ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 2, Spring 1995

ON AN ALTMAN TYPE FIXED POINT THEOREM ON CONVEX CONES

G. ISAC

1. Introduction. The principal result of this paper is a fixed point theorem on a convex cone in a Hilbert space. This result is obtained using the complementarity theory.

This fact is not surprising, since in our paper [**20**] we showed that there exist interesting implications from the fixed point theory to the complementarity theory, and reciprocally the complementarity theory can be used to obtain new fixed point theorems.

These relations are very interesting since the complementarity theory is in development and it has many and important applications in optimization, game theory, engineering, mechanics, elasticity theory, economics, etc., [**19, 13, 20**].

Let $(H, \langle , \rangle)$ be a separable Hilbert space. If $r > 0$ we denote $B_r = \{x \in H \mid ||x|| \leq r\}$ and $S_r = \{x \in H \mid ||x|| = r\}$.

In 1957 Altman proved the following fixed point theorem.

Theorem (Altman, [**1**]). *Let f be a weakly closed operator defined on B_r with range in H. If f maps the set B_r into a bounded set and the following condition is satisfied*

$$(A) \qquad \langle f(x), x\rangle \leq \langle x, x\rangle \quad \text{for every } x \in S_r,$$

then f possesses a fixed point in B_r.

Another version of this theorem was proved by Shinbrot in 1965 [**32**] without the assumption that $f(B_r)$ is bounded but supposing that f is continuous from the weak to the weak topology.

Shinbrot's theorem has interesting applications to partial differential equations.

Received by the editors on May 15, 1991.

We note that Altman proved his theorem using the topological degree and Shinbrot using a long proof.

In this paper we use a generalization of condition (A). Our condition is more flexible, and our principal result is a fixed point theorem on a convex cone in a Hilbert space for an operator of the form $T_1 + T_2$, where T_2 is a compact operator and $I - T_1$ satisfies condition $(S)_+$. It is well known that condition $(S)_+$ is an important condition used in nonlinear analysis and it was intensively studied by Browder [**7, 8, 9, 10**], Hess [**17**], Petryshyn [**29**], etc.

As a consequence of the principal result we obtain a fixed point theorem on a convex cone for an operator of the form $S + T$ where S is a contraction and T a compact operator, similar to a classical result proved by Krasnoselskii on bounded sets [**23**].

2. Preliminaries. Let $(H, \langle,\rangle)$ be Hilbert space and $K \subset H$ a closed convex cone, that is, a closed subset of H such that i) $\mathbf{K} + \mathbf{K} \subseteq \mathbf{K}$

ii) $\lambda\mathbf{K} \subseteq \mathbf{K}$ for every $\lambda \in \mathbf{R}_+$

iii) $\mathbf{K} \cap (-\mathbf{K}) = \{0\}$.

We denote by $\mathbf{K}^*$ the dual of $\mathbf{K}$, that is,

$$\mathbf{K}^* = \{y \in H \mid \langle x, y\rangle \geq 0; \forall\, x \in \mathbf{K}\}.$$

If $\mathbf{C} \subset H$ is a closed convex subset, we say that a continuous operator $P : H \to H$ is a *projection* on $\mathbf{C}$ if $P(H) = \mathbf{C}$ and $P(x) = x$, for every $x \in \mathbf{C}$.

We say that $\mathbf{K}$ is a *Galerkin cone* if there exists a countable family of convex subcones $\{\mathbf{K}_n\}_{n\in\mathbf{N}}$ of $\mathbf{K}$ such that:

1) $\mathbf{K}_n$ is locally compact for every $n \in \mathbf{N}$,

2) $\mathbf{K}_n \subseteq \mathbf{K}_m$ whenever $n \leq m$,

3) $\mathbf{K} = \overline{\cup_{n\in\mathbf{N}}\mathbf{K}_n}$.

If $\mathbf{K}$ is a Galerkin cone we denote it by $\mathbf{K}(\mathbf{K}_n)_{n\in\mathbf{N}}$.

In practice, a closed convex cone which has an approximation by the finite element method, or which has a Schauder base is a Galerkin cone.

In [**20**] it is proved that if $\mathbf{K}(\mathbf{K}_n)_{n\in\mathbf{N}}$ is a Galerkin cone in H, then there exists a family $\{P_n\}_{n\in\mathbf{N}}$ of projections such that P_n is a projection on $\mathbf{K}_n$, for every $n \in \mathbf{N}$ and $\lim_{n\to\infty} P_n(x) = x$, for every $x \in \mathbf{K}$.

