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ON THE EIGENVALUES AND EIGENFUNCTIONS OF AN INTERFACE STURM-LIOUVILLE SYSTEM

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1. Interface Sturm-Liouville systems have been studied extensively since 1960. See, for example, [**3, 8, 11, 13** and **4, 5, 6, 7**] on eigenvalue and eigenfunction problems discussed in this paper.

In this paper we will deal with interface Sturm-Liouville systems related to the vibrating string:

$$y''(x) + \lambda p(x)y(x) = 0, \quad -1 \le x \le 1 \tag{1}$$

where $p(x) > 0$ is piecewise continuous over $-1 \le x \le 1$.

The boundary conditions we will consider in connection with (1) are

$$y(-1) = y(1) = 0 \tag{2}$$

and the interface conditions are

$$sy'(0) = y'_+(0), \quad s > 0 \tag{3}$$

together with

$$ty_-(0) = y_+(0), \quad t > 0. \tag{4}$$

A solution of an interface BVP is a function $y(x) \in C^2$ on $-1 \le x < 0$ and $0 < x \le 1$, $y(x)$ and $y'(x)$ have one sided limits at $x = 0$, and $y(x)$ satisfies (1), (2), (3) and (4), [**7**].

The main object of the paper is to show some properties of the first eigenfunction $y(x) > 0$, $-1 < x < 1$, to show monotonicity of the frequencies of the string under symmetrization methods applied to $p(x)$, and to show monotonicity of the frequencies as related to the

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monotonicity of s/t. The results can be extended to the interface BVP (1) (5) (3) (4), where

$$y'(-1) = y(1) = 0. \tag{5}$$

2. Representation of the interface BVP as an ordinary BVP. Let $y(x)$ be a solution of the interface BVP. Then the function $\tilde{z}(\zeta)$ defined as

$$z^{\sim}(\zeta) = \begin{cases} y(\zeta), & -1 < \zeta < 0 \\ \frac{1}{t} y(\frac{t}{s}\zeta), & 0 < \zeta \leq \frac{s}{t} \end{cases} \tag{6}$$

is a solution of the DE

$$\tilde{z}''(\zeta) + \lambda \tilde{q}(\zeta) \tilde{z}(\zeta) = 0 \tag{1'}$$

with

$$\tilde{q}(\zeta) = \begin{cases} p(\zeta), & -1 \leq \zeta < 0 \\ \frac{t^2}{s^2} p(\frac{t}{s}\zeta), & 0 < \zeta \leq \frac{s}{t} \end{cases} \tag{6*}$$

where the boundary conditions in (2) are replaced by

$$\tilde{z}(-1) = \tilde{z}(s/t) = 0. \tag{2'}$$

As the system (1′) (2′) is a self-adjoint problem, it has a first eigenvalue greater than zero and a first eigenfunction greater than zero on $(-1, s/t)$; therefore the same holds for the problem (1)–(4) with $s > 0$, $t > 0$, as (1)–(4) and (1′) (2′) are related to each other by (6) and (6*), and z is extended naturally to be continuously differentiable at $\zeta = 0$.

Let $\tilde{w}(\zeta)$ be the first eigenfunction of this Sturm-Liouville system (1′), (2′), and let λ be the first eigenvalue. Then [**9**]

$$\lambda = \frac{\int_{-1}^{s/t} \tilde{w}'(\zeta)\, d\zeta}{\int_{-1}^{s/t} \tilde{q}(\zeta) \tilde{w}^2(\zeta)\, d\zeta} = \min_{\tilde{u}} \frac{\int_{-1}^{s/t} \tilde{u}'^2(\zeta)\, d\zeta}{\int_{-1}^{s/t} \tilde{q}(\zeta) \tilde{u}^2(\zeta)\, d\zeta} \tag{7}$$

where the minimum is taken over the class of continuous functions $\tilde{u}(x)$ with piecewise continuous derivative satisfying

$$\tilde{u}(-1) = \tilde{u}(s/t) = 0.$$

Going back to the variable x defined by

$$\zeta = \frac{s}{t}x = kx, \quad 0 < x < 1, \tag{8}$$

and denoting by $z(x)$ the first eigenfunction of the interface BVP

$$\begin{gathered} z'' + \lambda p z = 0 \\ z(-1) = z(1) = 0 \\ k z'_-(0) = z'_+(0), \qquad k > 0 \end{gathered} \tag{9}$$

