ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 1, Winter 1995

HIERARCHICALLY STRUCTURED BRANCHING POPULATIONS WITH SPATIAL MOTION

KENNETH J. HOCHBERG

ABSTRACT. We consider hierarchically structured systems of individuals undergoing birth-and-death or branching processes at each level, with spatial motion at the lowest level. The measure-valued continuous diffusion limit process is formulated, and the carrying dimension of the topological support of the underlying random measure is analyzed.

1. Introduction. Systems of particles that undergo simultaneous diffusion and birth-and-death or branching have long been used to model random phenomena in several fields. Early development of the Galton-Watson branching process was stimulated by studies of the spread of wealth and royalty among the gentry in turn-of-the-century England. Years later, such processes were applied to models describing the spread of mutant genes through natural populations, the spread of diseases among susceptible individuals, and the distribution of both neutral and selectively advantageous allelic types in population genetics.

It has, in fact, been the field of population biology which has provided both the motivation and direction for much of the recent work on both interacting and noninteracting branching-diffusing systems. This has certainly been true in the case of the Dawson-Watanabe superprocess, a measure-valued stochastic process that arises as the high-density continuous diffusion limit of an infinite system of noninteracting branching-diffusing individuals or particles. It is even more evident in the case of the measure-valued Fleming-Viot processes, which arise as the diffusion approximations to certain models originally formulated in population genetics.

In this paper we will study certain aspects of hierarchically structured birth-and-death and branching populations with spatial motion.

Received by the editors on June 13, 1992, and in revised form on January 21, 1993.
AMS 1980 *Subject Classification.* Primary 60G57, Secondary 60J80.
Keywords and phrases. Measure-valued process, superprocess, branching process, diffusion process, multilevel process, Hausdorff dimension.

Thus, we are dealing with populations of individuals or particles which not only undergo, individually, both diffusion and birth-and-death or branching, but which also undergo an additional birth-death or branching process which acts upon groups of particles simultaneously. Thus, for example, individual particles within a colony may multiply or disappear as a result of branching or birth-and-death at the individual level, and whole colonies or groups of individuals may disappear or reproduce according to another such process, with splitting rates perhaps depending upon the size of the colony. Between splits, each particle is assumed to undergo some spatial motion independent of the motion of the other particles. Clearly, additional levels of branching and birth-death can be added to the model as well.

Such multilevel processes were originally introduced in Dawson and Hochberg [**1**] and Dawson, Hochberg and Wu [**2**]. They arise naturally in population biology as models for mitochondrial DNA, where sampling takes place at both the individual and organelle levels. They arise as well in models describing the spread of species in competitive environments, where each species attempts to create copies of itself so as to become established in neighboring territory, while the possibility of simultaneous extinction by natural calamity or nearby competitors exists for all members present in a single colony. Applications to the spread of viruses in computerized data collections and environmental damage caused by propellants in the atmosphere, which can effect individuals (e.g., asthmatic individuals) as well as whole colonies (say, via destruction of the ozone-layer over an entire land-mass) can easily be detailed.

Mathematical analysis of the properties of such multilevel processes is complicated not only by the intrinsic fact that several processes at different levels are affecting the number and locations of individual particles, but even more so by the fact that the particles no longer exhibit independent behavior, since higher-level birth-death and branching affect groups of particles simultaneously. The absence of independence is crucial and, in some instances, necessitates a finer analysis of the path behavior than is needed for ordinary branching-diffusing systems.

Existence and uniqueness of a measure-valued process that describes the two-level birth-and-death situation, and the two-level branching process as a special case, were proved in the papers cited above. In addition, the moment structure of these processes was described via analysis of the infinite system of moment equations. Asymptotic growth rates were derived in [**1**], as was a Yaglom-type conditional limit law, conditioned on nonextinction of the process.

In this paper we will look more closely at the topological support and long-term properties of such hierarchically structured processes with spatial diffusion added. The resulting continuous limit is then a "super-2 process," or a measure-valued process defined on the space of Borel measures on $\mathbf{R}^d$. Thus, at each fixed time, the process has a value given by a measure on the space of measures $M(\mathbf{R}^d)$. Such processes can be generalized to "super-n processes" with values in the space $M^n(\mathbf{R}^d) = M(M^{n-1}(\mathbf{R}^d)) = M^{n-1}(M(\mathbf{R}^d))$.