We denote by "(w)-lim" the limit with respect to the weak topology. We recall now the concept of *duality mapping*.

Let $(E, || \, ||)$ be a Banach space and $(E^* || \, ||_*)$ the dual of E, where $|| \, ||_*$ is the dual norm of $|| \, ||$. We denote by $\langle E, E^* \rangle$ the natural duality defined by E and E^*.

We say that a continuous and strictly increasing function $\Phi : \mathbf{R}_+ \to \mathbf{R}_+$ is *a weight* if $\Phi(0) = 0$ and $\lim_{r\to\infty} \Phi(r) = +\infty$.

Given a weight Φ, a *duality mapping* on E associated to Φ is a mapping $J : E \to 2^{E^*}$ such that

$$J(x) = \{x^* \in E^* \mid \langle x, x^* \rangle = ||x|| \cdot ||x^*||_* \text{ and } ||x^*||_* = \Phi(||x||)\}.$$

A consequence of the Hahn-Banach theorem is the fact that for every $x \in E$, $J(x)$ is nonempty.

The properties of duality mapping are well studied in [**12**] and [**28**] where we find also many interesting examples.

If $(E, || \, ||)$ is a smooth reflexive Banach space, then every duality mapping associated to a weight function Φ is norm-weak continuous and point to point mapping.

Our results presented here are deeply based on the condition $(S)_+$ introduced by Browder [**8**] and Skrypnik [**33**].

The condition $(S)_+$ is very important in nonlinear analysis [**6**–**10**, **17, 29**] and in variational calculus [**2, 3, 24**].

Let $(H, \langle , \rangle)$ again be a Hilbert space and $D \subseteq H$ a weakly closed subset.

Definition 1. A mapping $T : D \to H$ is said to *satisfy condition* $(S)_+$ if for any sequence $\{x_n\}_{n\in\mathbf{N}} \subset D$ which converges weakly to x_* in D and for which $\limsup \langle x_n - x_*, T(x_n) \rangle \leq 0$ we have that $\{x_n\}_{n\in\mathbf{N}}$ has a subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ norm convergent to x_*. (We have the same definition when $T : D \to E^*$, where D is a weakly closed subset in a Banach space E.)

Examples. 1) Every duality mapping associated to the weight Φ satisfies condition $(S)_+$, as a mapping from E into E^*.

2) Some nonlinear elliptic operators satisfy condition $(S)_+$ [**10**].

3) If T_1 satisfies condition $(S)_+$ and T_2 is a compact operator, then $T_1 + T_2$ satisfies condition $(S)_+$.

4) From example 3 we have that every Fredholm operator satisfies condition $(S)_+$.

The following two results also give two important examples of operators which are not duality mapping but which satisfy condition $(S)_+$.

We say that $T : H \to H$ is *strongly ρ-monotone* if there is a continuous strictly increasing function $\rho : R_+ \to R_+$ such that $\rho(0) = 0$ and $\langle x - y, T(x) - T(y)\rangle \geq \rho(||x - y||)$ for every $x, y \in H$.

Proposition 1. *Each strongly ρ-monotone mapping $T : H \to H$ satisfies condition $(S)_+$.*

Proof. Let $\{x_n\}_{n\in\mathbf{N}} \subset H$ be a sequence weakly convergent to x_* in H and such that $\limsup_{n\to\infty}\langle x_n - x_*, T(x_n)\rangle \leq 0$.

Since $\rho(||x_n - x_*||) \leq \langle x_n - x_*, T(x_n) - T(x_*)\rangle = \langle x_n - x_*, T(x_n)\rangle - \langle x_n - x_*, T(x_*)\rangle$, we have

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} \rho(||x_n - x_*||) \leq \limsup_{n\to\infty} \rho(||x_n - x_*||) \\ &\leq \limsup_{n\to\infty}\langle x_n - x_*, T(x_n)\rangle - \lim_{n\to\infty}\langle x_n - x_*, T(x_*)\rangle \leq 0, \end{aligned}$$

which implies that $\lim_{n\to\infty} \rho(||x_n - x_*||) = 0$ and because ρ is strictly increasing and continuous we obtain that $\lim_{n\to\infty}(||x_n - x_*||) = 0$. □

We say that $T : H \to H$ is a *φ-contraction* (in Boyd and Wong's sense [**5**]) if there is a mapping $\varphi : R_+ \to R_+$ satisfying:

i) $||T(x) - T(y)|| \leq \varphi(||x - y||)$, for all $x, y \in H$,

ii) $\varphi(t) < t$, for all $t \in R_+\backslash\{0\}$.

