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PLANCHEREL THEOREM FOR VECTOR VALUED FUNCTIONS AND BOEHMIANS

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ABSTRACT. The classical Plancherel theorem asserts that the Fourier-Plancherel transform is an isomorphism between $\mathcal{L}^2(\mathbf{R})$ onto $\mathcal{L}^2(\mathbf{R})$. On the other hand, in the literature the theory of Fourier transform is extended to the space of $\mathcal{L}^1$ Boehmians and also to the space of tempered Boehmians. In this paper we shall introduce two types of Boehmians, each of which contains vector valued square integrable functions on $\mathbf{R}$ as a dense subspace and extend the theory of Fourier transform to this set up. Finally we prove that this extended Fourier transform is a one-to-one continuous linear map of one space of Boehmians onto the other.

1. Introduction. The theory of Schwartz distributions, tempered distributions and their applications are well known in the literature. The concept of Boehmians which was motivated by Boehme's regular operators [**1**] was defined and systematically developed and their properties investigated in [**2, 4, 5, 7, 12, 13**]. Further several integral transforms were also introduced on various spaces of Boehmians and their properties studied in [**5, 6, 8, 9, 10, 14**].

In [**19**], Zemanian develops the theory of Laplace transform for a testing function space consisting of Banach space valued functions defined on $\mathbf{R}^k$. Motivated by the above theory, in this paper we shall develop a theory of Fourier transform on a certain space of Boehmians which contains vector valued functions defined on $\mathbf{R}^1$. Let us first consider a separable commutative Banach algebra A with identity e. It may still be possible to take just a separable Banach space instead of a Banach algebra but that may lead to more complications, and we shall return to this problem later. To develop our theory we need an analogue of the Plancherel theorem on the space $\mathcal{L}^2(A)$ where

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$\mathcal{L}^2(A)$ consists of A-valued Borel measurable functions on $\mathbf{R}$ such that $\int_{\mathbf{R}} \|f(x)\|^2\, dx < \infty$. Since we are not aware of any such theory in the literature, we have assumed that A is a complex Hilbert space and a separable commutative Banach algebra with identity e such that the norm induced by the inner product and the norm in the Banach algebra are equivalent and developed a Plancherel theorem for this set up. The example $A = \mathbf{C}^n$ tells us that it is possible to assume such restrictions on A. On the other hand, if A is a Hilbert space and also a complex algebra in which the left and right multiplications are continuous, we can introduce a Banach algebra structure such that the Banach algebra norm and the Hilbert space norm are equivalent, see [**15**]. Thus we strongly believe the existence of such spaces A other than $\mathbf{C}^n$. However, we are not able to produce any concrete example for the present. Thus in all discussions in Section 2 and in Section 3 we assume $A = \mathbf{C}^n$ for some positive integer n and proceed with our theory.

We develop a Plancherel theorem for $\mathcal{L}^2(A)$ and then use it to define Fourier-Plancherel transform for our space of Boehmians. Unlike the classical theories where in the Fourier transform of elements of Boehmians spaces are classical distributions, we shall define the Fourier-Plancherel transform as a continuous linear map from one space of Boehmians onto another.

In Section 2 we shall recall several testing function spaces consisting of Banach space valued functions, Banach space valued distributions and convolutions of functions which take their values in a Banach space and merely state their properties citing proper references. A few modifications which are necessary for our purposes are also worked out in detail. Since the theory of Fourier transform on Banach algebra valued functions has already been developed, see [**17, 18**], we present the required theory with minimum details in Section 2. In Section 3 we introduce two different vector valued Boehmian spaces. We shall also exhibit an imbedding of $\mathcal{L}^2(A)$, the space of all A-valued square integrable functions on $\mathbf{R}$, in these Boehmian spaces. In Section 4 we introduce Fourier-Plancherel transform on our space of Boehmians and obtain its properties. Finally, in Section 5 we shall make a comparative study of the theory developed here and those that are already available in the existing literature and known to us.

2. Preliminaries. In order to make this paper as much self-contained as possible, we shall briefly recall the basic definitions and notations of testing function spaces which are Banach space valued. For further detail we refer to [**19**].