2. Multilevel birth-and-death and branching. The two-level birth-and-death process was described in [**2**] as an $\mathcal{N}(\mathbf{Z}^+)$-valued pure jump Markov process, where $\mathcal{N}(\mathbf{Z}^+)$ denotes the set of integer-valued measures on $\mathbf{Z}^+$. Births and deaths can take place at each level, so the four possible transitions of state consist of birth or death of a level-1 particle in a level-2 particle of size i at rates $\lambda_1(n_i, i)$ and $\mu_1(n_i, i)$, respectively, and birth or death of a level-2 particle of size i at rates $\lambda_2(n_i, i)$ and $\mu_2(n_i, i)$, where n_i denotes the number of level-2 particles consisting of exactly i level-1 particles. Note that the birth and death rates $\lambda_j(n_i, i)$ and $\mu_j(n_i, i)$, $j = 1, 2$, may depend on the size i of the level-2 particle and on the number n_i of level-2 particles of size i.

For test functions of the form

$$F(\mu) = f\left(\sum \phi_i n_i\right) = f(\langle \phi, \mu \rangle) \tag{2.1}$$

where $\mu = \sum n_i \delta_i$ and $\langle \phi, \mu \rangle = \int \phi \, d\mu$, the infinitesimal generator G of the two-level birth-and-death process $\{X(t) : t \geq 0\}$ without spatial motion is given by

$$\begin{aligned} GF(\mu) &= \lim_{t \downarrow 0} \frac{E\{F(X_t) \mid X_0 = \mu\} - F(\mu)}{t} \\ &= \sum_k \lambda_1(\mu_k, k)[f(\langle \phi, \mu \rangle - \phi_k + \phi_{k+1}) - f(\langle \phi, \mu \rangle)] \\ &\quad + \sum_k \mu_1(\mu_k, k)[f(\langle \phi, \mu \rangle - \phi_k + \phi_{k-1}) - f(\langle \phi, \mu \rangle)] \end{aligned} \tag{2.2}$$

$$+ \sum_k \lambda_2(\mu_k, k)[f(\langle \phi, \mu \rangle + \phi_k) - f(\langle \phi, \mu \rangle)]$$
$$+ \sum_k \mu_2(\mu_k, k)[f(\langle \phi, \mu \rangle - \phi_k) - f(\langle \phi, \mu \rangle)].$$

Existence of the two-level process that is characterized as the unique solution of the $\mathcal{N}(\mathbf{Z}^+)$-valued martingale problem associated with the generator G follows from the absence of explosions. In [**2**], several sets of sufficient conditions are given that assure that no explosion takes place. For example, if there exists a function $\lambda_2(n)$ such that

$$\sup \lambda_2(n, i) \le \lambda_2(n) \quad \text{for each } n \in \mathbf{Z}^+ \tag{2.3}$$

and

$$n\lambda_2(1) \le \lambda_2(n) \le qn\lambda_2(1) \quad \text{for all } n \in \mathbf{Z}^+, \text{ for some } q < \infty, \tag{2.4}$$

while

$$\sum_i \frac{1}{\lambda_1(n, i)} = \infty \tag{2.5}$$

for each $n \in \mathbf{Z}^+$, then analysis of the Laplace transform of the random time T_n of the n-th birth under conditions (2.3)–(2.5) leads to the conclusion that $T_n \to \infty$ as $n \to \infty$ a.s., which assures that the two-level pure-birth process $\{X(t) : t \ge 0\}$ has at most a finite number of jumps in any finite interval, and therefore no explosion will take place. A coupling argument allows this result to be extended to a wider class of birth-and-death processes. It is worth noting here that the conditions (2.3)–(2.4) on the level-2 growth rate are stronger and more restrictive than condition (2.5) on the level-1 growth rate. In fact, the two-level birth-and-death process might even have an infinite number of level-1 jumps in any finite interval and still exist, as was shown in [**5**] for the case $\lambda_1(n, i) = \mu_1(n, i) = i^2$ for each $n \in \mathbf{Z}^+$, $i = 1, 2, \dots$, and $\lambda_1(n, 0) = \mu_1(n, 0) = 0$. Since in this case, if we denote $\lambda_1(m, j)$ by λ_j and $\mu_1(m, j)$ by μ_j, we have

$$\sum_{n=0}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \sum_{k=0}^{n} \frac{1}{k^2} \frac{\mu_1 \mu_2 \cdots \mu_k}{\lambda_0 \lambda_1 \cdots \lambda_{k-1}} = \sum_{n=0}^{\infty} \frac{1}{n^2} \sum_{k=0}^{n} 1 = \sum_{n=0}^{\infty} \frac{n+1}{n^2} = \infty, \tag{2.6}$$

it follows that there exists a level-1 birth-and-death process $\{X_1(t) : t \geq 0\}$ such that $P\{X_1(t) < \infty\} = 1$ for all t.