This class of operators was studied in [**5**] where an interesting fixed point theorem is also proved.

Proposition 2. *If $T : H \to H$ is a φ-contraction with φ continuous, then $I - T$ satisfies condition $(S)_+$.*

Proof. Since T is a φ-contraction, for every $x, y \in H$ we have

$$\begin{aligned}\langle (I-T)(x) - (I-T)(y), x-y\rangle &= \langle x - y - T(x) + T(y), x - y\rangle \\ &= ||x-y||^2 - \langle T(x) - T(y), x-y\rangle \\ &\geq ||x-y||^2 - ||x-y||\varphi(||x-y||) \\ &= ||x-y||[||x-y|| - \varphi(||x-y||)].\end{aligned}$$

Let $\{x_n\}_{n\in\mathbf{N}} \subset H$ be a sequence weakly convergent to $x_* \in H$ and such that $\limsup_{n\to\infty}\langle x_n - x_*, (I-T)(x_n)\rangle \leq 0$.

We have $||x_n - x_*||[||x_n - x_*|| - \varphi(||x_n - x_*||)] \leq \langle x_n - x_*, (I-T)(x_n) - (I-T)(x_*)\rangle = \langle x_n - x_*, (I-T)(x_n)\rangle - \langle x_n - x_*, (I-T)(x_*)\rangle$, and, as in the proof of Proposition 1, we obtain that

$$(1) \qquad \lim_{n\to\infty} ||x_n - x_*||[||x_n - x_*|| - \varphi(||x_n - x_*||) = 0.$$

Let $\{x_{n_k}\}_{k\in\mathbf{N}}$ be a subsequence of $\{x_n\}_{n\in\mathbf{N}}$ such that $\{||x_{n_k} - x_*||\}_{k\in\mathbf{N}}$ is convergent.

If $\alpha = \lim_{n\to\infty} ||x_{n_k} - x_*||$ then α must be equal to zero.

Indeed, if $\alpha > 0$ then from (1) and using the properties of φ we obtain that $\lim_{n\to\infty} ||x_{n_k} - x_*||[||x_{n_k} - x_*|| - \varphi(||x_{n_k} - x_*||)] = \alpha[\alpha - \varphi(\alpha)] > 0$, which is impossible. Hence $\{x_{n_k}\}_{k\in\mathbf{N}}$ is norm convergent to x_*. □

Lemma 4. *If a mapping $T : H \to H$ satisfies condition $(S)_+$, then every sequence $\{x_n\}_{n\in\mathbf{N}} \subset H$ with $(w) - \lim_{n\to\infty} x_n = x_*$, $(w) - \lim_{n\to\infty} T(x_n) = u \in H$ and $\limsup_{n\to\infty}\langle x_n, T(x_n)\rangle \leq \langle x_*, u\rangle$ has a subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ norm convergent to x_*.*

Proof. The lemma is obtained by elementary calculus using the definition of condition $(S)_+$. □

3. Principal results. Let $(H, \langle,\rangle)$ be a Hilbert space, and let $\mathbf{K} \subset H$ be a closed convex cone.

Given a mapping $f : \mathbf{K} \to H$ we recall that the complementarity problem associated to f and $\mathbf{K}$ is:

$$\text{C.P.}\,(f, \mathbf{K}) : \begin{cases} \text{find } x_0 \in \mathbf{K} \text{ such that} \\ f(x_0) \in \mathbf{K}^* \text{ and } \langle x_0, f(x_0)\rangle = 0. \end{cases}$$

For more details on this problem we recommend [**13, 19, 20**].

We use the following classical theorem.

Theorem (Hartman-Stampacchia [**18**]). *Let E be a locally convex space, E^* its topological dual, and let $\mathbf{C}$ be a compact convex set in E.*

If $f : \mathbf{C} \to E^*$ is a continuous mapping, then there exists $x_* \in \mathbf{C}$ such that $\langle x - x_*, f(x_*)\rangle \geq 0$ for every $x \in \mathbf{C}$.

With this theorem we prove the following result.