Let A be a complex Banach space and K a compact subset of $\mathbf{R}$. Let $\mathcal{D}_K(A)$ denote the linear space of all functions ϕ from $\mathbf{R}$ to A such that $\operatorname{supp}\phi \subseteq K$ and, for every integer k, the kth derivative of ϕ, namely $\phi^{(k)}$ is continuous. We assign to $\mathcal{D}_K(A)$ the topology generated by the collection $\{\gamma_k(\phi)/0 \le k < \infty\}$ of semi-norms where

$$\gamma_k(\phi) = \sup_{t\in K} \|\phi^k(t)\|_A.$$

Let $\{K_j\}_{j=1}^{\infty}$ be a sequence of compact subsets of $\mathbf{R}$ such that $K_1 \subset K_2 \subset \cdots$, $\cup_j K_j = \mathbf{R}$ and every compact subset of $\mathbf{R}$ is contained in some K_j. We define $\mathcal{D}(A) = \cup_j \mathcal{D}_{K_j}(A)$ to be the inductive limit of $\mathcal{D}_{K_j}(A)$. When $A = \mathbf{C}$, $\mathcal{D}(\mathbf{C}) = \cup_j \mathcal{D}_{K_j} = \mathcal{D}$ is the classical space of test functions.

$\mathcal{E}(A)$ is defined as the largest ρ-type test function space containing $\mathcal{D}(A)$, see [**19**]. When $A = \mathbf{C}$, $\mathcal{E}(A) = \mathcal{E}$ is the usual space of smooth functions on $\mathbf{R}$.

If B is any other complex Banach space, we define $[\mathcal{D}(A) : B]$ as the space of all continuous linear mappings from $\mathcal{D}(A)$ to B also called $[A, B]$-valued distributions. Let τ_t denote the usual translation operator given by $(\tau_t\phi)(x) = \phi(x-t)$. Then, for $y \in [\mathcal{D}(A) : B]$, $v \in [\mathcal{E} : A]$, their convolution denoted by $y * v$ is defined as a B-valued mapping on $\mathcal{D}$ by $(y * v)(\phi) = y(\psi)$ where $\psi(t) = v(\tau_{-t}\phi)$ for all $\phi \in \mathcal{D}$. It can be shown that $\psi \in \mathcal{D}(A)$, see [**19**, pp. 99–100]. Thus $y * v$ is well defined and the mapping $v \to y * v$ is a continuous linear mapping of $[\mathcal{E} : A]$ into $[\mathcal{D} : B]$. It can also be shown that $\mathcal{D}(A)$ can be identified as a subspace of $[\mathcal{E} : A]$ and, in particular, if $y \in [\mathcal{D}(A) : B]$ and $v \in \mathcal{D}(A)$, $y * v$ is well defined and, further, it can be identified with a smooth B-valued function $u \in \mathcal{E}(B)$ in the sense that

$$(y * v)(\phi) = \int_{\mathbf{R}} u(x)\phi(x)\,dx, \quad \forall\, \phi \in \mathcal{D}.$$

In fact, u can be explicitly defined by

$$u(x) = y(\tau_x \tilde{v}) \quad \text{where} \quad \tilde{v}(t) = v(-t),$$

see [**19**, p. 106].

Now we shall recall the theory of Bochner integrable functions as found in [**3, 11**].

Let $\mathbf{R}, \mathbf{C}$ denote the usual real and complex spaces. We shall assume that A is a separable complex Banach space. $\mathbf{R}$ is considered as a measure space equipped with the σ-algebra of Borel subsets and the usual Lebesgue measure. We shall use the concept of Bochner measurability as available in [**3**].

Lemma 2.1. *Let (T, Σ, μ) be a finite measure space where T is a nonvoid set, Σ is a σ-algebra of subsets of T, μ is a positive measure on Σ. A Bochner measurable mapping f from T to a Banach space X is Bochner integrable if and only if $\int_T \|f\| \, d\mu < \infty$.*

Proof. See [**11**].

Lemma 2.2. *If $f : \mathbf{R} \to A$ is Borel measurable, then f is Bochner measurable. The same is true for $f : \mathbf{R} \times \mathbf{R} \to A$.*

Proof. See [**3**, pp. 73–77]. □

Definition 2.3. Let A be a separable, commutative complex Banach algebra. Let $\mathbf{R}$ be the measure space described above. For $1 \leq p < \infty$,

$$\mathcal{L}^p(A) = \begin{cases} [f] | f : \mathbf{R} \to A & \text{is Borel measurable and} \\ & \int_{\mathbf{R}} \|f(x)\|^p \, dm(x) < \infty \\ & \text{where } dm(x) = dx/\sqrt{2\pi} \end{cases}$$

where $[f]$ denotes the equivalence class containing f with respect to the equivalence relation $f \sim g$ if and only if $f = g$ almost everywhere on $\mathbf{R}$ with respect to the Lebesgue measure. We denote $\|f\|_p = \{\int_{\mathbf{R}} \|f(x)\|^p \, dm(x)\}^{1/p}$.