The specific case where

$$\begin{aligned} \lambda_1(n,i) &= \lambda_1 ni = \gamma_1(1 + c_1/2)ni \\ \mu_1(n,i) &= \mu_1 ni = \gamma_1(1 - c_1/2)ni \\ \lambda_2(n,i) &= \lambda_2 n = \gamma_2(1 + c_2/2)n \\ \mu_2(n,i) &= \mu_2 n = \gamma_2(1 - c_2/2)n \end{aligned} \tag{2.7}$$

corresponds to the two-level branching process analyzed in [**1**]. Section 3 of that paper explains in detail the motivation for considering the continuous diffusion limit of such multilevel systems, in which increasing numbers of particles are considered at each level, while the mass of each such particle or cluster is assumed to decrease at an appropriate rate.

For the two-level birth-and-death process, if new parameters $a_i(j)$, $b_i(j)$, $i = 1, 2$, are defined by

$$\begin{aligned} a_i(j) &= [\lambda_i(k,j) + \mu_i(k,j)]/2k \\ b_i(j) &= [\lambda_i(k,j) - \mu_i(k,j)]/2k \end{aligned} \tag{2.8}$$

where $a_1(\cdot), b_1(\cdot) \in C(\mathbf{R}^+)$, $a_2(\cdot), b_2(\cdot) \in C_b(\mathbf{R}^+)$, and

$$a_1(cx) \leq ca_1(x) \qquad b_1(cx) \leq cb_1(x) \tag{2.9}$$

for all $0 < c < 1$, then the diffusion approximation is obtained by first letting

$$\begin{aligned} a_1^\varepsilon(j) &= a_1(j)/\varepsilon_1^2\varepsilon_2, \qquad & b_1^\varepsilon(j) &= b_1(j)/\varepsilon_1\varepsilon_2 \\ a_2^\varepsilon(j) &= a_2(j)/\varepsilon_2^2, \qquad & b_2^\varepsilon(j) &= b_2(j)/\varepsilon_2, \end{aligned} \tag{2.10}$$

and defining

$$X^\varepsilon(t) = X^{\varepsilon_1,\varepsilon_2}(t) \tag{2.11}$$

as the two-level process with birth rate $k\lambda_i^\varepsilon(j)$ and death rate $k\mu_i^\varepsilon(j)$ given by

$$k\lambda_i^\varepsilon(j) = k[a_i^\varepsilon(j) + b_i^\varepsilon(j)], \qquad i = 1, 2, \tag{2.12}$$

$$k\mu_i^\varepsilon(j) = k[a_i^\varepsilon(j) - b_i^\varepsilon(j)], \qquad i = 1, 2. \tag{2.13}$$

Then the rescaled process

$$Y^\varepsilon(t,A)=\varepsilon_2 X^\varepsilon(t,A/\varepsilon_1) \tag{2.14}$$

has infinitesimal generator G^ε given for test functions of the form $F(\mu)=f(\langle\phi,\mu\rangle)$ for $f\in C^\infty(\mathbf{R})$ and $\phi\in C_b^2(\mathbf{R}^+)$ by

$$\begin{aligned}G^\varepsilon F(\mu)=\sum_j&[f(\langle\phi,\mu\rangle-\varepsilon_2\phi(j\varepsilon_1)+\varepsilon_2\phi((j+1)\varepsilon_1))-f(\langle\phi,\mu\rangle)]\\&\qquad\cdot\mu(j\varepsilon_1)\lambda_1^\varepsilon(j\varepsilon_1)\\+\sum_j&[f(\langle\phi,\mu\rangle-\varepsilon_2\phi(j\varepsilon_1)+\varepsilon_2\phi((j-1)\varepsilon_1))-f(\langle\phi,\mu\rangle)]\\&\qquad\cdot\mu(j\varepsilon_1)\mu_1^\varepsilon(j\varepsilon_1)\\+\sum_j&[f(\langle\phi,\mu\rangle+\varepsilon_2\phi(j\varepsilon_1))-f(\langle\phi,\mu\rangle)]\mu(j\varepsilon_1)\lambda_2^\varepsilon(j\varepsilon_1)\\+\sum_j&[f(\langle\phi,\mu\rangle-\varepsilon_2\phi(j\varepsilon_1))-f(\langle\phi,\mu\rangle)]\mu(j\varepsilon_1)\mu_2^\varepsilon(j\varepsilon_1).\end{aligned} \tag{2.15}$$

By a criterion of Gorostiza and Lopez-Mimbella [**3**], the family of rescaled processes $\{Y^\varepsilon(t)\}$ is tight in $D_{[0,\infty)}(M_f(\mathbf{R}^+))$, the set of all mappings from $[0,\infty)$ into the collection $M_f(\mathbf{R}^+)$ of finite measures on $\mathbf{R}^+$ that are continuous from the right and possess limits from the left at every point (see [**5**]).

