ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 4, Fall 1995

OPERATORS ON C^*-ALGEBRAS INDUCED BY CONDITIONAL EXPECTATIONS

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ABSTRACT. This paper investigates the relationship between a unital C^*-algebra $\mathcal{A}$ and a C^*-subalgebra $\mathcal{B}$ which is the range of a conditional expectation operator on $\mathcal{A}$ by studying a certain algebra $\mathcal{D}$ of operators on $\mathcal{A}$. The investigation of $\mathcal{D}$ was suggested by previous work of A. Lambert and B. Weinstock in the case where the conditional expectation operators were the classical ones of probability theory.

The commutant of $\mathcal{D}$, the radical $\operatorname{Rad}\mathcal{D}$, the quotient $\mathcal{D}/\operatorname{Rad}\mathcal{D}$, the spectra of elements of $\mathcal{D}$ and the lattice of invariant subspaces for $\mathcal{D}$ are studied, as well as the questions of when $\mathcal{D}$ is closed in the norm and strong operator topologies.

Introduction. In [**5**] the relationship between a probability space (X, Σ, m) and a σ subalgebra Σ_1 of Σ is studied by using a certain algebra of bounded operators on $L^2(X, \Sigma, m)$. These operators are defined in terms of the classical conditional expectation $E(\,|\Sigma_1)$, and they have several natural analogues in which the classical conditional expectation is replaced by a conditional expectation operator defined on a C^*-algebra (see Section 1 below). The purpose of this paper is to study the relationship between a unital C^*-algebra $\mathcal{A}$ and a C^*-subalgebra $\mathcal{B}$ which is the range of a conditional expectation operator on $\mathcal{A}$ by investigating one such analogue which seems particularly natural from the perspective of the theory of associative operator algebras. In the noncommutative case our chosen analogue causes us to lose the relationship (which is present in [**5**]) to the notion of Banach-Lie algebra. That relationship is preserved, however, by other analogues of the operators in [**5**], which clearly deserve further study.

In Section 1 we define a nondegeneracy condition (called "type 0") for a C^*-subalgebra $\mathcal{B}$ which is modeled on an analogous property for σ subalgebras which plays a key role in [**5**]. We also introduce a modification of this condition ("restricted type 0") which is often more appropriate for the case that the subalgebra does not contain the

Received by the editors on August 11, 1993, and in revised form on November 5, 1993.

identity of $\mathcal{A}$. In particular, $\mathcal{B}$ is of restricted type 0 (if $\mathcal{B}$ contains the identity of $\mathcal{A}$, of type 0) if and only if $\mathcal{B}$ contains no nontrivial closed two-sided ideal of $\mathcal{A}$. These concepts turn out to play a fundamental role in the theory to be developed.

The algebra $\mathcal{D}$ being studied is an algebra of bounded linear operators on the C^*-algebra $\mathcal{A}$. $\mathcal{D}$ depends upon the choice of a conditional expectation operator from $\mathcal{A}$ onto $\mathcal{B}$. Some basic structure theory for $\mathcal{D}$ is developed in Section 2. In Section 3 the radical of $\mathcal{D}$, the questions of when $\mathcal{D}$ is closed in the norm and strong operator topologies, and the spectra of elements of $\mathcal{D}$ are investigated. Section 4 characterizes the commutant of $\mathcal{D}$ when either of the nondegeneracy conditions (type 0 or restricted type 0) applies, as well as in other cases of interest. Section 5 is a study of the invariant subspaces of $\mathcal{D}$.

The setting of this investigation, an algebra $\mathcal{D}$ of operators on a C^*-algebra, may seem unfamiliar to many readers. However, the topics treated are motivated by the theory of nonselfadjoint algebras of operators on Hilbert space. The results obtained suggest that mathematicians from other backgrounds also may be interested in the rich structure of algebras of this type.

1. Conditional expectations on C^*-algebras. Let $\mathcal{A}$ be a unital C^*-algebra with identity element 1 and $\mathcal{B}$ a proper C^*-subalgebra of $\mathcal{A}$. A *conditional expectation* from $\mathcal{A}$ onto $\mathcal{B}$ is a mapping $\Phi : \mathcal{A} \to \mathcal{B}$ satisfying

(i) $\Phi(b) = b$ for all $b \in \mathcal{B}$

(ii) $\Phi(ab) = \Phi(a)b$ and $\Phi(ba) = b\Phi(a)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$

(iii) $\Phi(a)$ is positive for all positive $a \in \mathcal{A}$,

i.e., a conditional expectation operator from $\mathcal{A}$ onto $\mathcal{B}$ is a positive, $\mathcal{B}$-linear projection from $\mathcal{A}$ onto $\mathcal{B}$. (Recall that if $x \in \mathcal{A}$, then x is called *positive* if $x = x^*$ and the spectrum of x lies on the nonnegative real axis, or equivalently, if $x = y^*y$ for some $y \in \mathcal{A}$.) Henceforth the notation $\Phi : \mathcal{A} \to \mathcal{B}$ will always imply that $\Phi(\mathcal{A}) = \mathcal{B}$.

A linear mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is called a *Schwarz mapping* if $\Phi(a^*)\Phi(a) \leq \Phi(a^*a)$ for all $a \in \mathcal{A}$. The following proposition summarizes some of the well-known properties of conditional expectations. We refer to Stratila [**7**, Section 9] for a complete discussion.

Proposition 1.1 (i) (Tomiyama). *Every projection of norm* 1 *from* $\mathcal{A}$ *onto* $\mathcal{B}$ *is a conditional expectation.*

(ii) *Every projection from* $\mathcal{A}$ *onto* $\mathcal{B}$ *which is also a Schwarz mapping is a conditional expectation.*

(iii) *Every conditional expectation* Φ *from* $\mathcal{A}$ *onto* $\mathcal{B}$ *is a Schwarz mapping and has norm* 1. *Also,* $\mathcal{B}$ *is unital with identity element* $\Phi(1)$.

(iv) *If* $\Phi : \mathcal{A} \to \mathcal{B}$ *is a conditional expectation, then* $\Phi(a^*) = \Phi(a)^*$ *for all* $a \in \mathcal{A}$.

Remark. Henceforth we write e for $\Phi(1)$, unless otherwise indicated.

There are two further properties which a conditional expectation may possess and to which reference will be made below. A positive linear mapping $\Phi : \mathcal{A} \to \mathcal{B}$ is called *faithful* if $\Phi(a^*a) = 0$ implies $a = 0$. A Schwarz mapping of one von Neumann algebra onto another is called *normal* if it is continuous in the ultraweak topology.

Definition 1.2. (i) $\mathcal{S}_1(\mathcal{A} \mid \mathcal{B}) = \{x \in \mathcal{A} : \mathcal{A}x \subseteq \mathcal{B}\}$.

(ii) $\mathcal{S}_0(\mathcal{A} \mid \mathcal{B}) = \{x \in \mathcal{A} : \mathcal{A}ex \subseteq \mathcal{B}\}$.

When $\mathcal{A}$ and $\mathcal{B}$ are understood, we write these sets as $\mathcal{S}_1$ and $\mathcal{S}_0$, respectively. It is clear that $\mathcal{S}_1 \subseteq \mathcal{S}_0$ and that each of these sets is closed in $\mathcal{A}$. It is important to note that Φ plays no role in the definition of these sets.

It is useful to observe that, since $e(1-e) = 0$, we have $(1-e) \in \mathcal{S}_0$. When $e = 1$, we write $\mathcal{S}$ for $\mathcal{S}_1 = \mathcal{S}_0$.

Lemma 1.3. (i) $\mathcal{S}_1$ *is a closed two-sided ideal in* $\mathcal{A}$.

(ii) $\mathcal{S}_1 \subseteq \mathcal{B}$.

(iii) $\mathcal{S}_1$ *contains every left ideal of* $\mathcal{A}$ *which is contained in* $\mathcal{B}$.

Proof. If $a \in \mathcal{A}$, $x \in \mathcal{S}_1$ and $b \in \mathcal{B}$, then $\mathcal{A}ax \subseteq \mathcal{A}x \subseteq \mathcal{B}$ and $\mathcal{A}xb \subseteq \mathcal{B}b \subseteq \mathcal{B}$ so that $\mathcal{S}_1$ is a left ideal of $\mathcal{A}$ and a right ideal of $\mathcal{B}$. In particular, [**3**, page 252], $\mathcal{S}_1$ is a two-sided ideal of the C^*-algebra

$\mathcal{B}$, hence $\mathcal{S}_1$ is self-adjoint. Thus $\mathcal{S}_1$ is a self-adjoint left ideal of $\mathcal{A}$, so $\mathcal{S}_1$ is a two-sided ideal of $\mathcal{A}$. Finally, if $\mathcal{I}$ is a left ideal of $\mathcal{A}$ which is contained in $\mathcal{B}$, then $\mathcal{A}\mathcal{I} \subseteq \mathcal{B}$ so $\mathcal{I} \subseteq \mathcal{S}_1$. □

Remark. If $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, $\mathcal{S}_1$ is also closed in the weak operator topology.

The following lemma shows that the right (left) annihilator of $\ker \Phi$ in $\mathcal{B}$ depends only on $\mathcal{B}$ and is independent of the choice of Φ.

Lemma 1.4.

$$\mathcal{S}_1 = \{x \in B : (\ker \Phi)x = \{0\}\} = \{x \in \mathcal{B} : x(\ker \Phi) = \{0\}\}.$$

Proof.

$$\begin{aligned} x \in \mathcal{S}_1 &\Longleftrightarrow ax \in \mathcal{B} \quad \text{for all } a \in \mathcal{A} \\ &\Longleftrightarrow \Phi(ax) = ax \quad \text{for all } a \in \mathcal{A} \\ &\Longleftrightarrow x \in \mathcal{B} \quad \text{and} \quad \Phi(a)x = ax \quad \text{for all } a \in \mathcal{A} \\ &\Longleftrightarrow (\Phi(a) - a)x = 0 \quad \text{for all } a \in \mathcal{A} \\ &\Longleftrightarrow (\ker \Phi)x = 0. \end{aligned}$$

If $x(\ker \Phi) = \{0\}$, then by taking adjoints we have $(\ker \Phi)x^* = \{0\}$ so $x^* \in \mathcal{S}_1$, and hence $x \in \mathcal{S}_1$. Conversely, $x \in \mathcal{S}_1 \Rightarrow x^* \in \mathcal{S}_1 \Leftrightarrow (\ker \Phi)x^* = \{0\} \Rightarrow x(\ker \Phi) = \{0\}$. □

Remark. $(1-e)\mathcal{A} \subseteq \ker \Phi$ and thus $\mathcal{B} \cap (1-e)\mathcal{A} = \{0\}$. Also, $\mathcal{S}_1 + (1-e)\mathcal{A} \subseteq \mathcal{S}_0$. We will see below that in fact $\mathcal{S}_1 + (1-e)\mathcal{A} = \mathcal{S}_0$.

We shall be concerned principally with subalgebras which are of type 0 in the sense of the following definition. The terminology follows that of Lambert and Weinstock [**5**]. We note that since it is always the case that $(1-e) \in \mathcal{S}_0$, it follows that if $\mathcal{B}$ is of type 0, then $e = 1$.

Definition 1.5. (i) $\mathcal{B}$ is a subalgebra of type zero if $\mathcal{S}_0 = \{0\}$.

(ii) $\mathcal{B}$ is a subalgebra of restricted type zero if $e \neq 1$ and $\mathcal{S}_1 = \{0\}$.

The following examples will be instructive to the reader.

Example 1.6. Let (X, Σ, m) be a probability space, and let Γ be a sub σ algebra of Σ. In this measure theoretic setting we will refer to the conditional expectation $\Lambda : L^\infty(X, \Sigma) \to L^\infty(X, \Gamma)$ satisfying $\int_A \Lambda(f)\, dm = \int_A f\, dm$ for every Γ-set A as the probabilistic expectation. The usual notation is $\Lambda(f) = E(f \mid \Gamma)$. A slight modification of Lambert and Weinstock [**5**] shows that $L^\infty(X, \Gamma)$ is type 0 in $L^\infty(X, \Sigma)$ if and only if there is an f in $L^\infty(X, \Sigma)$ such that $f \neq 0$ almost everywhere dm but $\Lambda(f) = 0$ almost everywhere dm.

When $L^\infty(X, \Gamma)$ is not type 0, the ideal $\mathcal{S}$ is in fact a principal ideal. Indeed, there is a maximum set B_0 in Σ (modulo null sets) such that every Σ-set contained in B_0 is a Γ-set. It follows that $\mathcal{S} = \mathcal{A}\chi_{B_0}$ (see [**5**]).