$z > 0$ is continuous on $-1 < x < 1$, and we get

$$\begin{aligned} \lambda &= \frac{\int_{-1}^0 y'^2\,dx + (1/st)\int_0^1 y'^2\,dx}{\int_{-1}^0 p(x)y^2(x)\,dx + (1/st)\int_0^1 p(x)y^2(x)\,dx} \\ &= \frac{\int_{-1}^0 z'^2\,dx + (t/s)\int_0^1 z'^2\,dx}{\int_{-1}^0 p(x)z^2(x)\,dx + (t/s)\int_0^1 p(x)z^2(x)\,dx} \\ &= \min_u \frac{\int_{-1}^0 u'^2\,dx + (t/s)\int_0^1 u'^2\,dx}{\int_{-1}^0 p(x)u^2(x)\,dx + (t/s)\int_0^1 p(x)u^2(x)\,dx}. \end{aligned} \tag{10}$$

The minimum in (10) is taken over all continuous functions $u(x)$ with piecewise continuous derivative satisfying the boundary condition

$$u(-1) = u(1) = 0. \tag{2''}$$

We can summarize in the following lemma.

Lemma 1. *Let $p(x) > 0$ be piecewise continuous in $-1 \le x \le 1$, and let the interface* BVP (1) (2) (3) (4) *be given. Then the first eigenfunction $y(x) > 0$ satisfies* (10) *where $y(x) = z(x)$, $-1 \le x < 0$, $y(x) = tz(x)$, $0 < x \le 1$, and $u(x)$ denotes any continuous function with a piecewise derivative that satisfies* $(2'')$. *λ is the first eigenvalue of* (1) (2) (3) (4) *and of* (9) *for $k = s/t$, and also of* $(1')$, $(2')$.

3. Properties of the first eigenfunction of the interface BVP. We will discuss some properties of

$$z(x) = \begin{cases} y(x)/t, & 0 < x \le 1 \\ y(x), & -1 \le x < 0. \end{cases}$$

Here $y(x)$ is a solution of the interface BVP (1) (2) (3) (4). In other words, we discuss the first interface BVP solution satisfying (9) for $k = s/t$, where $z(x)$ is continuous on $-1 \le x \le 1$, and $z(x) > 0$, $-1 < x < 1$.

In Section 4 we discuss monotonicity properties of eigenvalues of interface BVP as a result of symmetrization methods applied to $p(x)$. The symmetrization method is called continuous symmetrization [**12**, p. 200]. The results hold under some restrictions applied to $p(x)$, one of them as defined here.

Definition. A function $p(x)$ in $[-1, 1]$ is called left-balanced (lb) if for every $x \in [0, 1]$, $p(-x) \ge p(x)$ [**1**]. If $p(-x) \le p(x)$ for every $x \in [0, 1]$ the function is called right balanced (rb).

Examples. Every decreasing function is (lb) and every increasing function is (rb).

Lemma 2. *Let $z(x) > 0$, $-1 < x < 1$, be a continuous solution of the interface* BVP

$$(9) \qquad \begin{aligned} z'' + \lambda p(x) z(x) &= 0 \\ z(-1) = z(1) &= 0 \\ k z'_-(0) = z'_+(0), &\quad k > 0 \end{aligned}$$

*and let $p(x) > 0$ be piecewise continuous in $-1 \le x \le 1$. If $p(x)$ is (*lb*) that means if*

$$p(x) \ge p(-x), \quad -1 \le x < 0,$$

then $z(x)$ is (lb)*, too. In other words,*

$$z(x) \ge z(-x), \quad -1 \le x < 0.$$

Proof of Lemma 2. Define

$$w(x) = z(-x), \quad -1 < x < 0.$$

Then, for $-1 < x < 0$,

$$\int_{-1}^{x} (w''z - z''w)\, dr = \lambda \int_{-1}^{x} (p(r) - p(-r)) wz\, dr.$$