If one first takes $\varepsilon_2\to 0$, then one gets a super-birth-and-death process with state space $M(\varepsilon_1\mathbf{Z}^+)$. One can then let $\varepsilon_1\to 0$ and obtain an $M_f(\mathbf{R}^+)$-valued process. Alternatively, taking $\varepsilon_1\to 0$ first yields an atomic random measure $M_a(\mathbf{R}^+)$. Then, taking $\varepsilon_2\to 0$, one gets a continuous measure on $\mathbf{R}^+$. If we set $\varepsilon_1=\varepsilon_2=\varepsilon$ and let ε approach zero, the resulting $M_f(\mathbf{R}^+)$-valued Markov process $\{Y(t):t\ge 0\}$ is characterized as the unique solution of the martingale problem associated with the infinitesimal generator G_c given by

$$\begin{aligned}G_cF(\mu)=f'(\langle\phi,\mu\rangle)(\langle a_1\phi'',\mu\rangle)+2f'(\langle\phi,\mu\rangle)\langle b_1\phi',\mu\rangle\\+f''(\langle\phi,\mu\rangle)\langle a_2\phi^2,\mu\rangle+2f'(\langle\phi,\mu\rangle)\langle b_2\phi,\mu\rangle\end{aligned} \tag{2.16}$$

for test functions of the form $F(\mu)=f(\langle\phi,\mu\rangle)$. The transition probabilities on $M_f(\mathbf{R}^+)$ are determined by the Laplace functional

$$E\{\exp(-\langle\phi,Y(t)\rangle)\mid Y(0)=\mu\}=\exp(-\langle u(t),\mu\rangle) \tag{2.17}$$

where the cumulant generating function $u(\cdot,\cdot)$ satisfies the nonlinear initial-value problem

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= a_1(x)\frac{\partial^2 u(t,x)}{\partial x^2} + 2b_1(x)\frac{\partial u(t,x)}{\partial x} \\ &\quad - a_2(x)u^2(t,x) + 2b_2(x)u(t,x) \\ u(0,x) &= \phi(x).\end{aligned} \tag{2.18}$$

It is clear from (2.7) that the two-level branching studied in [**1**] corresponds to

$$\begin{aligned} a_1(x) &= \gamma_1 x \qquad b_1(x) = \frac{1}{2}\gamma_1 c_1 x \\ a_2(x) &= \gamma_2 \qquad\ \ b_2(x) = \frac{1}{2}\gamma_2 c_2,\end{aligned} \tag{2.19}$$

under which (2.18) becomes

$$\frac{\partial u}{\partial t} = \gamma_1 x\frac{\partial^2 u}{\partial x^2} + \gamma_1 c_1 x\frac{\partial u}{\partial x} - \gamma_2 u^2 + \gamma_2 c_2 u. \tag{2.20}$$

3. Addition of spatial motion. We now add spatial motion to the hierarchically structured processes described above. Thus, particles are assumed to undergo a random walk or some spatial diffusion, independently, between splits. The number of particles at any specific time is then still determined by the birth-death or branching processes, but the description of the particle configuration must now include the spatial locations as well. In this situation, it is natural to study the topological support properties of the underlying random measure at fixed times t and the long-term behavior of the multilevel branching-diffusion process, of its continuous limit, and of some related processes, as we shall describe below.

We consider first the case where the spatial motion is a d-dimensional Brownian motion. Then, the hierarchically structured system of branching-diffusing particles in $\mathbf{R}^d$ can be represented as a random atomic measure $Y_a(t)$ given by

$$Y_a(t) = n_0(t)\delta_{\delta_\phi} + \sum_{i=1}^{\infty}\sum_{k=1}^{n_i(t)} \delta_{\sum_{r=1}^{i}\delta_{x_{i,k,r}(t)}}, \tag{3.1}$$

where $n_i(t)$ is the number of "superparticles" or level-2 clusters of size i at time t and $x_{i,k,r}(t)$ denotes the location in $\mathbf{R}^d$ of the rth particle in the kth superparticle $X_{i,k}(t)$ of size i at time t. Note that $Y_a(t)$ keeps track as well of the level-2 superparticles that contain no level-1 particles at time t.

In addition to the four possible transitions of state in the previous models–birth or death of particles at levels 1 and 2–there is now a fifth possible state transition, that of spatial diffusion according to the d-dimensional Brownian motion.