Lemma 2.4. *$\mathcal{L}^p(A)$ is a Banach space under this norm $\| \ \|_p$.*

Proof. See [**19**, p. 220]. □

Lemma 2.5. *Let A be a separable Banach algebra over $\mathbf{C}$. If f and g are A-valued Borel measurable functions on $\mathbf{R}$, then so is their product.*

Lemma 2.6. *Let A be a separable Banach algebra over $\mathbf{C}$. If f and g are A-valued Borel measurable functions on $\mathbf{R}$, then $F(x,y) = f(x-y)g(y)$ for all $x, y \in \mathbf{R}$ is Borel measurable on $\mathbf{R} \times \mathbf{R}$.*

The proofs of Lemmas 2.5 and 2.6 follow their analogues in the classical case wherein $A = \mathbf{C}$. The details are omitted.

Theorem 2.7. *If $f \in \mathcal{L}^p(A)$, $g \in \mathcal{D}(A)$, then $(f * g)(x) = \int_{\mathbf{R}} f(x-y)g(y)\,dm(y)$ exists as a Bochner integral.*

Proof. In fact, by Lemma 2.2 and Lemma 2.6, $f(x-y)g(y)$ as a function of y is Bochner measurable. If $K = \operatorname{supp} g$ and $\|g\|_0 = \sup_{x \in K} \|g(x)\|$, then $\int_{\mathbf{R}} f(x-y)g(y)\,dm(y) = \int_K f(x-y)g(y)\,dm(y)$. Now $\int_K \|f(x-y)g(y)\|\,dm(y) \leq \|g\|_0 \int_K \|f(x-y)\|\,dm(y) \leq c\|g\|_0 < \infty$ where $c = \|f\|_1$ if $p = 1$ and $c = \|f\|_p m(K)^{1/q}$ with $1/p + 1/q = 1$ if $p > 1$. Thus, by Lemma 2.1, $\int_K f(x-y)g(y)\,dm(y)$ exists for each $x \in \mathbf{R}$ as a Bochner integral. $\square$

Theorem 2.8. *Let $1 \leq p < \infty$. If $f \in \mathcal{L}^p(A)$, $g \in \mathcal{D}(A)$, then $f * g \in \mathcal{L}^p(A)$ and $\|f * g\|_p \leq \|f\|_p \|g\|_1$.*

Proof. Let $K = \operatorname{supp} g$ and $\|g\|_0 = \sup_{x \in K} \|g(x)\|$.

$$\begin{aligned} \|f * g\|_p^p &= \int_{\mathbf{R}} \|f * g(x)\|^p\,dm(x) \\ &\leq \int_{\mathbf{R}} \left(\int_K \|f(x-y)g(y)\|\,dm(y) \right)^p dm(x) \\ &\leq \int_{\mathbf{R}} \left(\int_K \|f(x-y)\|\|g(y)\|\,dm(y) \right)^p dm(x). \qquad (1) \end{aligned}$$

Let $\lambda = \int_K \|g(y)\|\,dm(y)$ and $d\mu(y) = (1/\lambda)\|g(y)\|\,dm(y)$. Then μ is a positive Borel measure with $\int_K d\mu(y) = 1$ and

$$\lambda \int_K \|f(x-y)\| \, d\mu(y) = \int_K \|f(x-y)\| \|g(y)\| \, dm(y)$$
$$\leq \|g\|_0 \int_K \|f(x-y)\| \, dm(y) < \infty.$$

Hence, by Jensen's inequality (1) becomes

$$\begin{aligned}\|f * g\|_p^p &\leq \lambda^p \int_{\mathbf{R}} \left(\int_K \|f(x-y)\|^p \, d\mu(y) \right) dm(x) \\ &= \lambda^{p-1} \int_{\mathbf{R}} \int_K \|f(x-y)\|^p \|g(y)\| \, dm(y) \, dm(x) \\ &= \lambda^{p-1} \int_K \|g(y)\| \left(\int_{\mathbf{R}} \|f(x-y)\|^p \, dm(x) \right) dm(y) \\ &= \lambda^{p-1} \lambda \|f\|_p^p \\ &= \lambda^p \|f\|_p^p \\ &= \|f\|_p^p \|g\|_1^p.\end{aligned}$$