Example 1.7. This example illustrates the fact that the type 0 condition depends only on $\mathcal{B}$ and not on the conditional expectation onto $\mathcal{B}$. The relationship between the type 0 condition and the $\mathcal{D}$ algebras (to be introduced in Section 2 below) arising from different expectations onto $\mathcal{B}$ will be made clear in Theorem 2.7. Let (X, Σ, m) be the standard Lebesgue measure space on $[-1, 1]$ ($dm = dx/2$) and let Γ be the sub σ algebra of Σ generated by all sets of the form $(-a, a)$. It follows that the probabilistic expectation is given by $\Lambda(f)(x) = (f(x) + f(-x))/2$. Now consider the mapping $\Phi : L^\infty(X, \Sigma) \to L^\infty(X, \Gamma)$ given by $\Phi(f)(x) = f(|x|)$. A moment's reflection (literally) shows that Φ is indeed a conditional expectation onto $L^\infty(X, \Gamma)$. The kernel of Φ is precisely $\{f \in L^\infty(X, \Sigma) : \chi_{[0,1]} \cdot f = 0\}$. Of course, this expectation is far from being faithful, and every function f vanishing on $[-1, 0]$ is an annihilator of $\ker \Phi$. However, no Γ-measurable function other than 0 annihilates $\ker \Phi$ since each such function is an even function vanishing to the left of 0.

Example 1.8. Let $\mathfrak{H}$ be a separable Hilbert space with orthonormal basis $\{e_1, e_2, \dots\}$. Viewing operators on $\mathfrak{H}$ as matrices with respect to this basis, we consider the conditional expectation Φ onto the algebra of diagonal matrices by setting $\Phi(A) = \operatorname{diag} A$. Then the kernel of Φ consists precisely of those matrices (corresponding to bounded operators) with zero diagonal. It is easily verified that the diagonal matrices are type 0 in the algebra of all bounded operators.

Example 1.9. Let $\mathcal{U}$ be a C^*-algebra, and let $\mathcal{I}$ be a (closed, two sided) ideal in $\mathcal{U}$. Define $\mathcal{A}$ to be the set of all matrices of the form

$$\begin{bmatrix} X & K \\ L & Y \end{bmatrix}$$

with X and Y in $\mathcal{U}$ and K and L in $\mathcal{I}$. With conjugate transpose as involution, $\mathcal{A}$ is seen to be a C^*-algebra, where the C^*-norm is constructed by representing $\mathcal{U}$ as an algebra of operators on a Hilbert space $\mathfrak{H}$ and viewing $\mathcal{A}$ as an algebra of operators on $\mathfrak{H} \oplus \mathfrak{H}$. Let $\mathcal{B}$ be the C^*-subalgebra consisting of all such matrices with $K = L = 0$, and take Φ to be the diagonal conditional expectation onto $\mathcal{B}$. It follows that $\mathcal{B}$ is type 0 if and only if 0 is the only annihilator of $\mathcal{I}$.

As a special case of this last construction, take $\mathcal{U}$ to be the algebra of all bounded operators on a separable Hilbert space $\mathfrak{H}$, and let $\mathcal{I}$ be the set of all compact operators on $\mathfrak{H}$. Since 0 is the only annihilator of the compact operators, this gives an example of a type 0 subalgebra.

Remark. Let $\mathcal{A}$ be a factor and $\mathcal{B}$ a von Neumann subalgebra of $\mathcal{A}$ which is the range of a conditional expectation operator defined on $\mathcal{A}$. Since, as was remarked above, $\mathcal{S}_1$ is closed in the weak operator topology and since, by [**3**, pages 442–444], each nonzero ideal in $\mathcal{A}$ is weak-operator dense, it follows that $\mathcal{S}_1 = \{0\}$. Thus $\mathcal{B}$ is either of restricted type 0 (if $e \neq 1$) or of type 0 (if $e = 1$).

2. The algebras $\mathcal{D}(\mathcal{A}|\mathcal{B};\Phi)$. Given C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{B}$ a subalgebra of $\mathcal{A}$, and Φ a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$, for each $a \in \mathcal{A}$ we define the linear operator $D_a : \mathcal{A} \to \mathcal{A}$ by

$$D_a(x) = \Phi(a)x - a\Phi(x).$$

Notice that $\mathcal{A}$ is the direct sum of the kernel of Φ with $\mathcal{B}$, and the kernel of Φ is the set of elements of $\mathcal{A}$ of the form $a - \Phi(a)$ with $a \in \mathcal{A}$. Let M_a denote the operator of left multiplication by a acting on $\mathcal{A}$. With respect to this direct sum decomposition,

$$D_a = \begin{bmatrix} M_{\Phi(a)} & M_{\Phi(a)-a} \\ 0 & 0 \end{bmatrix}.$$

Definition 2.1. $\mathcal{D} = \{D_a : a \in \mathcal{A}\}$.

Since Φ is linear, $\mathcal{D}$ is a linear space of operators. Since Φ is contractive, for each a we have $||D_a|| \leq 2||a||$, and in particular, $\mathcal{D}$ consists of bounded operators. One notes immediately that the mapping $\Delta : \mathcal{A} \to \mathcal{D}$ given by $\Delta(a) = D_a$ is linear, with norm no greater than two.

For the sake of clarity we shall on occasion describe $\mathcal{D}$ as $\mathcal{D}(\mathcal{A}|\mathcal{B})$ or $\mathcal{D}(\mathcal{A}|\mathcal{B};\Phi)$. This last form will be used in situations where we must consider more than one conditional expectation onto $\mathcal{B}$. We now show that $\mathcal{D}$ is an algebra of operators:

Proposition 2.2. (i) *For every x and y in $\mathcal{A}$, $D_x D_y = D_{\Phi(x)y}$; in particular, $\mathcal{D}$ is an algebra.*

(ii) $\Delta(\ker\Phi)\mathcal{D} = \{0\}$; *i.e., for each k in $\ker\Phi$, D_k is a left annihilator of $\mathcal{D}$.*

Proof. Note first that each D_a maps $\mathcal{A}$ into $\ker\Phi$. Let x, y and z be members of $\mathcal{A}$. Then

$$\begin{aligned} D_x D_y z &= \Phi(x) D_y z - x\Phi(D_y z) = \Phi(x)(\Phi(y)z - y\Phi(z)) \\ &= \Phi(\Phi(x)y)z - (\Phi(x)y)\Phi(z) = D_{\Phi(x)y} z. \qquad \square \end{aligned}$$

Although the conditional expectation Φ is used to define the algebra $\mathcal{D}(\mathcal{A}|\mathcal{B})$, this algebra is algebraically and topologically independent of Φ, as the following proposition shows.

Proposition 2.3. *Let $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$, and let Φ and Ψ be conditional expectations of $\mathcal{A}$ onto $\mathcal{B}$. Define $G = \Psi + 1 - \Phi$. Then*

(i) *G is an invertible operator on $\mathcal{A}$, and $G^{-1} = \Phi + 1 - \Psi$. Also, $\Phi G = \Psi = G\Psi$, $\Psi G^{-1} = \Phi = G^{-1}\Phi$, and $(1-\Psi)(1-\Phi) = 1 - \Psi$.*

(ii) *For each $x \in \mathcal{A}$, let $F_x \in \mathcal{D}(\mathcal{A}|B;\Phi)$ and $S_x \in \mathcal{D}(\mathcal{A}|\mathcal{B};\Psi)$ be defined by $F_x y = \Phi(x)y - x\Phi(y)$ and $S_x y = \Psi(x)y - x\Psi(y)$. Then $G^{-1}F_x G = S_{G^{-1}x}$, and the mapping $\Gamma : F_x \to G^{-1}F_x G$ is an algebra* *isomorphism of* $\mathcal{D}(\mathcal{A}|\mathcal{B};\Phi)$ *onto* $\mathcal{D}(\mathcal{A}|\mathcal{B};\Psi)$ *which is a homeomorphism (norm-norm or strong-strong topologies).*

Proof. (i) is just straightforward calculation. To establish (ii), let x and y be elements of $\mathcal{A}$. Then

$$G^{-1}F_xGy = G^{-1}((\Phi x)(\Psi y + y - \Phi y) - x(\Psi y)).$$

Now $\Phi F_x = 0$, so

$$\begin{aligned} G^{-1}F_xGy &= (1-\Psi)((\Phi x)(\Psi y + y - \Phi y) - x(\Psi y)) \\ &= (1-\Psi)((\Phi x)(\Psi y)) + (1-\Psi)((\Phi x)(1-\Phi)y) \\ &\quad - [(1-\Psi)x]\Psi y \\ &= 0 + (\Phi x)(1-\Psi)y - [(1-\Psi)x]\Psi y \\ &= (\Phi x)y - (\Phi x + x - \Psi x)\Psi y = (\Psi G^{-1}x)y - (G^{-1}x)\Psi y \\ &= S_{G^{-1}x}y. \end{aligned}$$

Thus Γ is an algebra isomorphism. Since it is conjugation by the bounded invertible operator G, it is a homeomorphism with respect to any of the operator topologies. □

Lemma 2.4. *Let* $\Phi : \mathcal{A} \to \mathcal{B}$ *be a conditional expectation, and let* $\mathcal{K}$ *denote the kernel of* Φ. *Then*

(i) $\mathcal{K} = (1-e)\mathcal{A}(1-e) \oplus (\mathcal{K}e + e\mathcal{K})$

(ii) $\mathcal{K}e \cap e\mathcal{K} = e\mathcal{K}e$

(iii) $\mathcal{K}e \cap e\mathcal{K} = \mathcal{K}$ *if and only if* $e = 1$

(iv) $\mathcal{K}e + e\mathcal{K} = \vee_{a\in\mathcal{A}}$ *range* D_a (*where* $\vee$ *denotes the algebraic span*).

Proof. (i) The set to the right of the equality symbol is clearly contained in the kernel of Φ. If $a \in \mathcal{A}$, then

$$a = eae + ea(1-e) + (1-e)ae + (1-e)a(1-e),$$

so that $a - \Phi(a)$ satisfies

$$\begin{aligned} a - \Phi(a) &= e(a - \Phi(a)) + (1-e)ae + (1-e)a(1-e) \\ &\in e\mathcal{K} + \mathcal{K}e + (1-e)\mathcal{A}(1-e). \end{aligned}$$

But every element of $\mathcal{K}$ has the form $a-\Phi(a)$ for some a. The fact that $(1-e)\mathcal{A}(1-e)\cap(\mathcal{K}e+e\mathcal{K})=\{0\}$ follows immediately from $e(1-e)=0$.

(ii) If $k,k'\in\mathcal{K}$ and $ke=ek'=x$, then $(1-e)x=0$, so that $x=ex=eke$.

(iii) If $e\mathcal{K}e=\mathcal{K}$, then $(1-e)=eae-\Phi(a)$ for some a. Multiplying by e, we have $0=eae-\Phi(a)$, i.e., $e=1$. The converse follows immediately.

(iv) For each a, $x\in\mathcal{A}$,

$$\begin{aligned}D_a(x)&=\Phi(a)(x-\Phi(x))-(a-\Phi(a))\Phi(x)\\&=e\Phi(a)(x-\Phi(x))-(a-\Phi(a))\Phi(x)e\in e\mathcal{K}+\mathcal{K}e.\end{aligned}$$

Conversely, if $k\in\mathcal{K}$, then $ek=D_1(k)$ and $ke=-D_k(1)$. □

The following lemma shows that a subalgebra can be the range of a unique conditional expectation. The subsequent proposition explores this case in greater detail in relation to the algebra $\mathcal{D}$. In particular, although Proposition 2.3 implies that if Φ and Ψ are conditional expectations from $\mathcal{A}$ onto $\mathcal{B}$, then $\mathcal{D}(\mathcal{A}\mid\mathcal{B};\Phi)=\{0\}$ if and only if $\mathcal{D}(\mathcal{A}\mid\mathcal{B};\Psi)=\{0\}$, Proposition 2.6 shows that when $\mathcal{D}=\{0\}$ there is in fact a unique conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$, of the form described by Lemma 2.5.