Proposition 3. *Let $\mathbf{K} \subset H$ be a locally compact convex cone, and let $f : \mathbf{K} \to H$ be a continuous mapping.*

If there is an element $u_0 \in \mathbf{K}$ and a number $r > ||u_0||$ such that $(M) : \langle x - u_0, f(x)\rangle \geq 0$, for all $x \in \mathbf{K}$ with $||x|| = r$, then the problem C.P. $(f, \mathbf{K})$ *has a solution x_* such that $||x_*|| \leq r$.*

Proof. By theorem (Hartman-Stampacchia) there exists $x_* \in \mathbf{K}_r = \{x \in \mathbf{K} \mid ||x|| \leq r\}$ such that

$$\langle x - x_*, f(x_*)\rangle \geq 0; \quad \text{for all } x \in \mathbf{K}_r. \tag{2}$$

We have two cases.

Case 1. $||x_*|| < r$. If $x \in \mathbf{K}$, then there exists $\lambda \in]0, 1[$ sufficiently small such that $w = \lambda x + (1 - \lambda)x_* \in \mathbf{K}_r$, and from (2) we have $\langle w - x_*, f(x_*)\rangle = \lambda\langle x - x_*, f(x_*)\rangle \geq 0$, that is, $\langle x - x_*, f(x_*)\rangle \geq 0$, for all $x \in \mathbf{K}$, which implies that x_* is a solution of the problem C.P. $(f, \mathbf{K})$.

Case 2. $||x_*|| = r$. In this case we have $\langle x_* - u_0, f(x_*)\rangle \geq 0$, (from (M)), and since $\langle x - x_*, f(x_*)\rangle \geq 0$ for all $x \in \mathbf{K}_r$, we obtain

$$\begin{aligned} \langle x - u_0, f(x_*)\rangle &= \langle x - x_* + x_* - u_0, f(x_*)\rangle \\ &= \langle x - x_*, f(x_*)\rangle + \langle x_* - u_0, f(x_*)\rangle \geq 0, \end{aligned}$$

that is, we have

$$(3) \qquad \langle x - u_0, f(x_*)\rangle \geq 0, \quad \text{for all } x \in \mathbf{K}_r.$$

If $x \in \mathbf{K}$, then there is a $\lambda \in]0,1[$ such that $v = \lambda x + (1-\lambda)u_0 \in \mathbf{K}_r$ (since $||u_0|| < r$).

If we put $x = v$ in (3) we have $\lambda\langle x - u_0, f(x_*)\rangle \geq 0$, that is,

$$(4) \qquad \langle x - u_0, f(x_*)\rangle \geq 0, \quad \text{for all } x \in \mathbf{K}.$$

Since $||u_0|| < r$, from (2) we have

$$(5) \qquad \langle u_0 - x_*, f(x_*)\rangle \geq 0.$$

Now from (4) and (5) we deduce $\langle x - x_*, f(x_*)\rangle \geq 0$, for all $x \in \mathbf{K}$, that is, x_* is a solution of the problem C.P. $(f, \mathbf{K})$ with $||x_*|| \leq r$. □

We know [**13, 19**] that if $T : \mathbf{K} \to \mathbf{K}$ then T has a fixed point in $\mathbf{K}$ if and only if the problem C.P. $(I - T, \mathbf{K})$ has a solution.

So, from Proposition 3 we have the following variant of Altman's theorem.

Corollary 1. *If* $\mathbf{K} \subset H$ *is a locally compact convex cone,* $T : \mathbf{K} \to \mathbf{K}$ *a continuous operator and there is a number* $r > 0$ *such that* $\langle T(x), x\rangle \leq ||x||^2$*, for every* $x \in \mathbf{K}$ *with* $||x|| = r$ *then* T *has a fixed point in* $\mathbf{K}$*.*

We remark that condition (M) was considered by Moré for the cone $\mathbf{R}^n_+$ [**27**].

We introduce now a more general condition specific for Galerkin cones.

Definition 2. Let $\mathbf{K}(\mathbf{K}_n)_{n\in\mathbf{N}}$ be a *Galerkin cone* in H and $f : \mathbf{K} \to H$ a mapping.

We say that f *satisfies condition* (GM) if there exist a bounded sequence of positive numbers $\{r_n\}_{n\in\mathbf{N}}$ and a sequence $\{u_n\}_{n\in\mathbf{N}} \subset \mathbf{K}$ such that for every $n \in \mathbf{N}$ we have

i) $r_n \geq ||u_n||$,

ii) $\langle x - u_n, f(x)\rangle \geq 0$, for all $x \in \mathbf{K}_n$ with $||x|| = r_n$.

We introduced condition (GM) to have more flexibility about condition ii) on every approximate cone $\mathbf{K}_n$.

Since $\mathbf{K}_n$ can be constructed by the finite element method it is important to verify condition ii) independent on every $\mathbf{K}_n$ and not on the cone $\mathbf{K}$.