Thus $\|f * g\|_p \leq \|f\|_p \|g\|_1$. $\square$

Remark 2.9. If $f \in \mathcal{L}^2(A)$, then f is locally integrable since $\int_K \|f(x)\| \, dx \leq (\int_K \|f(x)\|^2 \, dx)^{1/2} (m(k))^{1/2}$ for any compact subset K of $\mathbf{R}$ and m denotes the Lebesgue measure. Hence f can be considered as an $[A : A]$ valued distribution, i.e., a map from $\mathcal{D}(A)$ to A, given by $\Lambda_f(\phi) = \int_{\mathbf{R}} f(t)\phi(t) \, dt$, for all $\phi \in \mathcal{D}(A)$. The righthand side as a Bochner integral exists since $f\phi$ is Bochner measurable and, if $K = \operatorname{supp} \phi$,

$$\int_K \|f(t)\| \|\phi(t)\| \, dt \leq \|\phi\|_0 \int_K \|f(t)\| \, dt < \infty.$$

Λ_f is clearly linear and Λ_f is continuous as $\|\Lambda_f(\phi)\| \leq M \|\phi\|_0$ where $M = \int_K \|f(t)\| \, dt < \infty$. So $\Lambda_f \in [\mathcal{D}(A) : A]$. Let $g \in \mathcal{D}(A)$. As in [**19**] we can define the convolution $\Lambda_f * g$ which is regularized by $u \in \mathcal{E}(A)$ so that

$$u(x) = \Lambda_f(\tau_x \tilde{g}) = \int_{\mathbf{R}} f(t) g(x-t) \, dm(t) = (f * g)(x).$$

As $f \in \mathcal{L}^2(A)$, $g \in \mathcal{D}(A)$ by Theorem 2.8, $u = f * g \in \mathcal{L}^2(A)$. In effect we have $\Lambda_f * g = \Lambda_{f*g}$. Thus the convolution production $f * g$ with $f \in \mathcal{L}^2(A)$, $g \in \mathcal{D}(A)$ coincides with the convolution $\Lambda_f * g$ as defined in [**19**].

Lemma 2.10. *Let $f \in \mathcal{L}^1(A)$, $g \in \mathcal{D}(A)$. For compact subsets K_1, K_2 of $\mathbf{R}$, we have*

$$\int_{K_1} \int_{K_2} f(t)g(x-t)\,dt = \int_{K_2} \int_{K_1} f(t)g(x-t)\,dt.$$

Proof. Let A' denote the dual of A. Since A' is a normed space, it is sufficient to prove that for all $\Lambda \in A'$,

$$\Lambda\bigg(\int_{K_1} \int_{K_2} f(t)g(x-t)\,dx\,dt \bigg) = \Lambda\bigg(\int_{K_2} \int_{K_1} f(t)g(x-t)\,dt\,dx \bigg). \tag{2}$$

Now

$$\begin{aligned} \Lambda\bigg(\int_{K_1} \int_{K_2} f(t)g(x-t)\,dx\,dt \bigg) &= \int_{K_1} \Lambda\bigg(\int_{K_2} f(t)g(x-t)\,dx \bigg)\,dt \\ &= \int_{K_1} \int_{K_2} \Lambda(f(t)g(x-t))\,dx\,dt. \end{aligned}$$

A quick calculation shows that Fubini's theorem is applicable, and so we have (2). $\square$

Definition 2.11. If $f \in \mathcal{L}^1(A)$, we define

$$\hat{f}(t) = \lim_{n\to\infty} \int_{-n}^{n} f(x)e^{-itx}\,dm(x). \tag{3}$$

Remark 2.12. It is easily seen that this limit exists in the Banach space A.

Lemma 2.13. *If $f \in \mathcal{L}^1(A) \cap \mathcal{L}^2(A)$, then $(\Lambda \circ f)^\wedge = \Lambda \circ \hat{f}$ for any arbitrary linear functional Λ on A.*

Proof. Can be easily obtained using the definitions. □

Lemma 2.14. *If $f \in \mathcal{L}^1(A) \cap \mathcal{L}^2(A)$, $\phi \in \mathcal{D}(A)$, then $(f * \phi)^\wedge = \hat{f}\hat{\phi}$.*

Proof. Let $K = \operatorname{supp} \phi$.

$$\begin{aligned}(f*\phi)^\wedge(t) &= \lim_{n\to\infty} \int_{-n}^{n} (f*\phi)(x) e^{-itx}\, dm(x)\\ &= \lim_{n\to\infty} \int_{-n}^{n} \left(\int_K f(x-y)\phi(y)\, dm(y) \right) e^{-itx}\, dm(x)\\ &= \lim_{n\to\infty} \int_K \left\{ \left(\int_{-n}^{n} f(x-y) e^{-it(x-y)}\, dm(x) \right) \phi(y) e^{-ity} \right\} dm(y)\end{aligned}$$

by Lemma 2.10.