Lemma 2.5. *Let f be a Hermitian idempotent in $\mathcal{A}$, and let $\Phi:\mathcal{A}\to f\mathcal{A}f$ be a conditional expectation. Then $\Phi(a)=faf$.*

Proof. If $a\in\mathcal{A}$, then $faf=\Phi(faf)=f\Phi(a)f=\Phi(a)$. □

Proposition 2.6. *Let $\Phi:\mathcal{A}\to\mathcal{B}$ be a conditional expectation, and let $\mathcal{K}$ denote the kernel of Φ. Then the following are equivalent*:

(i) $\mathcal{K}e+e\mathcal{K}=\{0\}$

(ii) $\mathcal{K}e=\{0\}$

(iii) $e\mathcal{K}=\{0\}$

(iv) *e is central and $\mathcal{B}=\mathcal{A}e$, (and hence $\Phi(a)=ae$ for all $a\in\mathcal{A}$ by Lemma* 2.5)

(v) $\mathcal{D}=\{0\}$

(vi) $D_a|\mathcal{B} = 0$ for all $a \in \mathcal{A}$

(vii) $\mathcal{B}$ *is an ideal of* $\mathcal{A}$

(viii) $\mathcal{B} = \mathcal{S}_1$

(ix) $\mathcal{A} = \mathcal{S}_0$.

Proof. By Lemma 2.4 (iv), the first and fifth statements are equivalent. The first clearly implies the second and the third. If $\mathcal{D} = \{0\}$, then for all $x, y \in \mathcal{A}$, $\Phi(x)y = x\Phi(y)$. Replacing x and y in turn by 1 and z yields that for all $z \in \mathcal{A}$, $ez = \Phi(z) = ze$. Conversely, if $\Phi(a) = ae$ for all $a \in \mathcal{A}$ with e central, then $D_x(y) = (xe)y - x(ye) = 0$. Thus the fourth and fifth statements are equivalent. For any a, $D_a(e) = (\Phi(a) - a)e$. Since $(\Phi(a) - a)$ is the generic element of $\mathcal{K}$, the second and sixth statements are equivalent. Since both $\mathcal{K}$ and e are $*$-invariant, the second and third statements are equivalent. If the sixth statement is true, then for all a, $0 = D_a e = \Phi(a) - ae$, and by taking adjoints we have that e is central, i.e., the fourth statement is true. Thus, the first six statements are equivalent. Now (iv) $\Rightarrow$ (vii) $\Rightarrow$ (viii) $\Rightarrow$ (ii) since, if $\mathcal{B} = \mathcal{A}e$ with e central, then $\mathcal{B}$ is an ideal of $\mathcal{A}$; hence by Lemma 1.3 $\mathcal{B} = \mathcal{S}_1$ and, if $\mathcal{B} = \mathcal{S}_1$, then $e \in \mathcal{S}_1$ so by Lemma 1.4 $\mathcal{K}e = 0$. Finally, (ix) $\Rightarrow$ (iv) since $\mathcal{A} = \mathcal{S}_0$ implies that $aex \in \mathcal{B}$ for all $a, x \in \mathcal{A}$. In particular, $ae, ea \in \mathcal{B}$ so that $\Phi(a) = ae = ea$ for all a. Conversely, if $\Phi(a) = ea$ with e central, then as we have seen, $\mathcal{S}_1 = \mathcal{B}$, so since $a = ea + (1 - e)a \in \mathcal{S}_1 + (1 - e)\mathcal{A}$ we have $\mathcal{A} = \mathcal{S}_0$. □

Remark. One class of examples to which Proposition 2.6 applies is the case where $\mathcal{A}$ is a C^*-algebra of bounded operators on a Hilbert space and $\mathcal{B} = \mathcal{A}Q$ where Q is a central orthogonal projection. It follows from Lemma 2.5 that $\Phi(T) = TQ$ defines the unique conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$ in this case.

Note that the situation leading to the collapse of $\mathcal{D}$ to $\{0\}$ is antithetical to the condition "$\mathcal{S}_1 = \{0\}$." We now explore the strong relationship between $\mathcal{D}$, $\mathcal{S}_0$ and $\mathcal{S}_1$ in the general case.

Theorem 2.7. *Let* $\Phi : \mathcal{A} \to \mathcal{B}$ *be a conditional expectation, with* $e = \Phi(1)$. *Then*

(i) $\mathcal{S}_0 = \cap_{a\in\mathcal{A}} \ker D_a = \mathcal{S}_1 + (1-e)\mathcal{A} = \{a \in \mathcal{A} : D_{a^*} = 0\}$.

(ii) *If e is central, then $\mathcal{S}_0 = \{a \in \mathcal{A} : D_a = 0\}$ and $\mathcal{S}_0$ is a closed, two-sided ideal of $\mathcal{A}$. (Thus, $\mathcal{B} + \mathcal{S}_0$ is a C^*-subalgebra of $\mathcal{A}$ and $(\mathcal{B}+\mathcal{S}_0)/\mathcal{S}_0$ can be canonically identified with a C^*-subalgebra of $\mathcal{A}/\mathcal{S}_0$.)*

Proof. (i). We show that

$$\mathcal{S}_0 \subseteq \bigcap_{a\in\mathcal{A}} \ker D_a \subseteq \mathcal{S}_1 + (1-e)\mathcal{A} \subseteq \{a \in \mathcal{A} : D_{a^*} = 0\} \subseteq \mathcal{S}_0.$$

Let $x \in \mathcal{S}_0$. Then aex is in $\mathcal{B}$ for every a in $\mathcal{A}$ and, in particular, $ex \in \mathcal{B}$. Now $ex = \Phi(ex) = e\Phi(x) = \Phi(x)$, so that for each a in $\mathcal{A}$,

$$\begin{aligned} D_a x &= \Phi(a)x - aex = \Phi(a)ex - aex \\ &= \Phi(aex) - \Phi(aex) = 0. \end{aligned}$$

Thus $\mathcal{S}_0 \subseteq \cap_{a\in\mathcal{A}} \ker D_a$.

Now suppose that $x \in \cap_{a\in\mathcal{A}} \ker D_a$. Since $D_1 y = ey - \Phi(y)$ for every y, it follows that $ex = \Phi(x)$. Consequently, for every $a \in \mathcal{A}$, $0 = D_a x = \Phi(a)x - aex$, i.e., $[\Phi(a) - a]ex = 0$. It follows from Lemma 1.4 that $ex \in \mathcal{S}_1$. Since $x = ex + (1-e)x$, we have shown that $\cap_{a\in\mathcal{A}} \ker D_a \subseteq \mathcal{S}_1 + (1-e)\mathcal{A}$.

Let $c \in \mathcal{S}_1 + (1-e)\mathcal{A}$. Then ec is in $\mathcal{S}_1$ and $\Phi(c) = ec$. Taking the adjoints of both sides of this last equality yields the equation $\Phi(c^*) = c^*e$. Then for each y in $\mathcal{A}$,

$$\begin{aligned} D_{c^*} y &= \Phi(c^*)y - c^*\Phi(y) = c^*ey - c^*\Phi(y) \\ &= c^*ey - c^*e\Phi(y) = c^*e(y - \Phi(y)) = 0, \end{aligned}$$

since $c^*e \in \mathcal{S}_1$. Thus $\mathcal{S}_1 + (1-e)\mathcal{A} \subseteq \{a \in \mathcal{A} : D_{a^*} = 0\}$.

Finally, suppose $D_{c^*} = 0$. Then, for each y, $\Phi(c^*)y = c^*\Phi(y)$ and, in particular, $\Phi(c^*) = c^*e$. But the penultimate equation then becomes $c^*ey = c^*\Phi(y)$. Since $c^*\Phi(y) = c^*e\Phi(y)$, $(c^*e)\mathrm{Ker}\,\Phi = 0$. Thus c^*e is a member of $\mathcal{S}_1$ and consequently so is ec, which shows that $c \in \mathcal{S}_0$. This completes the proof of (i).

(ii) We assume that e is a central idempotent. Then $\mathcal{S}_1 + (1-e)\mathcal{A}$ is easily seen to be a closed, two-sided ideal in $\mathcal{A}$. Thus, if $c \in \mathcal{S}_1 + (1-e)\mathcal{A}$, then so is c^*. But we may then apply part (i) to see that $D_c = 0$. Conversely, if $D_c = 0$, then the argument in part (i) shows that $ce \in \mathcal{S}_1$, hence $ec \in \mathcal{S}_1$, which proves that $c \in \mathcal{S}_1 + (1-e)\mathcal{A}$. This establishes the equation $\mathcal{S}_0 = \{a \in \mathcal{A} : D_a = 0\}$. □

Example 2.8. Let $\mathcal{A}$ be the algebra of 2×2 matrices with complex entries and $\mathcal{B}$ the subalgebra consisting of matrices all of whose entries except possibly the (1,1) entry are zero. Let E be the unit element of $\mathcal{B}$, and define Φ on $\mathcal{A}$ by $\Phi(A) = EAE$. Then Φ is a conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$. It is easily verified that, in this case, $\mathcal{S}_1 = 0$ and that $\mathcal{S}_0 = (1-E)\mathcal{A}$ consists of those matrices whose first row is zero. Thus, $\mathcal{B}$ is of restricted type zero but not type zero. Also, E is not a central idempotent in $\mathcal{A}$. If $A = (a_{ij})$ and $X = (x_{ij})$ are elements of $\mathcal{A}$, then

$$D_A(X) = \begin{bmatrix} 0 & a_{11}x_{12} \\ -a_{21}x_{11} & 0 \end{bmatrix}.$$

Thus, $\{A : D_A = 0\}$ is the set of matrices whose first column is zero, i.e., $\mathcal{S}_0^*$. This illustrates that $\mathcal{S}_0$ need not be self-adjoint if $\Phi(1)$ is not in the center of $\mathcal{A}$.

Let $\mathcal{A}_1$ be a second C^*-algebra and $\pi : \mathcal{A} \to \mathcal{A}_1$ a $*$-homomorphism of $\mathcal{A}$ onto $\mathcal{A}_1$. Then $\mathcal{B}_1 = \pi(\mathcal{B})$ is a C^*-subalgebra of $\mathcal{A}_1$. If $\Phi(\ker \pi) \subseteq \ker \pi$, then Φ induces a mapping $\Phi_1 : \mathcal{A}_1 \to \mathcal{B}_1$ defined by $\Phi_1(\pi(x)) = \pi(\phi(x))$. We will denote $\mathcal{S}_0(\mathcal{A}_1 \mid \mathcal{B}_1)$ by $\mathcal{S}_0'$. It follows immediately from Definition 1.2 that $\pi(\mathcal{S}_0) \subseteq \mathcal{S}_0'$.

Lemma 2.9. *Suppose that $\mathcal{A}_1$ is a C^*-algebra and $\pi : \mathcal{A} \to \mathcal{A}_1$ a unital $*$-homomorphism of $\mathcal{A}$ onto $\mathcal{A}_1$ such that $\Phi(\ker \pi) \subseteq \ker \pi$. Let $\mathcal{B}_1 = \pi(\mathcal{B})$. Then*

(i) *Φ_1 is a conditional expectation of $\mathcal{A}_1$ onto $\mathcal{B}_1$.*

(ii) *The map $D_a \to D_{\pi(a)}$ is well-defined and is an algebra homomorphism T of $\mathcal{D}(\mathcal{A}|\mathcal{B};\Phi)$ onto $\mathcal{D}(\mathcal{A}_1|\mathcal{B}_1;\Phi_1)$*

(iii) *T is an isomorphism if and only if π also satisfies $\pi^{-1}(\mathcal{S}_0') \subseteq \mathcal{S}_0$.*

Proof. (i) If $x \in \mathcal{A}$, $\Phi_1(\Phi_1(\pi(x))) = \Phi_1(\pi(\Phi(x))) = \pi(\Phi(\Phi(x))) =$

$\pi(\Phi(x)) = \Phi_1(\pi(x))$, so Φ_1 is a projection. Also, if $b, c \in \mathcal{B}$,

$$\begin{aligned}\Phi_1(\pi(b)\pi(x)\pi(c)) &= \Phi_1(\pi(bxc) = \pi(\Phi(bxc)) \\ &= \pi(b\Phi(x)c) = \pi(b)\pi(\Phi(x))\pi(c) \\ &= \pi(b)\Phi_1(\pi(x))\pi(c).\end{aligned}$$

Thus Φ_1 is $\mathcal{B}_1$-linear. Finally, $\Phi_1(\pi(x)^*\pi(x)) = \Phi_1(\pi(x^*x)) = \pi(\Phi(x^*x))$. Since Φ is positive and π is a homomorphism, we see that Φ_1 is a positive map. Thus, Φ_1 is a conditional expectation.

(ii) $D_a = 0 \Rightarrow a^* \in \mathcal{S}_0 \Rightarrow \pi(a)^* = \pi(a^*) \in \mathcal{S}'_0 \Rightarrow D_{\pi(a)} = 0$. Thus, T is well defined. Also, T is clearly a linear surjection. Moreover, $D_a D_c = D_{\Phi(a)c}$. But $\pi(\Phi(a)c) = \pi(\Phi(a))\pi(c) = \Phi_1(\pi(a))\pi(c)$. Therefore, $D_{\pi(\Phi(a)c)} = D_{\pi(a)}D_{\pi(c)}$.