Hence,

$$w'(x)z(x) - z'(x)w(x) \geq 0, \quad -1 < x \leq 0 \tag{11}$$

and

$$z^2(w/z)' > 0, \quad -1 < x < 0.$$

As $w(0)/z(0) = 1$ we get

$$w/z < 1, \quad -1 < x < 0$$

which means that $z(x)$ is (lb). □

4. Symmetrization methods and monotonicity of eigenvalues. The object of this section is to show the monotonicity of λ, the first eigenvalue of an interface BVP (1) (2) (3) (4) related to a positive eigenfunction over $-1 < x < 1$, under a symmetrization method applied to $p(x)$.

A positive function $p(x)$ is said to be of class A in $[-1, 1]$ if $p(x) \in C$, nonincreasing in $[-1, l]$ and nondecreasing in $[l, 1]$ for some $l-1 \leq l \leq 1$. For a function $p(x)$ belonging to class A, we denote the inverse function of $p(x)$ by $x_1(y)$ for $x \in [-1, l)$ and by $x_2(y)$ for $x \in [l, 1]$.

We define a class of functions $p(x, a)$, $0 \leq a \leq 1$, $-1 \leq x \leq 1$, by a method called continuous symmetrization [**12**, p. 200; **1**, p. 350; and **2**].

For $x \in [-1, l(1-a)]$ we denote the inverse function of $p(x, a)$ by $x_1(y, a)$ and for $x \in [l(1-a), 1]$ we denote the inverse function by $x_2(y, a)$:

$$\begin{aligned} x_1(y, a) &= (1 - a/2)x_1(y) - (a/2)x_2(y) \\ x_2(y, a) &= (1 - a/2)x_2(y) - (a/2)x_1(y). \end{aligned} \tag{12}$$

If $p(-1) > p(1)$ we add to $x_2(y)$ an interval of definition $p(1) \leq y \leq p(-1)$ for which $x_2(y) = 1$.

To complete the definition of $p(x, a)$, we agree that if $p(x)$ attains the same constant value k in two intervals $[b, c]$ and $[d, f]$, $b \leq c \leq d \leq f$, then if

$$x_1(k) = (1 - m)b + mc, \quad 0 \leq m \leq 1,$$

then for a symmetrization procedure defined by (12) we choose

$$x_2(k) = md + (1-m)f.$$

We extend the symmetrization procedure to parameters $a \in [-1, 0]$ by

$$\begin{aligned} x_1(y,a) &= x_1(y) - a(1 - x_2(y)) \\ x_2(y,a) &= x_2(y) - a(1 - x_2(y)) \end{aligned} \tag{13}$$

or by

$$\begin{aligned} x_1(y,a) &= x_1(y) + a(1 + x_1(y)) \\ x_2(y,a) &= x_2(y) + a(1 + x_1(y)). \end{aligned} \tag{14}$$

If $p(-1) > p(1)$ we extend the interval of definition of $x_2(y)$ by adding the interval $p(1) \le y \le p(-1)$ on which we define $x_2(y) = 1$.

The functions $p(x,a)$ are equimeasurable; i.e., for each y,

$$m(x, p(x,a) \ge y, -1 \le x \le 1) = m(x, p(x) \ge y, -1 \le x \le 1).$$

See [**10**, Chapter X] and [**12**, Chapter VII].

Obviously, $p(x,0) = p(x), p(x,1)$ is the symmetrically increasing rearrangement $p^+(x)$ of $p(x)$ [**5, 6, 10**].

In [**1**] the following was proven as parts of Theorems 1 and 2.