Since the integrands within parentheses are pointwise convergent to $\hat{f}(t)\phi(y)e^{-ity}$ as $n \to \infty$ and bounded by $\|f\|_1 \|\phi(y)\|_0$ which is in $\mathcal{L}^1(\mathbf{R})$, by dominated convergence theorem we get

$$\begin{aligned}(f*\phi)^\wedge(t) &= \int_K \hat{f}(t)\phi(y)e^{-ity}\, dm(y)\\ &= \hat{f}(t)\hat{\phi}(t) = (\hat{f}\hat{\phi})(t).\end{aligned}$$

In the following we take $A = \mathbf{C}^n$ for some positive integer n.

We shall denote the Banach algebra norm in A by $\| \; \|_A$, the innerproduct of z and w by $\langle z, w \rangle$ and the norm in A induced by the innerproduct by $\| \; \|_H$, i.e., for $z = (z_1, z_2, \dots, z_n)$, $w = (w_1, w_2, \dots, w_n) \in \mathbf{C}^n$,

$$\begin{aligned}\|z\|_H &= \left(\sum_{i=1}^{n} |z_i|^2 \right)^{1/2}\\ \langle z, w \rangle &= \sum_{i=1}^{n} z_i \bar{w}_i\\ \|z\|_A &= \max_{1\le i\le n} (|z_i|).\end{aligned}$$

If $f \in \mathcal{L}^2(A)$ we denote $(\int_{\mathbf{R}} \|f(x)\|_H^2\, dm(x))^{1/2}$ by $\|f\|_H$ and $(\int_{\mathbf{R}} \|f(x)\|_A^2\, dm(x))^{1/2}$ by $\|f\|_A$ or $\|f\|_2$. □

Definition 2.15. For $f, g \in \mathcal{L}^2(A)$ we define an innerproduct

$$\langle f, g\rangle = \int_{\mathbf{R}} \langle f(x), g(x)\rangle\, dm(x). \tag{4}$$

Theorem 2.16. *$\mathcal{L}^2(A)$ is a Hilbert space under the innerproduct* (4).

Proof. The innerproduct is well defined since

$$\begin{aligned}\int_{\mathbf{R}} |\langle f(x), g(x)\rangle|\, dm(x) &\le \int_{\mathbf{R}} \|f(x)\|_H \|g(x)\|_H\, dm(x)\\ &\le \left(\int_{\mathbf{R}} \|f(x)\|_H^2\, dm(x)\right)^{1/2}\\ &\quad\cdot \left(\int_{\mathbf{R}} \|g(x)\|_H^2\, dm(x)\right)^{1/2}\\ &\le c^2\left(\int_{\mathbf{R}} \|f(x)\|_A^2\, dm(x)\right)^{1/2}\\ &\quad\cdot \left(\int_{\mathbf{R}} \|g(x)\|_A^2\, dm(x)\right)^{1/2}\end{aligned}$$

(where $\|a\|_H \le c\|a\|_A$ for all $a \in A$)

$$\begin{aligned}&\le c^2\|f\|_2\|g\|_2\\ &< \infty.\end{aligned}$$

It is easy to verify that $\mathcal{L}^2(A)$ is an innerproduct space with respect to the innerproduct given by (4). Since $\mathcal{L}^2(A)$ with $\|\ \|_A$ is complete, see [**19**], and since $\|\ \|_A$ and $\|\ \|_H$ are equivalent in A, it is easy to see that $\mathcal{L}^2(A)$ is a Hilbert space with respect to the innerproduct given by (4). ◻

Lemma 2.17. *If $f \in \mathcal{L}^1(A)$, then*

(i) $\int_E \langle f(x), y\rangle\, dx = \langle \int_E f(x)\, dx, y\rangle$ *for all* $y \in A$

(ii) $\langle y, \int_E f(x)\, dx\rangle = \int_E \langle y, f(x)\rangle\, dx$ *for all* $y \in A$

for any Borel set $E(\subseteq \mathbf{R})$ of finite measure.