(iii) T is an isomorphism if and only if $D_{\pi(a)} = 0 \Rightarrow D_a = 0$. If $\pi^{-1}(\mathcal{S}'_0) \subseteq \mathcal{S}_0$, then $D_{\pi(a)} = 0 \Leftrightarrow \pi(a)^* \in \mathcal{S}'_0 \Leftrightarrow \pi(a^*) \in \mathcal{S}'_0 \Rightarrow a^* \in \mathcal{S}_0 \Leftrightarrow D_a = 0$. Conversely, if $D_{\pi(a)} = 0 \Rightarrow D_a = 0$, the double implications in the previous line show that $\pi^{-1}(\mathcal{S}'_0) \subseteq \mathcal{S}_0$. □

The next proposition shows that for the purpose of studying an algebra $\mathcal{D}(\mathcal{A} \mid \mathcal{B})$ when e is a central idempotent in $\mathcal{A}$, one may assume without loss of generality that $\mathcal{B}$ is type 0 in $\mathcal{A}$.

Proposition 2.10. *Suppose that $e = \Phi(1)$ is a central idempotent in $\mathcal{A}$. Define*

$$\hat{\Phi} : \frac{\mathcal{A}}{\mathcal{S}_0} \longrightarrow \frac{\mathcal{B} + \mathcal{S}_0}{\mathcal{S}_0}$$

by $\widehat{\Phi(a)} = \Phi\ (a)$ (where $a \to \hat{a}$ is the canonical quotient map.)

Then

(i) *$\hat{\Phi}$ is a conditional expectation*

(ii) *$\mathcal{A}/\mathcal{S}_0$ and $(\mathcal{B} + \mathcal{S}_0)/\mathcal{S}_0$ have the same unit*

(iii) *$(\mathcal{B} + \mathcal{S}_0)/\mathcal{S}_0$ is type 0 in $\mathcal{A}/\mathcal{S}_0$*

(iv) *the map $D_a \to D_{\hat{a}}$ is an algebra isomorphism of $\mathcal{D}(\mathcal{A}/\mathcal{B})$ onto $\mathcal{D}(\mathcal{A}/\mathcal{S}_0|(\mathcal{B} + \mathcal{S}_0)/\mathcal{S}_0)$.*

Proof. Since $\Phi(\mathcal{S}_0) \subseteq \mathcal{S}_0$ we may apply Lemma 2.9 with $\mathcal{A}_1 = \mathcal{A}/\mathcal{S}_0$ and π the canonical projection to prove (i).

(ii) Each element of $(\mathcal{B}+\mathcal{S}_0)/\mathcal{S}_0$ has the form $b+\mathcal{S}_0$ for some $b \in \mathcal{B}$. Since $1-e$ is in $\mathcal{S}_0$, $\hat{1} = \hat{e}$. But

$$\hat{e}(b+\mathcal{S}_0) = eb+\mathcal{S}_0 = b+\mathcal{S}_0 = be+\mathcal{S}_0 = (b+\mathcal{S}_0)\hat{e};$$

while $\hat{1}$ is also the unit in $\mathcal{A}/\mathcal{S}_0$.

(iii) In order to prove that $(\mathcal{B}+\mathcal{S}_0)/\mathcal{S}_0$ is type 0 in $\mathcal{A}/\mathcal{S}_0$ it suffices, via Theorem 2.7(i), to show that $\cap_{a\in\mathcal{A}}\ker D_{\hat{a}} = \{0\}$. To this end, let $x \in \mathcal{A}$ for which $\hat{x} \in \cap_{a\in\mathcal{A}}\ker D_{\hat{a}}$. For all $a \in \mathcal{A}$, $\hat{D_a x} = 0$, so that $D_a x \in \mathcal{S}_0$. Thus $D_a x \in (1-e)\mathcal{A}$. In particular, $D_1 x \in (1-e)\mathcal{A}$. But $D_1 x = e(x-\Phi(x))$, so that $D_1 x = 0$, i.e., $\Phi(x) = ex$. But this shows that for all a in $\mathcal{A}$, $D_a x = \Phi(a)x - aex$. Now if $a \in \ker\Phi$, then $D_a x = -aex = e(-ax)$ by the centrality of e. Since $D_a x \in (1-e)\mathcal{A}$, $D_a x = 0$ in this case. On the other hand, if $a \in \mathcal{B}$, then $D_a x = \Phi(a)ex - aex = 0$ and so, by decomposing each $a \in \mathcal{A}$ along $\mathcal{B}$ and $\ker\Phi$, we see that $x \in \cap_{a\in\mathcal{A}}\ker D_a = \mathcal{S}_0$, so that $\hat{x} = 0$.

(iv) Since $\ker\pi = \mathcal{S}_0$ and $\mathcal{S}_0' = \{0\}$, Lemma 2.9 shows that this map is an isomorphism. □

Remark. (i) The hypothesis that e is central can be replaced by the assumption that the principal left ideal generated by e is two-sided, without otherwise affecting the validity of the preceding proposition.

(ii) Since $\mathcal{B}\cap\mathcal{S}_0 = \mathcal{S}_1$, and for any closed (two-sided) ideal $\mathcal{J}$ of $\mathcal{A}$ we have $(\mathcal{B}+\mathcal{J})/\mathcal{J}$ $*$-isomorphic to $\mathcal{B}/\mathcal{B}\cap\mathcal{J}$ (see [**3**, Corollary 10.1.9]), it follows that $(\mathcal{B}+\mathcal{S}_0)/\mathcal{S}_0$ is $*$-isomorphic to $\mathcal{B}/\mathcal{S}_1$. (Of course, when $e=1$, so that $\mathcal{S} = \mathcal{S}_1 = \mathcal{S}_0$, we have $\mathcal{B}+\mathcal{S}_0 = \mathcal{B}$.)

Finally, we consider the center of $\mathcal{D}$. In general, $\mathcal{D}$ can be abelian. It is easily verified that Example 2.8 above is an instance of this, and this example can be generalized to the case of 2×2 matrices over any commutative C^*-algebra. However, if e is central, the center of $\mathcal{D}$ is just $\{0\}$.

Proposition 2.11. *If e is central, then the center of $\mathcal{D}$ is $\{0\}$.*

Proof. Let $\mathcal{Z}$ denote the center of $\mathcal{D}$. $D_a \in \mathcal{Z} \Leftrightarrow D_{\Phi(a)x} = D_{\Phi(x)a}$ for all $x \in \mathcal{A}$, by Proposition 2.2(i). By Theorem 2.7(ii), $D_a \in \mathcal{Z} \Leftrightarrow$

$\Phi(a)x - \Phi(x)a \in \mathcal{S}_0$. Since $e(\Phi(a)x - \Phi(x)a) = \Phi(a)x - \Phi(x)a$, we have $D_a \in \mathcal{Z} \Leftrightarrow \Phi(a)x - \Phi(x)a \in \mathcal{S}_1$ for all x. Suppose that $D_a \in \mathcal{Z}$. Then, taking $x = 1$ shows that $\Phi(a) - ea \in \mathcal{S}_1 \cap \ker\Phi = \{0\}$, so that $\Phi(a) = ea = ae$. Taking $x \in \ker\Phi$ now shows that $aex \in \mathcal{S}_1 \cap \ker\Phi$, hence $aek = 0$ for all $k \in \ker\Phi$, i.e., $a \in \mathcal{S}_0$ (by Lemma 1.4). But this implies that $D_a = 0$ by Theorem 2.7(ii). □

Remark. Another proof of Proposition 2.11, based on Proposition 2.10, will be given below in Section 4. (See the remark following Theorem 4.3.)

3. Analytic properties of $\mathcal{D}(\mathcal{A} \mid \mathcal{B}; \Phi)$. In this section we investigate various analytic properties of $\mathcal{D}(\mathcal{A}|\mathcal{B};\Phi)$ such as closure in the norm and strong operator topologies, spectrum of elements of $\mathcal{D}$, and radical. Henceforth we use $\mathcal{K}$ to denote the kernel of Φ, and if X is a Banach space, we use $\mathbf{B}(X)$ to denote the space of bounded linear transformations on X.

For each $b \in \mathcal{B}$ let L_b be the operator of left multiplication by b, *restricted to* $\mathcal{K}$, and let $\mathcal{L} = \{L_b : b \in \mathcal{B}\}$. (Note that L_b maps $\mathcal{K}$ to $\mathcal{K}$.) We give $\mathcal{L}$ the norm topology it inherits as a subspace of the space of all bounded operators on $\mathcal{K}$.

Proposition 3.1. *$\mathcal{L}$ is isometrically isomorphic to $\mathcal{B}/\mathcal{S}_1$. In particular, $\mathcal{L}$ is closed in the norm topology.*

Proof. By Lemma 1.4, $L_b = 0$ if and only if $b \in \mathcal{S}_1$. Thus, $\mathcal{L}$ is isomorphic to $\mathcal{B}/\mathcal{S}_1$. For $b \in \mathcal{B}$ left $[b]$ denote the coset of b in $\mathcal{B}/\mathcal{S}_1$. Let $|[b]| = ||L_b||$. Then $|\cdot|$ defines a norm on $\mathcal{B}/\mathcal{S}_1$. For each b in $\mathcal{B}$ and k in $\mathcal{K}$,

$$\begin{aligned} ||bk||^2 &= ||k^*b^*bk|| \le ||k||\,||b^*bk|| \le ||k||\,|[b^*]|\,||bk|| \\ &\le ||k||^2|[b^*]|\,|[b]|. \end{aligned}$$

It follows that $|[b]|^2 \le |[b^*]|\,|[b]|$ and so $|[b]| \le |[b^*]|$. Since the same argument applies to b^* we have proven that $|[b]| = |[b^*]|$ for every $b \in \mathcal{B}$. Now if we take the supremum over vectors k of length 1 in the displayed expression above, we deduce that $|[b]|^2 \le |[b^*b]| \le |[b^*]|\,|[b]| = |[b]|^2$.

This shows that $|\cdot|$ exhibits all the properties of a C^*-algebra norm on $\mathcal{B}/\mathcal{S}_1$ with the possible exception of completeness.

Let $\mathcal{C}$ be the completion of $\mathcal{B}/\mathcal{S}_1$ in this norm. Then $\mathcal{C}$ is a C^*-algebra and the identity map on $\mathcal{B}/\mathcal{S}_1$ is an injective $*$-homomorphism of $\mathcal{B}/\mathcal{S}_1$ into $\mathcal{C}$. It follows that this map is in fact isometric [**3**, page 242], i.e., for all b in $\mathcal{B}$, $|[b]| = ||[b]||$. This shows that the map $[b] \to L_b$ is an isometric algebra isomorphism from $\mathcal{B}/\mathcal{S}_1$ onto $\mathcal{L}$. □

Corollary 3.2. *$\mathcal{D}$ is closed in the norm topology on $\mathbf{B}(\mathcal{A})$, the space of all bounded linear operators on $\mathcal{A}$.*

Proof. Throughout this argument "$\to$" refers to convergence in the norm topologies, context dictating whether applied to operators or vectors. Let $\{D_{a_n}\}$ be a sequence in $\mathcal{D}$, and let D be an operator on $\mathcal{A}$ for which $D_{a_n} \to D$. Since each operator from $\mathcal{D}$ maps $\mathcal{K}$ into itself, D does as well. Moreover, $D_{a_n}|\mathcal{K} \to D|\mathcal{K}$, where $|$ denotes restriction. Now $D_{a_n}|\mathcal{K}$ is precisely $L_{\Phi(a_n)}$ (as defined above), so by Proposition 3.1 there exists $b \in \mathcal{B}$ such that $D \mid \mathcal{K} = L_b$. For $c \in \mathcal{B}$, $D_{a_n}c = (\Phi(a_n) - a_n)c \to Dc$. Let $a = b - De$. Then, for any $x \in \mathcal{A}$,

$$\begin{aligned} D_a x &= \Phi(a)(x - \Phi(x)) + (\Phi(a) - a)\Phi(x) \\ &= b(x - \Phi(x)) + (De)\Phi(x) \\ &= D(x - \Phi(x)) + \lim_{n\to\infty}(D_{a_n}e)\Phi(x) \\ &= D(x - \Phi(x)) + \lim_{n\to\infty}\{(\Phi(a_n) - a_n)e\Phi(x)\} \\ &= D(x - \Phi(x)) + \lim_{n\to\infty}(\Phi(a_n) - a_n)\Phi(x) \\ &= D(x - \Phi(x)) + D\Phi(x) = Dx. \qquad \square \end{aligned}$$

Remark. In light of the preceding result it seems reasonable to ask if $\mathcal{D}$ need be closed in the strong operator topology. Note that if $\{D_{a_n}\}$ were a net converging strongly to the operator D, then the arguments presented in the preceding result would carry through, with the crucial exception of the existence of $b \in \mathcal{B}$, determined there by the norm closure of $\mathcal{L}$. There is, as Example 3.6 below will show, no reason to believe that $\mathcal{D}$ is strongly closed. Here we present one situation where $\mathcal{D}$ is in fact strongly closed.