The condition ii) is *currently* used in the complementarity theory [**27, 19**], etc.

Theorem 5. *Let $(H, \langle,\rangle)$ be a Hilbert space and $\mathbf{K}(\mathbf{K}_n)_{n\in\mathbf{N}}$ a Galerkin cone in H.*

Suppose, given two continuous operators $S, T : \mathbf{K} \to \mathbf{H}$ such that S is bounded, T is compact and $(S+T)(\mathbf{K}) \subseteq \mathbf{K}$.

If the following assumptions are satisfied:

1) *$I - S$ satisfies condition $(S)_+$,*

2) *$I - S - T$ satisfies condition* (G.M.),

then $S + T$ has a fixed point in $\mathbf{K}$.

Proof. The theorem is proved if we show that the problem C.P $(I - S - T, \mathbf{K})$ has a solution.

Since $I - S - T$ satisfies condition (G.M.), then from Proposition 3 we have that for every $n \in \mathbf{N}$ the problem C.P. $(I - S - T, \mathbf{K}_n)$ has a solution $x_n \in \mathbf{K}_n$ such that $||x_n|| \leq r_n$.

From condition (GM) we have that $\{x_n\}_{n\in\mathbf{N}}$ is bounded.

Since H is a reflexive space we have that $\{x_n\}_{n\in\mathbf{N}}$ has a subsequence, denoted also by $\{x_n\}_{n\in\mathbf{N}}$, which is weakly convergent to $x_* \in \mathbf{K}$.

The sequence $\{x_n\}_{n\in\mathbf{N}}$ being bounded and $I - S$ a bounded operator, we have that $\{(I - S)(x_n)\}_{n\in\mathbf{N}}$ is norm bounded and from reflexivity we obtain that $\{x_n\}_{n\in\mathbf{N}}$ has a subsequence, denoted again by $\{x_n\}_{n\in\mathbf{N}}$ such that $\{(I-S)(x_n)\}_{n\in\mathbf{N}}$ is weakly convergent to an element $u \in H$.

Since T is a compact operator and considered a subsequence, we may also suppose that $\{T(x_n)\}_{n\in\mathbf{N}}$ is norm convergent to an element $v \in H$.

Because x_n is a solution of the problem C.P. $(I-S-T, \mathbf{K}_n)$ we have

$$\langle x_n, x_n - S(x_n) - T(x_n)\rangle = 0; \quad \text{for every } n \in \mathbf{N}, \tag{6}$$

that is,

$$\langle x_n, x_n - S(x_n)\rangle = \langle x_n, T(x_n)\rangle; \quad \text{for every } n \in \mathbf{N}. \tag{7}$$

From (7) we deduce

$$\lim_{n\to\infty} \langle x_n, x_n - S(x_n)\rangle = \lim_{n\to\infty} \langle x_n, T(x_n)\rangle = \langle x_*, v\rangle. \tag{8}$$

Let $\{P_n\}_{n\in\mathbf{N}}$ be a sequence of projections such that for every $n \in \mathbf{N}$, P_n is a projection on $\mathbf{K}_n$, and for every $x \in \mathbf{K}\, \lim_{n\to\infty} P_n(x) = x$. We set $\hat{x}_n = P_n(x_*)$.

Since, for every $n \in \mathbf{N}$, x_n solves the problem C.P. $(I-S-T, \mathbf{K}_n)$ which is equivalent to a variational inequality [**13, 19**] and since denoting $z_n = \hat{x}_n + (1+1/n)x_n$, we have that $z_n \in \mathbf{K}_n$ (for every $n \in \mathbf{N}$) we obtain,

$$\begin{aligned} 0 &\le \langle z_n - x_n, x_n - S(x_n) - T(x_n)\rangle \\ &= \langle \hat{x}_n + x_n/n, x_n - S(x_n) - T(x_n)\rangle \\ &= \langle \hat{x}_n, x_n - S(x_n) - T(x_n)\rangle \\ &\quad + \frac{1}{n}\langle x_n, x_n - S(x_n) - T(x_n)\rangle \\ &= \langle \hat{x}_n, x_n - S(x_n) - T(x_n)\rangle, \end{aligned}$$

which implies

$$\langle \hat{x}_n, T(x_n)\rangle \le \langle \hat{x}_n, x_n - S(x_n), \rangle$$

and computing the limit in the last inequality, we deduce

$$\langle x_*, v\rangle \le \langle x_*, u\rangle. \tag{9}$$

From (8) and (9), we obtain $\lim_{n\to\infty}\langle x_n, x_n - S(x_n)\rangle \leq \langle x_*, u\rangle$.