Proof. We can easily prove (i) for any characteristic function of a measurable subset of finite measure. Since any simple Bochner integrable function is a finite linear combination of characteristic functions, and, since innerproduct is linear in the first variable, we get (i) for any simple Bochner integral function. If f is any Bochner integrable function, then there exists a sequence (f_n) of simple Bochner integrable functions such that

$$f_n \longrightarrow f \quad \text{a.e. and} \quad \int_E f_n \longrightarrow \int_E f.$$

Using the continuity of the innerproduct, we can get (i). Similarly (ii) can be proved. $\square$

Lemma 2.18. *If $f \in \mathcal{L}^1(A) \cap \mathcal{L}^2(A)$, then $\|f\|_H = \|\hat{f}\|_H$.*

Proof. Let $g(x) = \langle f, f_{-x} \rangle$ for all $x \in \mathbf{R}$. Then $g : \mathbf{R} \to \mathbf{C}$ and

$$g(x) = \int_{\mathbf{R}} \langle f(y), f(x+y) \rangle \, dm(y) \quad \forall\, x \in \mathbf{R}.$$

We first note that, as in the classical case, $x \to f_{-x}$ is uniformly continuous from $\mathbf{R}$ to $\mathcal{L}^2(A)$ and using the continuity of the innerproduct we see that g is continuous on $\mathbf{R}$. Now

$$\begin{aligned}
|g(x)| &\leq \int_{\mathbf{R}} |\langle f(y), f(x+y) \rangle| \, dm(y) \\
&\leq \int_{\mathbf{R}} \|f(y)\|_H \|f(x+y)\|_H \, dm(y) \\
&\leq c^2 \int_{\mathbf{R}} \|f(y)\|_A \|f(x+y)\|_A \, dm(y) \\
&\leq c^2 \|f\|_2^2,
\end{aligned}$$

using Holder's inequality and the fact that $\|f\|_2 = \|f_{-x}\|_2$. Thus g is bounded. As g is continuous, it is Borel measurable. Moreover,

$$\begin{aligned}\int_{\mathbf{R}} |g(x)|\, dx &= \int_{\mathbf{R}} \left| \int_{\mathbf{R}} \langle f(y), f(x+y)\rangle \, dm(y)\right| dm(x) \\ &\leq \int_{\mathbf{R}} \int_{\mathbf{R}} |\langle f(y), f(x+y)\rangle| \, dm(y)\, dm(x) \\ &\leq \int_{\mathbf{R}} \int_{\mathbf{R}} \|f(y)\|_H \|f(x+y)\|_H \, dm(y)\, dm(x) \\ &\leq c^2 \|f\|_1^2,\end{aligned}$$

by Fubini's theorem.

So we get $g \in \mathcal{L}^1(\mathbf{R})$. Using the classical techniques [**16**] we can get

$$\lim_{\lambda\to 0} (g * h_\lambda)(0) = g(0) \tag{5}$$

and

$$\lim_{\lambda\to 0} (g * h_\lambda)(0) = \int_{\mathbf{R}} \hat{g}(t)\, dm(t) \tag{6}$$

where $h_\lambda(x) = \int_{-\infty}^{\infty} e^{-\lambda|t|} e^{itx}\, dm(t)$, $\lambda > 0$.

Using the definition of g in (5), we get

$$\lim_{\lambda\to 0} (g * h_\lambda)(0) = \langle f, f\rangle = \|f\|_H^2. \tag{7}$$

Now, using Fubini's theorem, Lemma 2.17 and dominated convergence theorem wherever necessary, we get