Proposition 3.3. *If $\mathcal{K}$ contains a right invertible element of $\mathcal{A}$, then $\mathcal{D}$ is closed in the strong operator topology.*

Proof. By hypothesis, there exist $k \in \mathcal{K}$ and $c \in \mathcal{A}$ such that $kc = 1$. Suppose that $\{L_{b_n}\}$ is a net in $\mathcal{L}$ converging strongly to an operator L on $\mathcal{K}$. Then $b_n k \to Lk$, and so $b_n = b_n kc \to (Lk)c$. Since $\mathcal{B}$ is closed, $(Lk)c \in \mathcal{B}$. Now for any $h \in \mathcal{K}$, $L_{b_n} h = b_n h \to [(Lk)c]h = L_{(Lk)c} h$, so that $L = L_{(Lk)c} \in \mathcal{L}$. By the comments made in the paragraph preceding the statement of the proposition, we conclude that $\mathcal{D}$ is closed in the strong operator topology. □

Example 3.4. To see that $\mathcal{K}$ can contain a right invertible element, one need go no further than to take $\mathcal{A}$ to be the set of all block matrices of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where the entries are operators on a Hilbert space, and Φ is taken as the diagonal map

$$\Phi\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}.$$

Then

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

is an invertible member of $\ker \Phi$. (We have stated this example in sufficient generality to show that the strong closure of $\mathcal{D}$ does not require finite dimensionality.)

On the other hand, here is an example where $\mathcal{B}$ is type 0, Φ is faithful, $\mathcal{A}$ is abelian, and yet $\ker \Phi$ contains no invertible elements.

Example 3.5. Let X be the set of positive integers with probability distribution $m(2n-1) = 1/((n+1)2^n)$ and $m(2n) = n/((n+1)2^n)$ for $n \in X$, and let $\mathcal{A}$ be the l^∞ space $L^\infty(X, 2^X, m)$. Define Σ to be the sigma algebra of subsets of X generated by the set of 2-point atoms $\{\{2n-1, 2n\} : n \geq 1\}$, and let $\mathcal{B} = L^\infty(X, \Sigma, m|_\Sigma)$. For each bounded sequence α the (probabilistic) conditional expectation is given by

$$\Phi(\alpha) = E(\alpha \mid \Sigma) = \sum_{n=1} (m(2n-1)\alpha_{2n-1} + m(2n)\alpha_{2n})\chi_{\{2n-1,2n\}},$$

so that $\Phi(\alpha) = 0$ if and only if for all $n \geq 1$, $(\alpha_{2n-1} + n \cdot \alpha_{2n})/((n+1)2^n) = 0$. Since the sequence α is bounded, $\lim_{n\to\infty} \alpha_{2n} = 0$ and consequently α cannot be invertible in l^∞. Note that since Φ is a classical probabilistic conditional expectation, it is faithful and, of course, the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are abelian. Also, $\mathcal{B}$ is type 0 since if β is any sequence such that $\beta\alpha = 0$ for all $\alpha \in \ker \Phi$, then $\beta = 0$. Indeed, if $\beta_k \neq 0$ for some k, then we must have $\alpha_k = 0$ for all $\alpha \in \ker\Phi$. But $\ker\Phi$ obviously contains elements which do not satisfy this condition.

We note that for Φ as in the preceding example, although $\ker\Phi$ fails to contain an invertible element of $\mathcal{A}$, $\mathcal{D}$ is closed in the strong topology. Indeed, if $\{b^{(k)}\}$ is a net in $\mathcal{B}$ for which $b^{(k)}h$ converges in $\mathcal{A}$ for each member h of $\ker\Phi$, then by applying this to the sequence $g \doteq (1, -1, 1, -1/2, 1, -1/3, \dots) \in \ker\Phi$, we see that for each fixed integer n, $\lim_k\{b_n^{(k)}\} \doteq b_n$ exists. By hypothesis, $\lim_k\{b^{(k)}g\} \in \mathcal{A}$. Since $g_n = 1$ for odd n and since, for each $c \in \mathcal{B}$, we have $c_{2n-1} = c_{2n}$, b must be in $\mathcal{A}$ (in fact, in $\mathcal{B}$) and $b^{(k)} \to b$. It now follows from the paragraph preceding Proposition 3.3 that $\mathcal{D}$ is closed.

The following is the promised example where $\mathcal{B}$ is type 0 and $\mathcal{D}$ is not closed in the strong operator topology on $\mathcal{A}$.

Example 3.6. Consider the special case of Example 1.9 in which $\mathcal{U}$ is a proper unital C^*-subalgebra of $\mathbf{B}(\mathfrak{H})$ ($\mathfrak{H}$ a separable Hilbert space) such that $\mathcal{U}$ contains the ideal $\mathcal{I}$ of all compact operators on $\mathfrak{H}$. Then $\mathcal{K}$ consists of the set of all matrices of the form

$$\begin{bmatrix} 0 & K \\ L & 0 \end{bmatrix},$$

where K and L are compact operators on $\mathfrak{H}$.

Let A be an operator on $\mathfrak{H}$ which is not in $\mathcal{U}$. Since $\mathcal{I}$ is strongly dense in $\mathbf{B}(\mathfrak{H})$, we choose a net $\{A_n\}$ in $\mathcal{U}$ converging in the strong operator topology to A. Let

$$a_n = \begin{bmatrix} A_n & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and let} \quad \begin{bmatrix} C & K \\ L & B \end{bmatrix}$$

be an arbitrary member of $\mathcal{A}$. It is easily verified that

$$D_{a_n} x = \begin{bmatrix} 0 & A_n K \\ 0 & 0 \end{bmatrix}.$$

Since $\{A_n\}$ converges strongly to A and K is compact, $\{A_nK\}$ converges uniformly to AK. Thus, D_{a_n} converges in the strong operator topology to an operator D. However, for

$$y = \begin{bmatrix} E & M \\ N & F \end{bmatrix} \quad \text{in } \mathcal{A},$$

the (1,2) entry of the matrix $D_y x$ is $EK - MB$. If this were to equal AK for every choice of K and B (not to mention L and C), then M would equal 0 and E would equal A. But A is not in $\mathcal{U}$, so D cannot be D_y for any $y \in \mathcal{A}$.

We now turn to the radical, $\operatorname{Rad}\mathcal{D}$. We recall that $\Delta : \mathcal{A} \to \mathcal{D}$ is given by $\Delta(a) = D_a$.

Lemma 3.7. $\Delta(\mathcal{K})$ *is closed.*

Proof. By Corollary 3.2, $\mathcal{D}$ is norm closed. Let $D_{k_n} \to D_a$, $k_n \in \mathcal{K}$. Then, for every x in $\mathcal{K}$,

$$0 = D_{k_n} x \to D_a x = \Phi(a)x.$$

This implies that $\Phi(a) \in \mathcal{S}_1$, hence $D_{\Phi(a)} = 0$. Then $D_a = D_{a-\Phi(a)}$. Because $a - \Phi(a) \in \mathcal{K}$, $D_a \in \Delta(\mathcal{K})$. □

In the following lemma, attention is paid to the case $||1 - \Phi|| = 1$. This condition occurs in many, but by no means in all, examples.

Lemma 3.8. *Suppose* $||1 - \Phi|| = 1$. *Then*

(i) $||D_{\Phi(a)}|| = ||[\Phi(a)]||_{\mathcal{B}/\mathcal{S}_1}$

(ii) $\inf\limits_{k\in\mathcal{K}} ||D_{a+k}|| = ||[\Phi(a)]||_{\mathcal{B}/\mathcal{S}_1}$.

Proof. (i) For any x in the unit sphere of $\mathcal{A}$ and $s \in \mathcal{S}_1$,

$$\begin{aligned} ||D_{\Phi(a)}x|| &= ||\Phi(a)(x - \Phi(x))|| = ||(\Phi(a) + s)(x - \Phi(x))|| \\ &\le ||\Phi(a) + s||\,||1 - \Phi|| = ||\Phi(a) + s||, \end{aligned}$$

by Lemma 1.4. Thus $||D_{\Phi(a)}|| \leq ||[\Phi(a)]||_{\mathcal{B}/\mathcal{S}_1}$. On the other hand, from Proposition 3.1, we see that

$$||[\Phi(a)]||_{\mathcal{B}/\mathcal{S}_1} = \sup_{\substack{||x||=1 \\ x\in\mathcal{K}}} ||\Phi(a)x|| = \sup_{\substack{||x||=1 \\ x\in\mathcal{K}}} ||D_{\Phi(a)}x|| \leq ||D_{\Phi(a)}||.$$

(ii) By (i) $\inf_{k\in\mathcal{K}} ||D_{a+k}|| \leq ||D_{\Phi(a)}|| = ||[\Phi(a)]||_{\mathcal{B}/\mathcal{S}_1}$. Fix $k \in \mathcal{K}$. Then, by Proposition 3.1,

$$\begin{aligned} ||D_{a+k}|| &= \sup_{||x||=1} ||(a+k)\Phi(x) - \Phi(a)x|| \geq \sup_{\substack{||x||=1 \\ x\in\mathcal{K}}} ||\Phi(a)x|| \\ &= ||[\Phi(a)]||_{\mathcal{B}/\mathcal{S}_1}. \qquad \square \end{aligned}$$

When $\mathcal{B}$ is of type 0, the preceding arguments may be summarized in the following proposition, without the restriction that $||1-\Phi|| = 1$.

Proposition 3.9. *If $\mathcal{B}$ is of type 0, then for every a in $\mathcal{A}$,*

$$\begin{aligned} ||\Phi(a)|| = \sup_{\substack{||x||=1 \\ x\in\mathcal{K}}} ||\Phi(a)x|| &= \sup_{\substack{||x||=1 \\ x\in\mathcal{K}}} ||D_{\Phi(a)}x|| \\ &\leq \inf_{k\in\mathcal{K}} ||D_{a+k}|| \leq ||D_{\Phi(a)}|| \\ &\leq ||1-\Phi||\,||\Phi(a)||. \end{aligned}$$

Theorem 3.10. *$\Delta(\mathcal{K})$ is the radical of $\mathcal{D}$. Moreover, $\mathcal{D}/\Delta(\mathcal{K})$ is algebraically and topologically isomorphic to $\mathcal{B}/\mathcal{S}_1$. If $||1-\Phi|| = 1$, then this isomorphism may be taken to be isometric.*

Proof. We showed above that $\Delta(\mathcal{K})$ is closed. We now show that $\Delta(\mathcal{K})$ is a two-sided ideal. For $a \in \mathcal{A}$ and $k \in \mathcal{K}$, $D_aD_k = D_{\Phi(a)k}$ and $D_kD_a = D_{\Phi(k)a} = 0$. Since $\Phi(a)k \in \mathcal{K}$, $\Delta(\ker\Phi)$ is a two-sided ideal in $\mathcal{D}$. Define $\Gamma : \mathcal{D}/\Delta(\mathcal{K}) \to \mathcal{B}/\mathcal{S}_1$ by $\Gamma(D_a + \Delta(\mathcal{K})) = [\Phi(a)]$. Γ is linear and surjective. If $[\Phi(a)] = 0$, then $\Phi(a) \in \mathcal{S}_1$ so $D_a \in \Delta(\mathcal{K})$. Thus, Γ is one-to-one. Lemma 3.8 (ii) shows that Γ is isometric if $||1-\Phi|| = 1$.

Moreover, the proof of Lemma 3.8 (i) shows that $||\Gamma|| \leq ||1-\Phi||$. Also, Γ is multiplicative since

$$\begin{aligned}\Gamma((D_a+\Delta(\mathcal{K}))\cdot(D_c+\Delta(\mathcal{K}))) &= \Gamma(D_aD_c+\Delta(\mathcal{K}))\\ &= \Gamma(D_{\Phi(a)c}+\Delta(\mathcal{K})) = [\Phi(\Phi(a)c)]\\ &= [\Phi(a)\Phi(c)] = [\Phi(a)][\Phi(c)]\\ &= \Gamma(D_a+\Delta(\mathcal{K}))\cdot\Gamma(D_c+\Delta(\mathcal{K})).\end{aligned}$$

In order to prove that $\operatorname{Rad}\mathcal{D} \subseteq \Delta(\mathcal{K})$, we shall show that for a in $\mathcal{A}$, if D_aD_x is quasinilpotent for every x in $\mathcal{A}$, then $[\Phi(a)]$ is in $\operatorname{Rad}(\mathcal{B}/\mathcal{S}_1)$. Since $\mathcal{B}/\mathcal{S}_1$ is a C^*-algebra, $\operatorname{Rad}(\mathcal{B}/\mathcal{S}_1)$ is $\{0\}$, and the desired result will follow, since then $D_a = D_{a-\Phi(a)}$, so $D_a \in \Delta(\mathcal{K})$.