The random measure $Y_a(t)$ given by (3.1) is clearly an element in the space $M^2(\mathbf{R}^d) = M(M(\mathbf{R}^d))$. For $\overline{\mathbf{R}}^d = \mathbf{R}^d \cup \{\text{an isolated point}\}$, Iscoe [**4**] introduced the space $M_\rho(\overline{\mathbf{R}}^d)$ of ρ-tempered measures with the ρ-vague topology, the smallest topology that maps $\mu \to \langle \phi, \mu \rangle$ continuously for $\phi \in C_c(\mathbf{R}^d) \cup \{\phi_\rho\}$, where $d < \rho \leq d + 2$ and

$$\phi_\rho(e) = 1, \qquad \phi_\rho(x) = \frac{1}{1 + |x|^\rho} \tag{3.2}$$

for $x \in \mathbf{R}^d$. Then the space $M_\rho^2(\overline{\mathbf{R}}^d)$ is defined by

$$M_\rho^2(\overline{\mathbf{R}}^d) = \left\{ \nu \in M(M_\rho(\overline{\mathbf{R}}^d)) : \iint \phi_\rho(x)\mu(dx)\nu(d\mu) < \infty \right\}, \tag{3.3}$$

endowed with the smallest topology that maps $\nu \to \iint \phi(x)\mu(dx)\nu(d\mu)$ continuously for all $\phi \in C_c(\mathbf{R}^d) \cup \{\phi_\rho\}$.

For test functions $F(\nu)$ of the form

$$F(\nu) = f(\langle\langle h_1(\langle h_2, \cdot \rangle), \nu \rangle\rangle) \tag{3.4}$$

for $\nu \in M_\rho^2$; $f, h \in C_b^2(\mathbf{R})$; $h_2 \in C_c^2(\mathbf{R}^d)$; and $\langle\langle g(\mu), \nu \rangle\rangle = \int g(\mu)\nu(d\mu)$, the infinitesimal generator $G^{(2)}$ of the two-level branching-diffusing particle system can be expressed as a sum of five terms, one for each of the five types of state transitions just described. The diffusion approximation is obtained by letting

$$Y_n(t, A) = \frac{1}{n} Y_a\left(nt, \left\{ \mu : \frac{\mu}{n} \in A \right\}\right). \tag{3.5}$$

The process $\{Y_n(t)\} : t \geq 0\}$ is tight, and as $n \to \infty$, $Y_n(t)$ converges weakly to the M_ρ^2-valued process $Y(t)$ characterized by the unique solution to the martingale problem for the limiting infinitesimal generator $G_c^{(2)}$ given by

$$\begin{aligned} G_c^{(2)}F(\nu) &= \langle\langle \mathcal{L}\,F'(\nu,.),\nu\rangle\rangle \\ &\quad + \gamma_2\langle\langle F''(\nu),\delta_{\mu_1}(d\mu_2)\nu(d\mu_1)\rangle\rangle \end{aligned} \tag{3.6}$$

where $\mathcal{L}$ denotes the generator of the $M_\rho(R^d)$-valued branching process, i.e.,

$$\begin{aligned} \mathcal{L}\,F'(\nu,\mu) &= \mathcal{L}_1F'(\nu,\mu) + \mathcal{L}_2F'(\nu,\mu) \\ &= f'(\langle\langle h_1(\langle h_2,.\rangle),\nu\rangle\rangle)h_1'(\langle h_2,\mu\rangle)\langle\Delta h_2,\mu\rangle \\ &\quad + \gamma_1 f'(\langle\langle h_1(\langle h_2,.\rangle),\nu\rangle\rangle)h_1''(\langle h_2,\mu\rangle)\langle h_2^2,\mu\rangle \end{aligned} \tag{3.7}$$

where Δ is the d-dimensional Laplacian,

$$\mathcal{L}_1F'(\nu,\mu) = \int \Delta\frac{\delta F'(\nu,\mu)}{\delta\mu(x)}\mu(dx), \tag{3.8}$$

$$\mathcal{L}_2F'(\nu,\mu) = \gamma_1\iint \frac{\delta^2F'(\nu,\mu)}{\delta\mu(x)\delta\mu(y)}\delta_x(dy)\mu(dx), \tag{3.9}$$

and

$$\begin{aligned} F'(\nu,\mu) &= \frac{\partial F(\nu)}{\partial\nu(\mu)} = \frac{d}{d\varepsilon}[F(\nu+\varepsilon\delta_u)]_{\varepsilon=0} \\ &= f'(\langle\langle h_1(\langle h_2,.\rangle),\nu\rangle\rangle)h_1(\langle h_2,\mu\rangle). \end{aligned} \tag{3.10}$$