$$\begin{aligned}
\hat{g}(t) &= \int_{\mathbf{R}} g(x)e^{-itx}\,dm(x)\\
&= \int_{\mathbf{R}} \left(\int_{\mathbf{R}} \langle f(y), f(x+y)\rangle\,dm(y)\right) e^{-itx}\,dm(x)\\
&= \int_{\mathbf{R}} \left(\int_{\mathbf{R}} \langle f(y)e^{ity}, f(x+y)e^{it(x+y)}\rangle\,dm(x)\right) dm(y)\\
&= \lim_{n\to\infty} \int_{-n}^{n} \lim_{k\to\infty} \left(\int_{-k}^{k} \langle f(y)e^{ity}, f(x+y)e^{it(x+y)}\rangle\,dm(x)\right) dm(y)\\
&= \lim_{n\to\infty} \int_{-n}^{n} \lim_{k\to\infty} \left(\langle f(y)e^{ity}, \int_{-k}^{k} f(x+y)e^{it(x+y)}\,dm(x)\rangle\right) dm(y)\\
&= \lim_{n\to\infty} \int_{-n}^{n} \left(\langle f(y)e^{ity}, \lim_{k\to\infty} \int_{-k}^{k} f(x+y)e^{it(x+y)}\,dm(x)\rangle\right) dm(y)\\
&= \lim_{n\to\infty} \int_{-n}^{n} \langle f(y)e^{ity}, \hat{f}(-t)\rangle\,dm(y)\\
&= \|\hat{f}(-t)\|_H^2.
\end{aligned}$$

Now (6) gives

$$\lim_{\lambda\to 0} g * h_\lambda(0) = \|\hat{f}\|_H^2, \tag{8}$$

(7) and (8) together imply $\|f\|_H^2 = \|\hat{f}\|_H^2$. □

Lemma 2.19. *If $f \in \mathcal{L}^1(A)\cap\mathcal{L}^2(A)$, then $\hat{\hat{f}} = \tilde{f}$ where $\tilde{f}(x) = f(-x)$ for all $x \in \mathbf{R}$.*

Proof. If $f \in \mathcal{L}^1(A)\cap\mathcal{L}^2(A)$, then for any continuous linear functional Λ on A, $\Lambda\cdot f \in \mathcal{L}^1(\mathbf{R})\cap\mathcal{L}^2(\mathbf{R})$ and by classical Plancherel theorem on $\mathcal{L}^2(\mathbf{R})$ we get $(\Lambda\circ f)^{\sim} = (\Lambda\circ f)^{\wedge\wedge}$. By repeated application of Lemma 2.13, we get $\Lambda\circ\tilde{f} = \Lambda\circ\hat{\hat{f}}$. Since Λ is an arbitrary continuous linear functional on A, we get $\tilde{f} = \hat{\hat{f}}$. □

Theorem 2.20. *To each $f \in \mathcal{L}^2(A)$ we assign $\hat{f} \in \mathcal{L}^2(A)$ such that*

(i) *If* $f \in \mathcal{L}^1(A) \cap \mathcal{L}^2(A)$ *then* $\|f\|_H = \|\hat{f}\|_H$.

(ii) $f \to \hat{f}$ *is a Hilbert space isomorphism of* $\mathcal{L}^2(A)$ *onto* $\mathcal{L}^2(A)$.

Proof. (i) follows from Lemma 2.18.

Let $f \in \mathcal{L}^2(A)$ and $f_n = \chi_{[-n,n]}f$ for all n where $\chi_{[-n,n]}$ denotes the characteristic function on $[-n, n]$. it is clear that $(f_n) \in \mathcal{L}^1(A) \cap \mathcal{L}^2(A)$ and $\|f_n - f\|_A \to 0$ as $n \to \infty$. Since the norms are equivalent we get $\|f_n - f\|_H \to 0$ as $n \to \infty$. By (i) $\|\hat{f}_n\|_H = \|f_n\|_H$. Since (f_n) is Cauchy with respect to $\| \, \|_H$ we get that $(\hat{f}_n)$ is Cauchy with respect to $\| \, \|_H$ and therefore with respect to $\| \, \|_A$. Since $\mathcal{L}^2(A)$ is complete $(\hat{f}_n)$ converges in $\mathcal{L}^2(A)$, say, to $\hat{f}$ with respect to $\| \, \|_A$ and therefore with respect to $\| \, \|_H$. Moreover,

$$\|\hat{f}\|_H = \lim_{n\to\infty} \|\hat{f}_n\|_H = \lim_{n\to\infty} \|f_n\|_H = \|f\|_H. \tag{9}$$

Now using Lemma 2.19 and the continuity of Fourier transform, we can obtain $\hat{\hat{f}} = \tilde{f}$ for any $f \in \mathcal{L}^2(A)$ and this implies that the mapping $f \to \hat{f}$ from $\mathcal{L}^2(A)$ to $\mathcal{L}^2(A)$ is onto. $\square$

3. The Boehmian spaces $\mathcal{B}(\mathcal{L}^2(A), \Delta)$ and $\mathcal{B}(\mathcal{L}^2(A), \hat{\Delta})$. We take $A = \mathbf{C}^n$ for some n. Let $G = \mathcal{L}^2(A)$ and $S = \mathcal{D}(A)$. For $f \in G$, $g \in S$, we define $f * g$ as in Theorem 2.7. We now obtain a number of preliminary results for the construction of our Boehmian spaces.