Note that for a and x in $\mathcal{A}$,

$$(D_aD_x)^n = D^n_{\Phi(a)x} = D_{(\Phi(a)\Phi(x))^{n-1}\Phi(a)x},$$

and, in particular, if $\Phi(x)=x$,

$$(D_aD_x)^n = D_{(\Phi(a)\Phi(x))^n}.$$

Now it was shown above that for every $b\in\mathcal{B}$, $||[b]|| \leq ||D_b||$, so that

$$||([\Phi(a)x])^n||^{1/n} \leq ||D_{(\Phi(a)x)^n}||^{1/n} = ||(D_aD_x)^n||^{1/n}.$$

Consequently, if D_aD_x is quasinilpotent, then $[\Phi(a)x] = [\Phi(a)][x]$ is quasinilpotent. Since x is an arbitrary element of $\mathcal{B}$, $[\Phi(a)]$ belongs to $\operatorname{Rad}(\mathcal{B}/\mathcal{S}_1)$.

As for the reverse inclusion, we have already noted that for all $k\in\mathcal{K}$ and $x\in\mathcal{A}$, $D_kD_x=0$. Thus, $\Delta(\mathcal{K}) \subseteq \operatorname{Rad}\Delta$. □

Finally, we study the spectrum of elements of $\mathcal{D}$. Of course, since the range of each element of $\mathcal{D}$ is contained in $\ker\Phi$, no element of $\mathcal{D}$ is invertible in $\mathbf{B}(\mathcal{A})$. Nevertheless, it is possible for D_a to be one-to-one. To see this, take $\mathcal{A}$ and Φ to be as in Example 3.4. Let

$$M = \begin{bmatrix} u & v\\ w & x\end{bmatrix}$$

where u and v are isometries with mutually orthogonal ranges, and w and x are likewise isometries with mutually orthogonal ranges. If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad D_M A = \begin{bmatrix} 0 & ub - vd \\ xc - wa & 0 \end{bmatrix}.$$

Thus $D_M A = 0$ implies $A = 0$.

Lemma 3.11. *If $\lambda \neq 0$, $D_a - \lambda I$ is one-to-one on $\mathcal{A}$ if and only if $L_{\Phi(a)} - \lambda I$ is one-to-one on $\mathcal{K}$.*

Proof. If $D_a - \lambda I$ is one-to-one on $\mathcal{A}$ and $\Phi(a)k - \lambda k = 0$, then $D_a k - \lambda k = 0$, hence $k = 0$. Conversely, if $L_{\Phi(a)} - \lambda I$ is one-to-one on $\mathcal{K}$ and $D_a x - \lambda x = 0$, then applying Φ to this equation shows that $\Phi(x) = 0$, i.e., $x \in \mathcal{K}$. Hence $\Phi(a)x - \lambda x = 0$, so $x = 0$. □

Lemma 3.12. *If $\lambda \neq 0$, $D_a - \lambda I$ is invertible in the algebra of bounded operators on $\mathcal{A}$ if and only if $L_{\Phi(a)} - \lambda I$ is invertible in the algebra of bounded operators on $\mathcal{K}$.*

Proof. Suppose first that $L_{\Phi(a)} - \lambda I$ is invertible. Then $D_a - \lambda I$ is one-to-one, so it suffices to show that $D_a - \lambda I$ maps $\mathcal{A}$ onto $\mathcal{A}$. Given $x \in \mathcal{A}$, we may write $x = b + k$, where $b \in \mathcal{B}$ and $k \in \mathcal{K}$. By our hypothesis, we can find $h \in \mathcal{K}$ such that $L_{\Phi(a)}h - \lambda h = k$. But $L_{\Phi(a)}$ agrees with D_a on $\mathcal{K}$. Hence $D_a h - \lambda h = k$. Likewise, we can find $g \in \mathcal{K}$ such that $(\Phi(a) - \lambda)g = (\Phi(a) - a)(\lambda^{-1}b)$. Then

$$\begin{aligned}(D_a - \lambda I)(g - \lambda^{-1}b) &= \Phi(a)(g - \lambda^{-1}b) + \lambda^{-1}ab - \lambda g + b \\ &= (\Phi(a) - \lambda)g - (\Phi(a) - a)(\lambda^{-1}b) + b = b.\end{aligned}$$

Thus, x is in the range of D_a.

Conversely, if $D_a - \lambda I$ is invertible, then given $k \in \mathcal{K}$, we can find $x \in \mathcal{A}$ with $D_a x - \lambda x = k$. Applying Φ to this equation, we see that $x \in \mathcal{K}$, and since D_a agrees with $L_{\Phi(a)}$ on $\mathcal{K}$, we conclude that $L_{\Phi(a)} - \lambda I$ is onto, and hence invertible. □

For the remainder of this section, we use $\sigma(x)$ to denote the spectrum of x in the Banach algebra $\mathbf{B}(\mathcal{A})$, and for other Banach algebras $\mathcal{C}$, $\sigma_{\mathcal{C}}(x)$ to denote the spectrum of x in $\mathcal{C}$.

Theorem 3.13. *Let $a \in \mathcal{A}$.*

(i) $\sigma(D_a) \cup \sigma(D_{a^*})^- = \{0\} \cup \sigma_{\mathcal{B}/\mathcal{S}_1}([\Phi(a)])$. ($^-$ *denotes complex conjugation*).

(ii) *The spectral radius of D_a equals the spectral radius of $[\Phi(a)]$ in $\mathcal{B}/\mathcal{S}_1$.*

(iii) *If $[\Phi(a)]$ is a normal element of the C^*-algebra $\mathcal{B}/\mathcal{S}_1$, then $\sigma(D_a) = \{0\} \cup \sigma_{\mathcal{B}/\mathcal{S}_1}([\Phi(a)])$.*

Proof. As was remarked above, D_a is never invertible, so for all $a \in \mathcal{A}$ we have $0 \in \sigma(D_a)$. By Lemma 3.12, if $\lambda \neq 0$, then $\lambda \in \sigma(D_a) \Leftrightarrow \lambda \in \sigma_{\mathbf{B}(\mathcal{K})}(L_{\Phi(a)})$. Now $\Phi(a)^* = \Phi(a^*)$, hence

$$\begin{aligned}\sigma(D_a) \cup \sigma(D_{a^*})^- &= \{0\} \cup \sigma_{\mathbf{B}(\mathcal{K})}(L_{\Phi(a)}) \cup \sigma_{\mathbf{B}(H)}(L_{\Phi(a)^*})^- \\ &= \{0\} \cup \sigma_{\mathcal{L}}(L_{\Phi(a)}) = \{0\} \cup \sigma_{\mathcal{B}/\mathcal{S}_1}([\Phi(a)]),\end{aligned}$$

where the second equality follows from the application of Rickart's spectral permanence theorem for C^*-algebras [**6**, Corollary 1 of Theorem 2] to the C^*-algebra $\mathcal{L}$ and the third equality follows from Proposition 3.1. Assertions (ii) and (iii) follow similarly from [**6**, Theorem 3, and **6**, Corollary 1 of Theorem 2], respectively. □

Corollary 3.14. *If $\mathcal{B}$ is abelian, then $\sigma(D_a) = \sigma_{\mathcal{B}/\mathcal{S}_1}([\Phi(a)]) \cup \{0\}$.*

Remark. As the proof of Theorem 3.13 indicates, one has $\sigma(D_a) = \sigma_{\mathcal{B}/\mathcal{S}_1}([\Phi(a)]) \cup \{0\}$ whenever $L_{\Phi(a)}$ invertible in $\mathbf{B}(\mathcal{K})$ implies $L_{\Phi(a)}$ invertible in $\mathcal{L}$, at least in the case that $e = 1$. Suppose, for instance, that $\mathcal{B}$ is of type 0 and that for some closed ideal $\mathcal{I}$ of $\mathcal{B}$ the representation $L : \mathcal{B} \to \mathbf{B}(\mathcal{K})$ given by $b \to L_b$ is equivalent to the left regular representation λ of $\mathcal{B}$ on $\mathcal{I}$. If $\mathcal{I}$ has the property that λ_b is invertible in $\mathbf{B}(\mathcal{I})$ if and only if b is invertible in $\mathcal{B}$ then it is true that $\sigma(D_a) = \sigma_{\mathcal{B}/\mathcal{S}_1}([\Phi(a)]) \cup \{0\}$.

Consider specifically the modification of Example 3.6 where for $\mathcal{U}$ one takes $\mathbf{B}(\mathfrak{H})$. Then $\mathcal{B}$ is still of type 0. If

$$a = \begin{bmatrix} A & K \\ L & B \end{bmatrix},$$

then the invertibility of $L_{\Phi(a)}$ in $\mathbf{B}(\mathcal{K})$ is equivalent to the invertibility, in the algebra of bounded operators on the space of compact operators, of the operators "left multiplication by A" and "left multiplication by B." But the invertibility of these operators implies that A and B are invertible operators on $\mathfrak{H}$ since the space of compact operators on $\mathfrak{H}$ contains every rank one projection.

(In the situation described above, type 0 is equivalent to $\mathcal{I}$ being an essential ideal of $\mathcal{B}$. It would be interesting to know when an essential ideal $\mathcal{I}$ in a unital C^*-algebra $\mathcal{B}$ has the property that λ_b is invertible in $B(\mathcal{I})$ if and only if b is invertible in $\mathcal{B}$. It is true, for example, when $\mathcal{B}$ is abelian.)

4. The commutant of $\mathcal{D}$. It is easy to see that if $b \in \mathcal{B}$ then the operation R_b of right multiplication by b on $\mathcal{A}$ commutes with every element of $\mathcal{D}$. We denote $\{R_b : b \in \mathcal{B}\}$ by $\mathcal{R}$. In this section we determine $\mathcal{D}'$, the commutant of $\mathcal{D}$ in the algebra of all bounded operators on $\mathcal{A}$, under the assumption that either $e = 1$ or $\mathcal{B}$ is of restricted type 0. In particular, we show that $\mathcal{D}' = \mathcal{R}$ if and only if $\mathcal{B}$ is of type 0. Thus, when $\mathcal{B}$ is of type 0, $\mathcal{D}'$ depends only on $\mathcal{B}$ and not on Φ.

Let $\mathcal{A}_1 = \mathcal{B}$, $\mathcal{A}_2 = \mathcal{K}e$, $\mathcal{A}_3 = \mathcal{K}(1-e)$. Then $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$. Relative to this direct sum decomposition, any operator T on $\mathcal{A}$ has the matricial form (T_{ij}), $1 \le i, j \le 3$ where $T_{ij} : \mathcal{A}_j \to \mathcal{A}_i$. If $\mathcal{A}_3 = \{0\}$, then $\mathcal{A}_2 = \mathcal{K}$ and the matricial form of T is a 2×2 matrix.

Theorem 4.1. *Let $\mathcal{A}$ be a C^*-algebra with unit 1, $\mathcal{B}$ a C^*-subalgebra with unit e, and Φ a conditional expectation from $\mathcal{A}$ onto $\mathcal{B}$.*

(i) *If $\mathcal{S}_1 = \{0\}$ then $T \in \mathcal{D}'$ if and only if T has the matricial form:*

$$\begin{bmatrix} R_t & 0 & 0 \\ A & R_t & B \\ X & 0 & Y \end{bmatrix}$$

where $t = T1$, R_t is right multiplication by t, $L_{\Phi(a)}A = L_{\Phi(a)}X = 0$, and $L_{\Phi(a)}$ commutes with B and Y, for all $a \in \mathcal{A}$. (Here L_x denotes left multiplication by x on the appropriate space.)

(ii) *If $\mathcal{S}_1 = \{0\}$ and e is in the center of $\mathcal{A}$, $T \in \mathcal{D}'$ if and only if T*

has the matricial form

$$\begin{bmatrix} R_t & 0 & 0 \\ 0 & R_t & 0 \\ X & 0 & Y \end{bmatrix}$$

where X and Y are arbitrary operators between the appropriate spaces.