The Laplace functional of the M_ρ^2-valued process $Y(t)$ is then given by

$$\begin{aligned} L_{t,\nu}(H) &= E\left[\exp\left(-\int_{M_\rho(R^d)} H(\mu)Y(t,d\mu)\right)\Big|\,Y(0)=\nu\right] \\ &= \exp\left\{-\int u(t,\mu)\nu(d\mu)\right\}, \end{aligned} \tag{3.11}$$

where $u(t,\mu)$ satisfies the integral equations

$$\begin{aligned} u(t,\mu) &= T_t u(0,\mu) - \gamma_2 \int_0^t [T_{t-s} u^2(s,\cdot)](\mu)\, ds \\ u(0,\mu) &= H(\mu) \end{aligned} \tag{3.12}$$

for $H(\mu) = f(\langle \phi, \mu \rangle)$, $\phi \in C_c(R^d)$ and $f \in C_b(R)$. Here $\{T_t : t \geq 0\}$ is the expectation semigroup associated with the measure-valued branching process with generator $\mathcal{L}$.

The carrying dimension of the topological support of measure-valued branching diffusion processes is of practical interest, for it describes the extent to which there is an inherent tendency of individuals to cluster or disperse over the state space. In such processes, there are two factors that tend to have opposing influences on such behavior: branching tends to "thin out" the spatial distribution, as individuals may disappear at the time of a branch, whereas diffusion leads to a spread of the process to additional states. Since the mean distance between survivors in a d-dimensional single-level branching process is of the order $t^{1/d}$ while a stable symmetric diffusion of index α spreads at the rate $t^{1/\alpha}$, we tend to see changes of behavior at the critical dimension $d = \alpha$. Similarly, for two-level branching, we should expect to observe changes of behavior at critical dimension $d = 2\alpha$, since then the mean distance between surviving individuals is of the order $t^{2/d}$. To show this, we need the following result:

Theorem 3.1 (Zähle [**6**]). *Let X be a random measure with second moment measure $K(x, dy)$ and assume that for $E(X)$—almost every $x \in \mathbf{R}^d$,*

$$\int_{S(x,\varepsilon)} |x-y|^{-D} K(x,dy) < \infty \tag{3.13}$$

where $S(x,\varepsilon)$ is some sphere of radius $\varepsilon > 0$ centered at x. Then,

$$P\{X(B) > 0 \backslash \{\dim(\Phi \cap B) \geq D\}\} = 0, \tag{3.14}$$

where Φ denotes the closed support of X, and $\dim(A)$ denotes the Hausdorff-Besicovitch dimension of the set A.

In the case of the super-2 process $\{Y(t) : t \geq 0\}$ describing the two-level branching diffusion, we define the process

$$\{\overline{Y}(t) : t \geq 0\} = \int \mu Y(t, d\mu), \tag{3.15}$$

the process that counts all of the particles in all of the superparticles.

The nth moment measure $m_n(t, \mu_0; dx_1, \ldots, dx_n)$ of $\overline{Y}(t)$ is then defined by

(3.16)

$$m_n(t, \mu_0; dx_1, \ldots, dx_n) = E\left\{ \prod_{i=1}^{n} \overline{Y}(t, dx_i) \mid \overline{Y}(0) = \int \mu\nu_0(d\mu) = \mu_0 \right\}$$

where $\nu_0 \in \mathcal{M}_\rho^2(\mathbf{R}^d)$, so $\mu_0 \in \mathcal{M}_\rho(\mathbf{R}^d)$. According to Theorem 3.1, we can obtain a lower bound for the Hausdorff-Besicovitch dimension of the topological support of the process $\overline{Y}(t)$ via knowledge of the second moment measure.

Since the Laplace functional of $\overline{Y}(t)$ is given by

$$\begin{aligned} (3.17)\quad E\left\{ \exp\left(-\int \theta\phi(x)\overline{Y}(t, dx) \right) \mid \overline{Y}(0) = \mu_0 \right\} \\ = E\left\{ \exp\left(-\iint \theta\phi(x)\mu(dx)Y(t, \mu) \right) \mid Y(0) = \nu_0 \right\} \\ = \exp\{-\langle\langle u(t), \nu_0 \rangle\rangle\} \end{aligned}$$

where $u(t, \mu)$ solves

$$\begin{aligned} \frac{\partial u(t, \mu)}{\partial t} &= \mathcal{L}\, u(t, \mu) - \gamma_2 (u(t, \mu))^2 \\ u(0, \mu) &= \int \theta\phi(x)\mu(dx), \end{aligned} \tag{3.18}$$

the second moment measure can be obtained from the solution to (3.18). The function $u(t, \mu)$ is also a solution to

$$u(t) = T_t u(0) - \gamma_2 \int_0^t T_{t-s}(u(s))^2\, ds, \tag{3.19}$$

where $\{T_t : t \geq 0\}$ is the expectation semigroup defined earlier. We can iterate in (3.19) several times to obtain an expression for $u(t)$ in terms of $T_t(u), T_t^2(u), \ldots$, which can in turn be expressed in terms of the Brownian expectation semigroup $\{S_t : t \geq 0\}$ via