Since $I - S$ satisfies condition $(S)_+$, from Lemma 4 we have that $\{x_n\}_{n\in\mathbf{N}}$ has a norm convergent subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$.

We denote again $\{x_{n_k}\}_{k\in\mathbf{N}}$ by $\{x_n\}_{n\in\mathbf{N}}$ and $x_* = \lim_{n\to\infty} x_n$.

The proof is finished if we show that x_* is a solution of the problem C.P. $(I - S - T, \mathbf{K})$. Indeed, let $z \in \mathbf{K}$ be an arbitrarily element. If we denote $z_n = P_n(z)$ we have $\lim_{n\to\infty}(z_n - x_n) = z - x_*$. Since $z_n \in \mathbf{K}_n$, for every $n \in \mathbf{N}$, and x_n solves the problem C.P. $(I - S - T, \mathbf{K}_n)$, we obtain (using again that C.P. $(I - S - T, \mathbf{K}_n)$ is equivalent to a variational inequality).

$$(10) \qquad \langle z_n - x_n, x_n - S(x_n) - T(x_n)\rangle \geq 0.$$

Taking the limit in (10) as n tends to $+\infty$, we obtain $\langle z - x_*, x_* - S(x_*) - T(x_*)\rangle \geq 0$ for all $z \in \mathbf{K}$, which implies that x_* solves the problem C.P. $(I - S - T, \mathbf{K})$ and the theorem is proved. □

For the following corollaries we suppose that $\mathbf{K}(\mathbf{K}_n)_{n\in\mathbf{N}}$ is a Galerkin cone in Hilbert space $(H, \langle,\rangle)$.

Corollary 1. *Let $S, T : \mathbf{K} \to \mathbf{H}$ be two continuous operator such that S is a φ-contraction with φ continuous, T is a compact operator, and $(S + T)(\mathbf{K}) \subseteq \mathbf{K}$.*

If $I - S - T$ satisfies condition (G.M.)*, then $S + T$ has a fixed point in $\mathbf{K}$.*

Corollary 2. *Let $S, T : \mathbf{K} \to \mathbf{H}$ be two operators such that S is a contraction, T is a continuous compact operator and $(S + T)(\mathbf{K}) \subseteq \mathbf{K}$.*

If $I - S - T$ satisfies condition (G.M.)*, then $S + T$ has a fixed point in $\mathbf{K}$.*

Remark. A fixed point theorem for an operator of the form $S + T$ where S is a contraction and T is continuous and compact was proved on bounded sets by Krasnoselskii in [**23**]. Corollary 2 is similar to Krasnoselskii's theorem but on a convex cone which is an unbounded set.

Corollary 3. *If* $S, T : \mathbf{K} \to \mathbf{H}$ *are continuous* $(S+T)\mathbf{K}) \subseteq \mathbf{K}$, T *is compact and* S *is a contraction or a* φ*-contraction with* φ *continuous, and there is an* $r > 0$ *such that* $\langle x, S(x) + T(x)\rangle \leq ||x||^2$, *for every* $x \in K$ *with* $||x|| = r$, *then* $S + T$ *has a fixed point in* $\mathbf{K}$.

A class of mappings much studied in fixed point theory is the class of pseudo-contractive mappings [**4, 11, 14, 15, 16, 21, 22, 25, 26, 30, 31, 35**].

We recall that in a Hilbert space a mapping T is a pseudo-contractive if and only if $I - T$ is monotone.

In this sense we say that $S : \mathbf{K} \to \mathbf{K}$ is ρ-pseudo-contractive if $I - S$ is strongly ρ-monotone in the sense of the definition used before.

From Proposition 1 and Theorem 5 we have the following result.

Corollary 4. *Let* $S, T : \mathbf{K} \to \mathbf{H}$ *be two continuous operators such that* $(S+T)(\mathbf{K}) \subseteq \mathbf{K}$, T *is compact and* S *is* ρ*-pseudo-contractive and bounded.*

If $I - S - T$ *satisfies condition* (G.M.), *then* $S + T$ *has a fixed point in* $\mathbf{K}$.

The next result is a coincidence theorem on cones in Hilbert spaces.

We denote by "$\leq$" the ordering defined by $\mathbf{K}$.