(iii) *If $\mathcal{B}$ is of type 0, then $T \in \mathcal{D}'$ if and only if $T = R_t$.*

Proof. If $T \in \mathcal{D}'$, then for all $a, x \in \mathcal{A}$,

$$T(\Phi(a)x) - T(a\Phi(x)) = \Phi(a)Tx - a\Phi(Tx). \tag{4.1}$$

If $a, x \in \mathcal{K}$ we see that a $\Phi(Tx) = 0$ so that $\Phi(Tx) \in \mathcal{S}_1$. But $\mathcal{S}_1 = \{0\}$. Thus $T(\mathcal{K}) \subseteq \mathcal{K}$, which implies that $T_{12} = T_{13} = 0$.

If we take $x = 1$ in (4.1) we see that

$$T(\Phi(a) - ae) = \Phi(a)t - a\Phi(t).$$

If $a = e$ we conclude that $\Phi(t) = et$. Thus,

$$\begin{aligned} T((\Phi(a) - a)e) &= \Phi(a)t - aet = (\Phi(a) - a)et \\ &= (\Phi(a) - a)\Phi(t) = (\Phi(a) - a)\Phi(t)e. \end{aligned}$$

Since $\Phi((\Phi(a) - a)\Phi(t)) = 0$ we see that $T\mathcal{A}_2 \subseteq \mathcal{A}_2$ and that $T = R_t$ on $\mathcal{A}_2$. Thus, $T_{22} = R_t$ and $T_{32} = 0$.

Now for all $a, x \in \mathcal{A}$,

$$\begin{aligned} D_a x &= D_a[\Phi(x) + (x - \Phi(x))e + x(1 - e)] \\ &= (\Phi(a) - a)\Phi(x) + \Phi(a)(x - \Phi(x))e + \Phi(a)(x(1 - e)) \end{aligned}$$

so that the matricial form of D_a is

$$\begin{bmatrix} 0 & 0 & 0 \\ -L_{(1-\Phi)a} & L_{\Phi(a)} & 0 \\ 0 & 0 & L_{\Phi(a)} \end{bmatrix}.$$

Since left and right multiplication operators always commute, we conclude that $TD_a = D_aT$ implies

(a) $-L_{(1-\Phi)(a)}T_{11} + L_{\Phi(a)}T_{21} = -R_tL_{(1-\Phi)a}$

(b) $L_{\Phi(a)}T_{23} = T_{23}L_{\Phi(a)}$

(c) $L_{\Phi(a)}T_{31} = 0$

(d) $L_{\Phi(a)}T_{33} = T_{33}L_{\Phi(a)}$.

However, if $b, c \in \mathcal{B}$, (4.1) implies that

$$\begin{aligned} 0 &= T(D_c b) = c(Tb - \Phi(Tb)) \\ &= c(T_{21} + T_{31})b \end{aligned}$$

so that $L_{\Phi(a)}(T_{21} + T_{31}) = 0$. Condition (c) thus implies that $L_{\Phi(a)}T_{21} = 0$ so that condition (a) becomes $L_{(1-\Phi)(a)}(R_t - T_{11}) = 0$. Thus, $R_t b - T_{11}b \in \mathcal{S}_1$ for all $b \in \mathcal{B}$, so $T_{11} = R_t$.

Conversely, if T has the form given in part (i) of the theorem, direct calculation shows that $T \in \mathcal{D}'$.

If e is central the (3,3) entry of the matrix for D_a is 0 for all $a \in \mathcal{A}$. Thus there is no condition imposed on T_{31} or T_{33}. Also, for $b \in \mathcal{B}$, $x \in \mathcal{A}$,

$$\begin{aligned} T_{21}b &= (T(b) - \Phi(Tb))e = T(b)e - \Phi(Tb) \\ &= eT(b) - \Phi(Tb) = D_1 Tb \\ &= TD_1 b = 0, \end{aligned}$$

and, likewise,

$$\begin{aligned} T_{23}(x(1-e)) &= [T(x(1-e)) - \Phi(T(x(1-e)))]e \\ &= (Tx)e - T(xe)e - \Phi(Tx) + \Phi(T(xe)) \\ &= [eTx - \Phi(Tx)] - [eT(xe) - \Phi(T(xe))] \\ &= D_1 T_x - D_1 T(xe) \\ &= D_1 T(x - xe) = TD_1(x - xe) = 0 \end{aligned}$$

because e is central. Thus, $T_{21} = T_{23} = 0$.

Conversely, if e is central and T has the matricial form given in (ii) of the theorem then $T \in \mathcal{D}'$ by direct calculation. If $\mathcal{B}$ if of type 0, $e = 1$, so the matricial form given in part (i) of the theorem reduces to

$$\begin{bmatrix} R_t & 0 \\ A & R_t \end{bmatrix}.$$

Thus, the argument above shows that $T_{21} = 0$ for $T \in \mathcal{D}'$. Thus, $T \in \mathcal{D}' \Rightarrow T = R_t$. $\square$

Corollary 4.2. *$\mathcal{D}' = \mathcal{R}$ if and only if $\mathcal{B}$ is of type* 0.

Proof. Assume first that $\mathcal{S}_1 = \{0\}$. Then by Theorem 4.1, $\mathcal{D}'$ consists of right multiplications if and only if $\mathcal{B}$ is of type 0, in which case $T \in \mathcal{D}' \Rightarrow T = R_t$ where $t = T1$ and $\Phi(t) = et = t$. But then $t \in \mathcal{B}$, so that $\mathcal{D}' \subseteq \mathcal{R}$. Since the reverse inclusion is always true, we conclude that if $\mathcal{S}_1 = \{0\}$ then $\mathcal{D}' = \mathcal{R} \Leftrightarrow \mathcal{B}$ is of type 0.

Suppose now that $\mathcal{S}_1 \neq \{0\}$. Then by Lemma 1.4 there exists $b \in \mathcal{B}$ with $b \neq 0$ and $\mathcal{K}b = \{0\}$. We may assume that $b \neq \lambda e$ for any scalar λ: if $e \in \mathcal{S}_1$, Φ must be right multiplication by e, hence $\mathcal{D} = \{0\}$. If $b^2 \neq 0$, choose a continuous linear functional γ on $\mathcal{B}$ such that $\gamma(e) = 1$ and $\gamma(b) = 0$. Define T on $\mathcal{A}$ by the requirement that $T(a) = \gamma(\Phi(a))b$ for all $a \in \mathcal{A}$. Then for all $a, c \in \mathcal{A}$, $D_cT(a) = \gamma(\Phi(a))(\Phi(c) - c)b = 0$ by the choice of b. Also $TD_c = 0$ since the range of D_c lies in $\mathcal{K}$. Thus $T \in \mathcal{D}'$. But T is not right multiplication by any element of $\mathcal{B}$. Indeed, since $T(e) = b$, the only candidate is right multiplication by b, but the fact that $\gamma(b) = 0$ rules out this possibility. If $b^2 = 0$, repeat the argument with γ satisfying $\gamma(b) \neq 0$. □

Using Theorem 4.1 we can determine $\mathcal{D}'$ whenever $e = 1$.

Theorem 4.3. *Suppose that $e = 1$. Let $\mathcal{H}$ be the set of bounded operators $h : \mathcal{A} \to \mathcal{S}$ such that $h(\mathcal{K}) = 0$. Then $\mathcal{D}' = \mathcal{R} + \mathcal{H}$.*

Proof. If $T(x) = xb + h(x)$ where $b \in \mathcal{B}$ and h is as above, then a straightforward calculation shows that $T \in \mathcal{D}'$.

Suppose now that $T \in \mathcal{D}'$. Taking $a = 1$ in (4.1), we have

$$T(x) - T(\Phi(x)) = T(x) - \Phi(T(x))$$

so

$$T(\Phi(x)) = \Phi(T(x)). \tag{4.2}$$

Thus $T\mathcal{B} \subseteq \mathcal{B}$. Taking $x = 1$ in (4.2) we see that if $t = T(1)$ then $t \in \mathcal{B}$.

Next recall that $x \in \mathcal{S} \Leftrightarrow \Phi(a)x = ax$ for all $a \in \mathcal{A}$. Taking $x \in \mathcal{S}$ in (4.1) and recalling that $\mathcal{S} \subseteq \mathcal{B}$ and hence $T\mathcal{S} \subseteq \mathcal{B}$, we have

$$T[\Phi(a)x] - T[a\Phi(x)] = \Phi(a)T(x) - aT(x)$$

and

$$\Phi(a)x = \Phi(ax) = a\Phi(x).$$

Therefore, $\Phi(a)T(x) = aT(x)$, i.e., $T(x) \in \mathcal{S}$. Since $T\mathcal{S} \subseteq \mathcal{S}$, T induces $\hat{T}$ on $\mathcal{A}/\mathcal{S}$ defined by $\hat{T}(\hat{a}) = T(a)^\wedge$. Clearly, $\hat{T}$ commutes with each operator $D_{\hat{a}}$ defined in terms of the conditional expectation $\hat{\Phi} : \mathcal{A}/\mathcal{S} \to \mathcal{B}/\mathcal{S}$ of Theorem 2.10. But $\mathcal{B}/\mathcal{S}$ is of type 0 in $\mathcal{A}/\mathcal{S}$. Therefore, $\hat{T}(\hat{a}) = \hat{a}(\hat{T}\hat{1})$ by Theorem 4.1, i.e., $T(a) - at \in \mathcal{S}$. Thus, since $\mathcal{S} \subseteq \mathcal{B}$, $T(a) - at = \Phi(T(a)) - \Phi(a)t = T(\Phi(a)) - \Phi(a)t$. For $x \in \mathcal{A}$, let $h(x) = T(\Phi(x)) - \Phi(x)t$. Then $h \in \mathcal{H}$ and $T = R_t + h$. $\square$

Remark. As an application of Corollary 4.2 and Proposition 2.10 we can now give another proof that the center of $\mathcal{D}$ is trivial if e is central in $\mathcal{A}$. We note first that by Proposition 2.10 we need only consider the case where $\mathcal{B}$ is of type 0. Then, by Corollary 4.2, every element D_a in the center of $\mathcal{D}$ is of the form R_b for some $b \in \mathcal{B}$, so for all $x \in \mathcal{A}$ we have $D_a x = xb$. Applying Φ to both sides yields $0 = \Phi(x)b$. With $x = 1$ we conclude $0 = b$ so that $D_a = 0$. $\square$

One of the curious properties of $\mathcal{D}$ which we would like to point out is the fact that when $\mathcal{B}$ is type 0, "commutes with" is an equivalence relation on a certain subset of $\mathcal{D}$.

Theorem 4.4. *Let G denote the set of invertible elements of $\mathcal{B}'$. Define the relation R on G by aRc if and only if $\Phi(a)c = \Phi(c)a$. Then R is a transitive relation* (*and thus an equivalence relation*) *on G.*

Proof. Assume that a, c and d belong to G, $\Phi(a)c = \Phi(c)a$, and $\Phi(d)c = \Phi(c)d$. Then $c\Phi(a) = \Phi(c)a$ and $\Phi(d)c = d\Phi(c)$. This yields $\Phi(a)a^{-1} = c^{-1}\Phi(c) = \Phi(c)c^{-1} = d^{-1}\Phi(d)$, or $d\Phi(a) = \Phi(d)a$. One more application of the hypothesis that $d \in \mathcal{B}'$ gives aRd. $\square$

In view of Proposition 2.2(i) and Theorem 2.7, we immediately have the following

Corollary 4.5. *If $\mathcal{B}$ is type* 0, *"commutes with" is an equivalence relation on $\Delta(G)$.*

5. Invariant subspaces for $\mathcal{D}$. Throughout this section, we assume that $\mathcal{B}$ contains 1. We continue to denote $\ker \Phi$ by $\mathcal{K}$.

Theorem 5.1. *Let $\mathfrak{G}$ denote the collection of closed subspaces of $\mathcal{B}$. Let $\mathfrak{F}$ denote the collection of closed left $\mathcal{B}$-submodules of $\mathcal{K}$. Then, the collection of closed invariant subspaces of $\mathcal{D}$ is*

$$\mathfrak{M} = \{G + J : G \in \mathfrak{G}, J \in \mathfrak{F}, \text{ and } kG \subseteq J \text{ for all } k \in \mathcal{K}\}.$$

Proof. First consider $G + J \in \mathfrak{M}$ and $a \in \mathcal{A}$. For $g \in G$, $D_a g = [\Phi(a) - a]g \in \mathcal{K}G \subseteq J$. For $j \in J$, $D_a j = \Phi(a)j \in J$. Thus $G + J$ is invariant under $\mathcal{D}$. To see that $G + J$ is closed, consider a net $g_\alpha + j_\alpha$ with terms in G and J, respectively, which converges to $x \in \mathcal{A}$. $\Phi(g_\alpha)$ converges to $\Phi(x)$, so $\Phi(x)$ belongs to G. Likewise, $(I - \Phi)x$ belongs to J.

