. Theorem 2.2).

We now show that, for $0 < |\mu - \lambda_2^0| \ll 1$,

$$\operatorname{ind}(T(\lambda_1^0, \mu), (u_1^0, 0)) = \operatorname{ind}(\tilde{T}(\mu), (u_1^0, 0)). \tag{3.10}$$

Of course we have to show that $(u_1^0, 0)$ is an isolated fixed point of both $T(\lambda_1^0, \mu)$ and $\widetilde{T}(\mu)$ in order that the fixed point indices in (3.10) are well defined. We will present a proof which guarantees that the indices in (3.10) do make sense and they are equal.

Define, for $t \in [0,1]$,

$$H_1(t, u_1, u_2) = \begin{pmatrix} K_p[\phi_p(u_1)(\lambda_1^0 - tF_1(u_1^0, u_2) - (1-t)F_1(u_1, u_2))] \\ K_p[\phi_p(u_2)(\mu - F_2(u_1, u_2))] \end{pmatrix}.$$

We claim that $I - H_1(t)$ does not vanish on S^α for $t \in [0,1]$. Suppose not. Then $H_1(t, u_1, u_2) = \binom{u_1}{u_2}$ for some $t \in [0,1]$ and $(u_1, u_2) \in S^\alpha$. Assume $u_2 > 0$. Then $\int (u_2)^p \le \alpha^p$, $\int |u_1 - u_1^0|^p \le \alpha^p$, and (3.9) is satisfied. This contradicts the choice of α. Note that since our solutions (i.e., fixed points of $H_1(t)$) are nonnegative, $u_2 \not> 0$ implies $u_2 \equiv 0$. Consequently $u_2 \equiv 0$ and $0 < u_1 \in S_1^\alpha$. Then u_1 satisfies

$$-\Delta_p u_1 = \phi_p(u_1)(\lambda_1^0 - tf_1(x, u_1^0, 0) - (1-t)f_1(x, u_1, 0)). \tag{3.11}$$

Let $\int |u_1|^p = \gamma^p$, $\gamma > 0$. Multiplying (3.11) by $(u_1^p - (u_1^0)^p)/\phi_p(u_1)$ and (3.5) by $(u_1^p - (u_1^0)^p)/\phi_p(u_1^0)$, and integrating the difference of the two resulting equations, we have

$$0 \leq \int \left(-\Delta_p u_1, \frac{u_1^p - (u_1^0)^p}{u_1^{p-1}} \right) - \left(-\Delta_p u_1^0, \frac{u_1^p - (u_1^0)^p}{(u_1^0)^{p-1}} \right)$$
$$= -(1-t) \int \left(f_1(x, u_1, 0) - f_1(x, u_1^0, 0) \right) \left(u_1^p - (u_1^0)^p \right).$$

Using (H2), one can easily see that the right hand side of the above is nonpositive. We thus deduce (from [**2**, Proposition 2]) that $u_1 = \delta u_1^0$, for some $\delta > 0$. There are three possibilities.

(i) $\delta = 1$, then $u_1^0 = u_1 \in S_1^\alpha$, a contradiction.

(ii) $\delta \in (0,1)$, i.e. $\gamma < \beta$. Substituting $u_1 = \delta u_1^0$ into (3.11), we obtain (again using (H2))

$$-\Delta_p u_1^0 > \phi_p(u_1^0)(\lambda_1^0 - f_1(x, u_1^0, 0)),$$

which contradicts (3.5).

(iii) Finally, $\delta > 1$, i.e. $\gamma > \beta$. As before, replacing u_1 by δu_1^0 in (3.11), we obtain

$$-\Delta_p u_1^0 < \phi_p(u_1^0)(\lambda_1^0 - f_1(x, u_1, 0)),$$

which is again a contradiction.

Hence, $I - H_1(t)$ does not vanish on S^α for $t \in [0,1]$.

Notice that it follows from the above arguments that for $t = 0, 1$, $I - H_1(t)$ equals to zero only at $(u_1^0, 0)$. In other words, $(u_1^0, 0)$ is the only fixed point of $T(\lambda_1^0, \mu)$ and $\widetilde{T}(\mu)$ in the neighborhood B^α of $(u_1^0, 0)$. (3.10) now follows from the homotopy invariance of the degree.

Next, we prove,

$$\operatorname{ind}(\widetilde{T}(\mu), (u_1^0, 0)) = \operatorname{ind}(T^0(\mu), (u_1^0, 0)). \tag{3.12}$$

We proceed in a similar manner as before. Define, for $t \in [0,1]$,

$$H_2(t, u_1, u_2) = \begin{pmatrix} K_p[\phi_p(u_1)(\lambda_1^0 - F_1(u_1^0, tu_2))] \\ K_p[\phi_p(u_2)(\mu - F_2(tu_1 + (1-t)u_1^0, u_2))] \end{pmatrix}.$$

We claim that $I - H_2(t)$ does not vanish on S^α for $t \in [0,1]$. Indeed, suppose $H_2(t, u_1, u_2) = \binom{u_1}{u_2}$, for some $t \in [0,1]$ and $(u_1, u_2) \in S^\alpha$. Assume first that $u_2 > 0$. Then u_2 satisfies

$$-\Delta_p u_2 = \phi_p(u_2)(\mu - f_2(x, tu_1 + (1-t)u_1^0, u_2)).$$

Again our choice of α implies that this is impossible.

Thus, we must have $u_1 > 0$ and $u_2 \equiv 0$. In that case u_1 satisfies

$$-\Delta_p u_1 = \phi_p(u_1)(\lambda_1^0 - f_1(x, u_1^0, 0)).$$

But, as proved above, this equation has no positive solution on S_1^α. Thus we obtain another contradiction. Checking the cases $t = 0, 1$ further shows that $T^0(\mu)$ has no fixed point in B^α other than $(u_1^0, 0)$. Consequently (3.12) holds.

According to a result in the degree theory (cf. Theorems 8.5, 8.7 and Proposition 8.4 of Deimling [**6**])

$$\operatorname{ind}(T^0, (u_1^0, 0)) = \operatorname{ind}(T_1^0, u_1^0) \cdot \operatorname{ind}(T_2^0(\mu), 0). \tag{3.13}$$

We deduce from Lemma 5.1 and the proof of Theorem 5.1 of [**3**] that

$$\operatorname{ind}(T_2^0(\mu'), 0) \neq \operatorname{ind}(T_2^0(\tilde{\mu}), 0)$$

provided $\mu' < \lambda_2^0 < \tilde{\mu}$. On the other hand, it is easily seen that

$$\operatorname{ind}(T_1^0, u_1^0) = \operatorname{ind}(K_p(\lambda_1^0 \phi_p(\cdot)), u_1^0) = 1$$

provided $\lambda_1^0 > \lambda_0$.

Using the above two facts and together with (3.10), (3.12) and (3.13), we have shown that $\operatorname{ind}(T(\lambda_1^0, \mu), u_1^0)$ indeed changes its value as μ crosses λ_2^0. Our conclusion now follows from the global bifurcation theorem of Alexander and Antman [**1**]. This completes the proof. □

Remark. We can only deal with the "true" bifurcation case. If f_1 is not strictly increasing in u_1, for example, if f_1 is independent of u_1, then our method breaks down.

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