Corollary 5. *Let* $(H, \langle , \rangle)$ *be a Hilbert space and* $\mathbf{K}(\mathbf{K}_n)_{n\in\mathbf{N}}$ *a Galerkin cone in* H. *Suppose, given two continuous operators* $S_0, T : \mathbf{K} \to H$ *such that* S_0 *is bounded,* T *is compact and* $S_0(x) \leq T(x) + x$, *for all* $x \in \mathbf{K}$.

If the following assumptions are satisfied:

1) S_0 *satisfies condition* $(S)_+$,

2) $S_0 - T$ *satisfies condition* (G.M.),

then there exists an element $x_* \in \mathbf{K}$ *such that* $S_0(x_*) = T(x_*)$.

Proof. We apply Theorem 5 with $S = I - S_0$. □

Remark. In the proof of Theorem 5, the assumption that $\{r_n\}_{n\in\mathbf{N}}$ (defined in condition (G.M.)) is used to obtain that the sequence $\{x_n\}_{n\in\mathbf{N}}$ is bounded.

We can use the condition (G.M.) without the assumption that $\{r_n\}_{n\in\mathbf{N}}$ is bounded, but in this case it is necessary to suppose that S and T satisfy some supplementary conditions, as for example to be φ-asymptotically bounded [**34**].

Let $\varphi : \mathbf{R}_+ \to \mathbf{R}_+$ be a mapping such that $\varphi(t) > 0$ for every $t \geq \gamma$ where $\gamma \in \mathbf{R}_+$.

We say that a mapping $S_0 : \mathbf{K} \to H$ is *φ-asymptotically bounded* if there exist $r, c \in \mathbf{R}_+\backslash\{0\}$ such that $r \leq ||x||$, $(x \in \mathbf{K})$ implies $||S_0(x)|| \leq c\varphi(||x||)$.

If we suppose that S and T are asymptotically bounded, S with respect to φ_1 and T with respect to φ_2 and $\limsup_{r\to+\infty}[\varphi_1(r) + \varphi_2(r)] < +\infty$, then the sequence $\{x_n\}_{n\in\mathbf{N}}$ defined in the proof of Theorem 5 is bounded.

Indeed, if we suppose that $\{x_n\}$ is unbounded, then from (6) we have

$$\langle x_n, x_n\rangle = \langle x_n, S(x_n)\rangle + \langle x_n, T(x_n)\rangle : \forall\, n \in N$$

which implies

$$||x_n||^2 \leq [c_1\varphi_1(||x_n||) + c_2\varphi_2(||x_n||)]||x_n||, \tag{11}$$

for every $n \in \mathbf{N}$ such that $\max(r_1, r_2) \leq ||x_n||$, where c_1, c_2, r_1, r_2 are the constants defined by the assumption that S and T are φ-asymptotically bounded).

When $\{||x_n||\}_{n\in\mathbf{N}}$ tends from infinity, we obtain from (11) a contradiction since $\limsup_{||x_n||\to\infty} \varphi_1(||x_n||) + \varphi_2(||x_n||)] < +\infty$.

Comments. The principal result of this paper is probably the first fixed point theorem for a sum of two operators, on a convex cone, based on the condition $(S)_+$.

The connection between the complementarity theorem and the fixed point theorem seems to be interesting.

We intend to present some applications in another paper.

REFERENCES

1. M. Altman, *A fixed point theorem in Hilbert space*, Bull. Acad. Polon. Sci. **5** (1957), 19–22.

2. N.A. Bobylev and M.A. Krasnoselskii, *Normal deformations of variational problems*, Soviet Math. Dokl. **40** (1990), 141–144.

3. ———, *On bifurcation of the extremals of variational problems*, Soviet Math. Dokl. **40** (1991), 420–423.

4. J. Bogin, *On strict pseudo-contractions and a fixed point theorem*, Technion, Preprint Series **T-219**, Haifa, Israel, 1974.

5. D.W. Boyd and J.S.W. Wong, *A nonlinear contraction*, Proc. Amer. Math. Soc. **20** (1969), 458–464.

6. F.E. Browder, *Nonlinear elliptic boundary value problems and the generalized topological degree*, Bull. Amer. Math. Soc. **76** (1970), 999–1005.

7. ———, *Existence theorems for nonlinear partial differential equations*, Proc. Sympos. Pure Math. **16** (1970), 1–60.

8. ———, *Nonlinear eigenvalue problems and Galerkin approximations*, Bull. Amer. Math. Soc. **74** (1968), 651–656.

9. ———, *Existence theory for boundary value problems for quasilinear elliptic systems with strongly nonlinear lower order terms*, Proc. Sympos. Pure Math. **23** (1973), 269–286.