Theorem A. *Let $p(x) > 0$ be* (lb) *and of class A, and let $p(x,a)$ be the rearrangement of $p(x)$ as defined in* (12) *and* (13). *Then $p(x,a)$ is* (lb) *and of class A and*

$$p(x,a) \le p(x,b), \quad 1 \ge a \ge b \ge -1,\ -1 \le x \le 0 \tag{15}$$

and for every continuous (lb) *function $0 < z(x)$, $-1 < x < 1$ with one maximum*

$$\int_{-1}^{1} p(x,a) z^2 \, dx \le \int_{-1}^{1} p(x,b) z^2 \, dx, \quad 0 \le a \le b \le 1. \tag{16}$$

The analogous results hold for (rb) $p(x)$: let $p(x) > 0$, (rb), and of class A, and let $p(x,a)$ be the rearrangement of $p(x)$ as defined in (12) and (14). Then $p(x,a)$ is (rb) and of class A and

$$(15') \qquad p(x,a) < p(x,b), \quad 1 \geq a \geq b \geq -1,\ 0 \leq x \leq 1$$

and for every continuous (rb) function $z(x)$ with one maximum, (16) holds.

Theorem 1. *Let $p(x) > 0$ be of class A and* (lb)*. Then $\lambda(k,a)$ the first eigenvalue of the system*

$$(17) \qquad \begin{gathered} y'' + \lambda(a)p(x,a)y(x) = 0, \quad y(\pm 1) = 0 \\ sy'_-(0) = y'_+(0), \qquad ty_-(0) = y_+(0) \\ s/t = k \geq 1 \end{gathered}$$

is decreasing with a, $0 \leq a \leq 1$, when $p(x,a)$ is derived from $p(x)$ by the continuous symmetrization method, $y(x) > 0$, $-1 < x < 1$, the first eigenfunction of (17) *is continuous in $-1 \leq x < 0$, $0 < x \leq 1$, $y(x) \in C^2$ and $y(x)$ and $y'(x)$ have finite one sided limits at $x = 0$.*

Remark. Theorem 1 can be extended to right balanced $p(x)$ as follows: Let $p(x) > 0$ be of class A and (rb). Then $\lambda(k,a)$ is the first eigenvalue of the system

$$(17') \qquad \begin{gathered} y'' + \lambda(a)p(x,a)y(x) = 0, \quad y(\pm 1) = 0 \\ sy'_-(0) = y'_+(0), \qquad ty_-(0) = y_+(0) \\ s/t = k \leq 1 \end{gathered}$$

is decreasing with a, $-1 \leq a < 1$, when $p(x,a)$ is derived from $p(x)$ by the continuous symmetrization method, $y(x) > 0$, $-1 < x < 1$ is the first eigenfunction of (17′), $y(x) \in C^2$ and $y(x)$ and $y'(x)$ have finite one sided limits at $x = 0$.

Proof of Theorem 1. The proof relies on (15), (16), which is not true for non (lb) $p(x)$. $\lambda(k,a)$, the first eigenvalue of (17), is also the first eigenvalue of

$$(18) \quad z'' + \lambda p(x,a)z = 0, \qquad z(-1) = z(1) = 0, \qquad kz'_-(0) = z'_+(0)$$

and it satisfies:

$$\lambda(k,a) = \frac{\int_{-1}^{1} z'^2\,dx + (k-1)\int_{-1}^{0} z'^2\,dx}{\int_{-1}^{1} p(x,a)z^2\,dx + (k-1)\int_{-1}^{0} p(x,a)z^2\,dx}$$

where $z(x) > 0$ is the continuous solution of the interface BVP (17). Then for a given fixed $k \geq 1$,

$$\begin{aligned}\lambda(k,a) &= \frac{\int_{-1}^{1} z'^2\,dx + (k-1)\int_{-1}^{0} z'^2\,dx}{\int_{-1}^{1} p(x,a)z^2\,dx + (k-1)\int_{-1}^{0} p(x,a)z^2\,dx}\\ &\geq \frac{\int_{-1}^{1} z'^2\,dx + (k-1)\int_{-1}^{0} z'^2\,dx}{\int_{-1}^{1} p(x,b)z^2\,dx + (k-1)\int_{-1}^{0} p(x,b)z^2\,dx}\\ &\geq \min_u \frac{\int_{-1}^{1} u'^2\,dx + (k-1)\int_{-1}^{0} u'^2\,dx}{\int_{-1}^{1} p(x,b)u^2\,dx + (k-1)\int_{-1}^{0} p(x,b)u^2\,dx}\\ &= \lambda(b).\end{aligned}$$