$$T_t\langle \phi, \mu\rangle = \langle S_t\phi, \mu\rangle \tag{3.20}$$

and

$$T_t\langle \phi, \mu\rangle^2 = \langle S_t\phi, \mu\rangle^2 + 2\gamma_1 \int\int_0^t S_{t-s}(S_s\phi)^2\, ds\mu(dx) \tag{3.21}$$

for $t > 0$, $\phi \in C_c(\mathbf{R}^d)$, and $\mu \in M_\rho(\mathbf{R}^d)$. Expressions (3.20) and (3.21) follow from the fact that

$$\begin{aligned} T_t \exp\{-\langle \phi\theta, \mu\rangle\} &= \int \exp\{-\langle \phi\theta, m\rangle\} P(t, \mu, dm) \\ &= \exp\{-\langle v(t), \mu\rangle\} \\ &= 1 - \langle v(t), \mu\rangle + \frac{1}{2}\langle v(t), \mu\rangle^2 + \cdots \end{aligned} \tag{3.22}$$

where $P(t, \mu_0, d\mu)$ is the supertransition function over $\{T_t\}$ with generator $\mathcal{L}$, and the function $v(t)$ solves

$$\begin{aligned} v(t,x) &= S_t v(0,x) - \gamma_1 \int_0^t S_{t-s}(v(s,x))^2\, ds \\ v(0,x) &= \theta\phi(x). \end{aligned} \tag{3.23}$$

Therefore,

(3.24)

$$\begin{aligned} v(t,x) &= S_t\theta\phi(x) \\ &\quad - \gamma_1 \int_0^t S_{t-s}\left[S_s v(0)(x) - \gamma_1 \int_0^s S_{s-u}(v(u))^2(x)\, du\right]^2 ds \\ &= S_t\theta\phi(x) - \gamma_1\theta^2 \int_0^t S_{t-s}(S_s\phi)^2(x)\, ds + \cdots, \end{aligned}$$

and this expression can be substituted into (3.22). On the other hand,

$$\begin{aligned} T_t \exp\{-\langle \phi\theta, \mu\rangle\} &= \int \left[1 - \theta\langle \phi, m\rangle + \frac{1}{2}\theta^2\langle \phi, m\rangle^2 + \cdots\right] P(t, \mu, dm) \\ &= 1 - \theta T_t\langle \phi, \mu\rangle + \frac{1}{2}\theta^2 T_t\langle \phi, \mu\rangle^2 + \cdots. \end{aligned} \tag{3.25}$$

Equating coefficients of θ and θ^2 in these two expressions yields (3.20) and (3.21).

Returning to (3.18)–(3.19), we now have

(3.26)
$$u(t,\mu) = \theta \int S_t\phi(x)\mu(dx) - \gamma_2\theta^2 \int_0^t T_{t-s}\left[\iint S_s\phi(x_1)S_s\phi(x_2)\mu(dx_1)\mu(dx_2)\right] ds + \cdots ,$$

from which the Laplace functional of $\overline{Y}(t)$ with initial measure μ_0, given in (3.17) by $\exp\{-\langle\langle u(t), \nu_0\rangle\rangle\}$, can be expressed as a power series in θ. Differentiating twice with respect to θ at $\theta = 0$ yields

(3.27)
$$\begin{aligned} E\{[\phi(x)\overline{Y}(t,x)]^2 \mid \overline{Y}(0) = \mu_0\} &= \iint \phi(x_1)\phi(x_2)m_2(t,\mu_0; dx_1, dx_2) \\ &= 2\gamma_2 \iint_0^t T_{t-s}\left[\iint S_s\phi(x_1)S_s\phi(x_2)\mu(dx_1)\mu(dx_2)\right] ds\nu_0(d\mu) \\ &\quad + \left[\iint S_t\phi(x)\mu(dx)\nu_0(d\mu)\right]^2. \end{aligned}$$

Applying (3.20) and (3.21) to equation (3.27) in the case $Y(0) = \delta_{\delta_x} dx$, for which

$$d\mu_0 = \int \mu\delta_{\delta_x}\, dx = \delta_x\, dx, \tag{3.28}$$

we get an expression for the double integral (3.27) with respect to the second moment measure in terms of the Brownian semigroup $\{S_t\}$, which can be expressed in terms of the Brownian transition function

$$p(u, x-y) = (2\pi u)^{-d/2} \exp\left\{-\frac{|x-y|^2}{2u}\right\}. \tag{3.29}$$

After much manipulation, it follows that the second moment measure $m_2(t, \delta_{\delta_x} dx; dx_1, dx_2)$ converges weakly in $\mathcal{M}_\rho(\mathbf{R}^d) \times \mathcal{M}_\rho(\mathbf{R}^d)$ to the measure with density function

$$4\gamma_1\gamma_2(4\pi)^{d/2}2^{d-4}|x_2 - x_1|^{-d+4}\Gamma\left(\frac{d}{2} - 2\right) \tag{3.30}$$

as $t \to \infty$ for dimensions $d > 4$. Moreover, (3.30) also gives the behavior of m_2 for fixed times t and small $|x_2 - x_1|$.