10. ———, *Fixed point theory and nonlinear problems*, Bull. Amer. Math. Soc. **9** (1983), 1–39.

11. C.E. Chidume, *Iterative approximation of fixed point of Lipschitzian strictly pseudo-contractive mappings*, Proc. Amer. Math. Soc. **99** (1987), 283–288.

12. I. Ciorănescu, *Geometry of Banach spaces, duality mappings and nonlinear problems*, Kluwer Academic Publishers, Boston, 1990.

13. R.W. Cottle, *Complementarity and variational problems*, Symposia Math. **19** (1976), 177–208.

14. J.A. Gatica, *Fixed point theorems for k-set contractions and pseudo-contractive mappings*, J. Math. Anal. Appl. **46** (1974), 555–564.

15. J.A. Gatica and W.A. Kirk, *Fixed point theorems for Lipschitzian pseudo-contractive mappings*, Proc. Amer. Math. Soc. **36** (1972), 111–115.

16. ———, *Fixed point theorems for contraction mappings with applications to nonexpansive and pseudo-contractive mappings*, Rocky Mountain J. Math. **4** (1974), 69–79.

17. P. Hess, *On nonlinear mappings of monotone type homotopic to odd operators*, J. Func. Anal. **11** (1972), 138–167.

18. R.B. Holmes, *Geometrical functional analysis and its applications*, Springer-Verlag, New York-Berlin, 1975.

19. G. Isac, *Problèmes de complémentarité* (*En dimension infinie*), *Mincours*, Publ. Dép Math. Univ. Limoges, France, 1985.

20. ———, *Fixed point theory, coincidence equations on convex cones and complementarity problem*, Contemp. Math. **72** (1988), 139–155.

21. W.A. Kirk, *A fixed point theorem for local pseudo-contraction in uniformly convex spaces*, Manuscripta Math. **30** (1979), 89–102.

22. W.A. Kirk and R. Schönberg, *Some results on pseudo-contractive mappings*, Pacific J. Math. **71** (1977), 89–100.

23. M. Krasnoselskii, *Two remarks on the method of successive approximations*, Uspehi Math. Nauk **10** (1955), 123–127.

24. M.A. Krasnoselskii, N.A. Bobylev and E.M. Muhamadiev, *On a scheme for studying degenerate extremals of functionals in the classical calculus of variations*, Soviet Math. Dokl. **19** (1978), 634–638.

25. C. Morales, *Pseudo-contractive mappings and the Leray-Schauder boundary condition*, Comment. Math. Univ. Carolinae **20** (1979), 745–756.

26. ———, *On the fixed point theory for local k-pseudo-contractions*, Proc. Amer. Math. Soc. **81** (1981), 71–74.

27. J.J. Moré, *Coercivity conditions in nonlinear complementarity problems*, SIAM Rev. **16** (1974), 1–16.

28. D. Pascali and S. Sburlan, *Nonlinear mappings of monotone type*, Ed. Acad. Buc. Romania and Noordhoff Inter. Publ., 1978.

29. W.V. Petryshyn, *Antipodes theorem for A-proper mappings and its applications to mappings of the modified type (S) or $(S)_+$ and to mappings with the pm property*, J. Funct. Anal. **7** (1971), 165–221.

30. J. Reinermann and R. Schöneberg, *Some results and problems in the fixed point theory for nonexpansive and pseudo-contractive mappings in Hilbert space*, Academic Press, Seminar proceedings, Dalhousie University, Halifax, N.S., Canada, 1975.

31. R. Schöneberg, *On the structure of fixed point sets of pseudo-contractive mappings*, Comment. Math. Univ. Carolinae **18** (1977), 299–310.

32. M. Shinbrot, *A fixed point theorem and some applications*, Arch. Rational Mech. Anal. **17** (1965), 255–271.

33. I.V. Skrypnik, *Higher-order nonlinear elliptic equations*, Nauk. Dumka, Kiev, 1973.

34. V.H. Weber, *ϕ-asymptotisches spectrum and surjektivitätssätze von Fredholm type für nichtlineare operatoren mit anwendungen*, Math. Nachr. **117** (1984), 7–35.

35. M. Willem, *Perturbations non-linéaires d'opérateurs linéaires dont le noyau est de dimension infinie et applications*, Ph.D. thesis, Univ. Catholique de Louvain, Fac. Sci. Appliquées, 1979.

Department of Mathematics, Royal Military College of Canada, Kingston, Ontario, Canada K7K 5L0