Indeed, the first inequality follows from (15) and (16) and the second inequality follows from the characterization of the first eigenvalue of the interface BVP (10). This completes the proof of Theorem 1. □

5. Monotonicities of frequencies of the system (9) **as related to the monotonicity of** k**.**

Theorem 2. *Let* $p(x) > 0$ *be piecewise continuous on* $-1 < x < 1$. *Let* $y(x)$ *be the first eigenfunction of* (1) (2). *If* $y'(0) \neq 0$, *then for every* $0 < k < \infty$, *there is an eigenvalue* $\lambda(k)$ *and an eigenfunction* $z(x) > 0$, $-1 < x < 1$ *that satisfies the interface* BVP (9). *Moreover, for* $y'(0) < 0$,

$$\lambda(k_1) < \lambda(k_2), \quad 0 < k_2 < k_1 \tag{19}$$

and for $y'(0) > 0$

$$\lambda(k_1) < \lambda(k_2), \quad 0 < k_1 < k_2. \tag{20}$$

Proof of Theorem 2. Let $\bar{\lambda}$ be the first eigenvalue and $\bar{y}(x)$ the first eigenfunction of (9) for $k = 1$. We will prove (19) for $1 < k_2 < k_1$. The other parts of (19) and (20) are obtained similarly. Comparing

$$\bar{y}'' + \bar{\lambda} p \bar{y} = 0, \qquad \bar{y}(-1) = 0, \quad -1 \le x \le 0$$

with

$$v'' + (\bar{\lambda} - \varepsilon) p v = 0, \qquad v(-1) = 0, \quad \varepsilon > 0, \ -1 \le x \le 0,$$

we get that

$$\bar{y}'(x) v(x) - v'(x) \bar{y}(x) < 0, \quad -1 < x \le 0.$$

As $\bar{y}(x) > 0$, $-1 < x \le 0$ also $v(x) > 0$. Therefore, at $x = 0$ choosing $\bar{y}(0) = v(0) > 0$, we get

$$y(0)[\bar{y}'(0) - v'(0)] < 0.$$

Hence,

$$\bar{y}'(0) < v'(0).$$

Let us now compare

$$\bar{y}'' + \bar{\lambda} p \bar{y} = 0, \qquad y(1) = 0, \quad 0 \le x \le 1$$

with

$$u'' + (\bar{\lambda} - \varepsilon) p u = 0, \qquad u(1) = 0, \quad \varepsilon > 0, \ 0 \le x \le 1.$$

The same procedure leads to

$$u'(0) < \bar{y}'(0)$$

and hence

$$u'(0) < \bar{y}'(0) < v'(0).$$

We assume that $\bar{y}'(0) \le 0$, then as long as $v'(0) < 0$ we get that $\lambda = \bar{\lambda} - \varepsilon$ is the first eigenvalue and

$$z(x) = \begin{cases} u(x), & 0 < x \le 1, \\ v(x), & -1 \le x \le 0, \end{cases}$$

is the first eigenfunction of (9) with

$$k = \frac{u'(0)}{v'(0)} = \frac{z'_+(0)}{z'_-(0)}.$$

As $kz'(0) < \bar{y}'(0) < z'_-(0)$ we get k to satisfy

$$k > \frac{\bar{y}'(0)}{z'_-(0)} > 1.$$

This simple procedure shows us that while $k > 1$ increases, $\lambda = \bar{\lambda} - \varepsilon$ decreases. Immediate use of Lemma 2 brings us to the following corollary: Let $p(x) > 0$ be piecewise continuous, (lb), and not symmetric on $(-1, 1)$. Then the results of Theorem 2 hold. That follows from $p(x)$ being (lb), also $z(x)$ is (lb) and $z'(0) < 0$. $\square$

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