Similarly, if $Y(0) = \delta_\lambda$ where $\lambda(dx) = dx$ denotes Lebesgue measure on $\mathbf{R}^d$, then the second moment measure $m_2(t, \lambda; dx_1, dx_2)$ in dimensions $d > 4$ behaves like a constant times $|x_2 - x_1|^{4-d}$ for small $|x_2 - x_1|$. Returning to Theorem 3.1, we now have

$$
\begin{aligned}
\int_{S(x,\varepsilon)} |x-y|^{-D} K(x,y)\, dy &\sim c \int_{S(x,\varepsilon)} |x-y|^{-D} |x-y|^{4-d}\, dy \\
&= c \int_0^\varepsilon \frac{1}{r^{D+d-4}} r^{d-1}\, dr \\
&= c \int_0^\varepsilon \frac{1}{r^{D-3}}\, dr,
\end{aligned}
\tag{3.31}
$$

which converges if and only if $D < 4$. Thus, by Theorem 3.1, $\dim \Phi$ must be at least D for all $D < 4$, so $\dim \Phi \geq 4$. We have thus proved the following:

Theorem 3.2. *The Hausdorff-Besicovitch dimension of the topological support of the process* $\overline{Y}(t) = \int \mu Y(t, d\mu)$ *with* $Y(0) = \delta_\lambda$ *and Brownian spatial diffusion in four or more dimensions is at least* 4 *at each fixed time* $t > 0$.

It is clear that a similar argument in the case of a symmetric stable diffusion of index α leads to a lower bound of 2α for the dimension of support.

We have earlier noted that the asymptotic behavior of the second moment measure m_2 of $\overline{Y}(t)$ is given by (3.30). Wu [**5**] has applied (3.20) and (3.21) in proving that the two-level process $Y(t)$ with Brownian diffusion starting with initial measure $Y(0) = \delta_{\delta_x}\, dx$ in dimensions $d \leq 4$ suffers local extinction, in the sense that for every compact set $B \subset \mathbf{R}^d$ and every $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$,

$$\lim_{t\to\infty} P\{\langle\langle 1_{K_{B,\varepsilon_2}}, Y(t)\rangle\rangle > \varepsilon_1\} = 0,$$

where $K_{B,\varepsilon_2} := \{\mu \in M_\rho(\mathbf{R}^d) : \mu(B) > \varepsilon_2\}$ and $1_{K_{B,\varepsilon_2}}$ denotes the indicator function of the set K_{B,ε_2}. Thus, local extinction occurs in dimensions 3 and 4 in spite of the fact that the underlying motion process is transient and the one-level superprocess is stable.

Acknowledgments. Much of our work on hierarchically structured branching populations has been developed in collaboration with D.A. Dawson and Y. Wu. We acknowledge the support of the U.S. Army Research Office through the Mathematical Sciences Institute of Cornell University.

REFERENCES

1. D.A. Dawson and K.J. Hochberg, *A multilevel branching model*, Adv. in Appl. Probab. **23** (1991), 701–715.

2. D.A. Dawson, K.J. Hochberg and Y. Wu, *Multilevel branching systems*, in *White noise analysis: Mathematics and applications* (T. Hida, H.H. Kuo, J. Potthoff and L. Streit, eds.), World Scientific Publ., Singapore, 1990, 93–107.

3. L.G. Gorostiza and J.A. Lopez-Mimbela, *The multitype measure branching process*, Adv. in Appl. Probab. **22** (1990), 49–67.

4. I. Iscoe, *On the supports of measure-valued critical branching Brownian motion*, Ann. Probab. **16** (1988), 200–221.

5. Y. Wu, *Dynamic particle systems and multilevel measure branching processes*, Thesis, Carleton University, Canada, 1992.

6. U. Zähle, *Self-similar random measures* I. *Notion, carrying Hausdorff dimension and hyperbolic distribution*, Probab. Theory Related Fields **80** (1992), 79–100.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, BAR-ILAN UNIVERSITY, RAMAT-GAN 52900, ISRAEL