Now let $\mathcal{M}$ be any subspace of $\mathcal{A}$ which is invariant for $\mathcal{D}$. $D_1 = I - \Phi \in \mathcal{D}$, so $\Phi[\mathcal{M}] \subseteq \mathcal{M}$. Consequently, $\Phi[\mathcal{M}] = \mathcal{M} \cap \mathcal{B}$ by the idempotence of Φ. Thus, $\Phi[\mathcal{M}]$ is closed. Likewise, $(I - \Phi)[\mathcal{M}] = \mathcal{M} \cap \mathcal{K}$ is closed. Take $G = \Phi[\mathcal{M}]$ and $J = (I - \Phi)[\mathcal{M}]$. J is a left $\mathcal{B}$-module because for all $b \in \mathcal{B}$ and $j \in J$, $bj = D_b j \in \mathcal{M} \cap \mathcal{K}$. Similarly, for all $k \in \mathcal{K}$ and $g \in G$, $Kg = -D_k g \in \mathcal{M} \cap \mathcal{K}$. □

With basically the same argument, we have the following result.

Corollary 5.2. *Let $\mathfrak{G}_1$ denote the collection of closed right ideals of $\mathcal{B}$. Let $\mathfrak{F}_1$ denote the collection of closed $\mathcal{B}$-bimodules contained in $\mathcal{K}$. Let $\mathcal{F}$ be the algebra generated by $\mathcal{D}$ and $\{R_b : b \in \mathcal{B}\}$. Then the collection of closed invariant subspaces of $\mathcal{F}$ is $\mathfrak{N} = \{G + J : G \in \mathfrak{G}_1, J \in \mathfrak{F}_1$ and $kG \subseteq J$ for all $k \in \mathcal{K}\}$.*

Remark. (i) Theorem 5.1 and Corollary 5.2 can be generalized to other topologies on the space of bounded operators on $\mathcal{A}$. What is required is that $\mathcal{A}$ and $\mathcal{B}$ be closed and that Φ be continuous. For example, if $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras and Φ is normal, then the σ-weak topology satisfies these conditions (see [**2**, page 53]).

(ii) When $\mathcal{B}$ is of type 0, $\mathcal{F}$ is the algebra generated by $\mathcal{D} \cup \mathcal{D}'$.

In some cases the ideals and modules in the preceding results have explicit characterizations. Then we have useful descriptions of all of the invariant subspaces of $\mathcal{D}$ or $\mathcal{F}$. (We continue to use $\mathcal{F}$ to denote the algebra generated by $\mathcal{D}$ and $\{R_b : b \in B\}$.) We now present some of the most illuminating examples.

Example 5.3. Take $\mathcal{A} = L^\infty[0,1]$, and let $\Phi f(x) = \int_0^1 f(t)\,dt$ for all x in $[0,1]$. Then G could only be $\{0\}$ or $\mathbf{C}$. In the latter case, the condition $\mathcal{K}G \subseteq J$ implies that the invariant subspace $G + J$ is all of $\mathcal{A}$. In the case $G = \{0\}$, J can be any subspace of $\mathcal{K} = \{f \in L^\infty[0,1] : \int_0^1 f(t)\,dt = 0\}$. Thus $\mathcal{A}$ together with all subspaces of $\mathcal{K}$ represent all of the invariant subspaces for $\mathcal{D}$ or $\mathcal{F}$.

Example 5.4. Consider the following modification of Example 1.9. Let

$$A = \left\{ \begin{bmatrix} A & K \\ K & A \end{bmatrix} \in \mathbf{B}(\mathfrak{H} \oplus \mathfrak{H}) : A \in \mathbf{B}(\mathfrak{H}), K \text{ compact} \right\}.$$

Define Φ to be the diagonal map onto

$$\mathcal{B} = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \mathbf{B}(\mathfrak{H}) \right\}.$$

Proposition 5.5. *Under the hypotheses of Example* 5.4*, a norm-closed subspace* $\mathcal{M}$ *of* $\mathcal{A}$ *is invariant for* $\mathcal{D}$ *if and only if there exist a closed subspace* $\mathfrak{N}$ *of* $\mathfrak{H}$*, a closed subspace* $\mathcal{G}$ *of* $\mathbf{B}(\mathfrak{H})$*, and a subspace* $\mathcal{J}$ *of compact operators on* $\mathfrak{H}$ *such that*

(i) $Ax = 0$ *for all* $x \in \mathfrak{N}$ *and* $A \in \mathcal{G}$,

(ii) $\mathcal{J} = \{K : K$ *is compact and* $Kx = 0$ *for all* $x \in \mathfrak{N}\}$,

(iii) $\mathcal{M} = \left\{ \begin{bmatrix} A & K \\ K & A \end{bmatrix} : A \in \mathcal{G} \text{ and } K \in \mathcal{J} \right\}$.

Proof. Write the general closed invariant subspace for $\mathcal{D}$ as $G + J$, as in Theorem 5.1. Then G has the form

$$G = \left\{ \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \mathcal{G} \right\}$$

for some closed subspace $\mathcal{G}$ of $\mathbf{B}(\mathfrak{H})$, and J has the form

$$J = \left\{ \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} : K \in \mathcal{J} \right\}$$

where $\mathcal{J}$ is a closed left ideal in the algebra of compact operators such that for all compact operators L, $L\mathcal{G} \subseteq \mathcal{J}$. By a theorem of Kaplansky [**4**], there exists a closed subspace $\mathfrak{N}$ of $\mathfrak{H}$ such that $\mathcal{J} = \{K : K \text{ is compact and } Kx = 0 \text{ for all } x \in \mathfrak{N}\}$. The condition $L\mathcal{G} \subseteq \mathcal{J}$ for all compact operators L is equivalent to the condition that $Ax = 0$ for all $x \in \mathfrak{N}$ and all $A \in \mathcal{G}$. □

Remark. Under the hypotheses of Example 5.4, it follows immediately from Corollary 5.2 that a norm-closed subspace $\mathcal{M}$ of $\mathcal{A}$ is invariant for $\mathcal{F}$ if and only if $\mathcal{M} = \{0\}$ or there exists a norm-closed right ideal $\mathcal{G}$ in $\mathbf{B}(\mathfrak{H})$ such that

$$\mathcal{M} = \left\{ \begin{bmatrix} A & K \\ K & A \end{bmatrix} : A \in \mathcal{G}, K \text{ compact} \right\}.$$

Example 5.6. Assume that there exist minimal projections a_α in $\mathcal{B}'$ with α in a countable index set A such that $\sum_{\alpha \in A} a_\alpha = 1$, the identity operator. Fix $S \subseteq A$, and for each $s \in S$ fix T_s, another subset of A. Let $\mathcal{M} = \{m \in \mathcal{A} : \Phi(m)a_\alpha = 0 \text{ for all } \alpha \in S \text{ and } a_\alpha m a_\beta = 0 \text{ for all } (\alpha, \beta) \text{ such that } \beta \in S \text{ and } \alpha \in T_\beta\}$. Then we call $\mathcal{M}$ a *fundamental subspace* for $\mathcal{F}$.

Theorem 5.7. *Take $\mathcal{A} = \mathbf{B}(\mathfrak{H})$, and let $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{A}$ such that $\mathcal{B} = \mathcal{B}'$ and the identity operator is a sum of minimal projections from $\mathcal{B}$. The set of invariant subspaces for $\mathcal{F}$ which are closed in the σ-weak topology is the set of fundamental subspaces for $\mathcal{F}$. This is also the set of invariant subspaces for $\mathcal{F}$ which are closed in the weak operator topology.*

Proof. The fact that $\mathcal{B}'$ is atomic and commutative implies that Φ is normal [**1**, page 90]. From the same reference, note for use below that one necessarily has

$$\Phi(x) = \sum_{\alpha \in A} a_\alpha x a_\alpha \quad \text{for all} \quad x \in \mathbf{B}(\mathfrak{H}). \tag{5.1}$$

From the fact that the equation $\Phi(m)a_\alpha = 0$ can be rewritten as $a_\alpha m a_\alpha = 0$ in this case, it is clear that a fundamental subspace is closed in the weak operator topology (as well as the σ-weak topology). Let $\mathcal{N}$ denote a fundamental subspace, and write $G = \Phi[\mathcal{N}]$, $J = (I - \Phi)[\mathcal{N}]$. It is easily verified that G is an ideal in $\mathcal{B}$ and J is a $\mathcal{B}$-bimodule. Consider $k \in \mathcal{K}$ and $N \in \mathcal{N}$. $k\Phi(n) = (I - \Phi)[k\Phi(n)] \in (I - \Phi)[\mathcal{N}]$ because $\Phi(n)a_\beta = 0$ for all $\beta \in S$. Thus, $\mathcal{K}G \subseteq J$, so $\mathcal{N}$ is invariant under $\mathcal{F}$ by Corollary 5.2.

Let $\mathcal{M}$ be a subspace of $\mathbf{B}(\mathfrak{H})$ which is closed in the σ-weak topology and invariant for $\mathcal{F}$. Define S to be the subset of the index set A for the atoms such that $\Phi(\mathcal{M})a_\alpha = \{0\}$ for $\alpha \in S$. For each $\alpha \in S$, define T_α to be the subset of A such that $\beta \in T_\alpha$ implies that $a_\beta \mathcal{M} a_\alpha = \{0\}$. From Corollary 5.2 and the following remark, $\mathcal{M} = G + J$ where G is an ideal of $\mathcal{B}$ which is σ-weakly closed, J is a σ-weakly closed $\mathcal{B}$-sub-bimodule of $\mathcal{K}$ and $kG \subseteq J$ for all $k \in \mathcal{K}$.

There exists a projection $p \in \mathcal{B}$ such that $G = p\mathcal{B}$ [**8**, page 81]. For $\alpha \notin S$, there exists $m \in \mathcal{M}$ such that $\Phi(m)a_\alpha \neq 0$. $\Phi(m) \in G$, so $pa_\alpha \neq 0$. Then $a_\alpha p = pa_\alpha = a_\alpha$ because a_α is an atom in $\mathcal{B}$. Thus, $p = \sum_{\alpha \in A} a_\alpha p = \sum_{\alpha \in S^c} a_\alpha$. It follows that G is exactly $\{b \in \mathcal{B} : ba_\alpha = 0 \text{ for } \alpha \in S\}$.

Now we claim that $J = \{j \in \mathcal{K} : a_\alpha j a_\beta = 0 \text{ for all } (\alpha, \beta) \text{ such that } \beta \in S \text{ and } \alpha \in T_\beta\}$. Denote the latter set by L. Clearly, $J \subseteq L$. For each (α, β), $a_\alpha J a_\beta$ is a σ-weakly closed $\mathcal{B}$-bimodule of the set of all bounded linear operators from the range of a_β to the range of a_α. Denote the range of a_α by $\mathbf{H}_\alpha$ for each $\alpha \in A$. From (5.1), $a_\alpha \mathbf{B}(\mathfrak{H}) a_\alpha \subseteq \mathcal{B}$ (and likewise with α replaced by β). From this and the fact that J is a $\mathcal{B}$-bimodule it follows that if $a_\alpha J a_\beta$ contains a nonzero operator it contains every finite rank operator in $a_\alpha \mathbf{B}(\mathfrak{H}) a_\beta$. It then follows from σ-weak closure that $a_\alpha J a_\beta = a_\alpha \mathbf{B}(\mathfrak{H}) a_\beta$ except for those (α, β) such that $a_\alpha J a_\beta = 0$. In the latter case, $\alpha = \beta$ or $\beta \in S$ because otherwise $(a_\alpha \mathbf{B}(\mathfrak{H}) a_\beta)(a_\beta G a_\beta) \neq \{0\}$ would contradict the fact that $\mathcal{K}G \subseteq J$. If $\beta \in S$, then $\alpha \in T_\beta$ by definition of T_β. Thus, $J = L$.

It now follows from the fact that $\mathcal{M} = G + J$ that $\mathcal{M}$ is the fundamental subspace corresponding to the sets S and $\{T_\beta : \beta \in S\}$. $\square$

Note added in proof: The characterization of the spectrum of D_a as $\sigma_{\mathcal{B}/\mathcal{L}_1}([\Phi(a)]) \cup \{0\}$ has now been obtained in full generality, because the authors have shown that invertibility of $\mathcal{L}_{\Phi(a)}$ is always equivalent to invertibility in $\mathcal{L}$. (see the remark following Corollary 3.14).

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