Lemma 3.1. (i) *If* $g_1, g_2 \in S$, *then* $g_1 * g_2 \in S$.

(ii) *If* $f, g \in G$ *and* $h \in S$, *then* $(f + g) * h = f * h + g * h$.

(iii) $f * g = g * f$ *for all* $f, g \in S$.

(iv) *if* $f \in G, g, h \in S$, *then* $(f * g) * h = f * (g * h)$.

Proof. Proofs of (i)–(iv) are simple analogues of the classical cases and so we prefer to omit them. $\square$

Definition 3.2. A sequence of A-valued functions $(\delta_n) \in S$ is said to be in Δ if (i) $\int_{\mathbf{R}} \delta_n(x)\, dm(x) = e$.

(ii) $\int_{\mathbf{R}} \|\delta_n(x)\|\, dm(x) \le M$ for all n for some $M > 0$ and

(iii) $\operatorname{supp} \delta_n \to 0$ as $n \to \infty$.

Theorem 3.3. *Let $f, g \in G$ and $(\delta_i) \in \Delta$ be such that $f * \delta_i = g * \delta_i$ for all $i = 1, 2$. Then $f = g$ in $\mathcal{L}^2(A)$.*

Proof. We first claim that $f * \delta_i \to f$ in $\mathcal{L}^2(A)$. Let $\operatorname{supp} \delta_i \subseteq K$ for all i. Consider

$$\begin{aligned} \|f * \delta_i - f\|_2^2 &= \int_{\mathbf{R}} \left\| \int_K (f(x-y) - f(x))\delta_i(y)\, dm(y) \right\|^2 dm(x) \\ &\le \int_{\mathbf{R}} \left(\int_K \|f(x-y) - f(x)\| \|\delta_i(y)\|\, dm(y) \right)^2 dm(x). \end{aligned} \tag{10}$$

Let $\lambda = \int_K \|\delta_i(y)\|\, dm(y) \le M$ and $d\mu(y) = (1/\lambda)\|\delta_i(y)\|\, dm(y)$. By Jenson's inequality (10) becomes

$$\begin{aligned} \|f * \delta_i - f\|_2^2 &\le \lambda \int_{\mathbf{R}} \left(\int_K \|f(x-y) - f(x)\|^2 \|\delta_i(y)\|\, dm(y) \right) dm(x) \\ &\le M \int_K \|\delta_i(y)\| \left(\int_{\mathbf{R}} \|f(x-y) - f(x)\|^2 dm(x) \right) dm(y). \end{aligned} \tag{11}$$

If $f \in \mathcal{L}^2(A)$, the mapping $y \to f_y$ where $f_y(x) = f(x-y)$ is uniformly continuous from $\mathbf{R} \to \mathcal{L}^2(A)$. Since $\operatorname{supp} \delta_i \to 0$ as $i \to \infty$, we choose $\eta > 0$ such that $\operatorname{supp} \delta_i \subseteq [-\eta, \eta]$ for large i and $\|y\| < \eta \Rightarrow \|f_y - f\| < \varepsilon/M$. So (11) becomes

$$\begin{aligned} \|f * \delta_i - f\|_2^2 &\le M \int_{|y|<\eta} \|\delta_i(y)\| \|f_y - f\|_2^2\, dm(y) \\ &\le \varepsilon^2/M \int_{|y|<\eta} \|\delta_i(y)\|\, dm(y) \\ &< \varepsilon^2 \quad \text{for large } i. \end{aligned}$$

In a similar manner, $g * \delta_i \to g$ in $\mathcal{L}^2(A)$ as $i \to \infty$. The proof of the theorem now follows by taking $\mathcal{L}^2$ limits in the equality $f * \delta_i = g * \delta_i$. □

Theorem 3.4. *Let* $\delta = (\delta_1, \delta_2, \delta_3, \dots)$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots)$ *be in* Δ. *Then* $\delta * \varepsilon = (\delta_1 * \varepsilon_1, \delta_2 * \varepsilon_2, \delta_3 * \varepsilon_3, \dots) \in \Delta$.

Proof. we have (i) $\int_{\mathbf{R}} \delta_i(x)\, dm(x) = \int_{\mathbf{R}} \varepsilon_i(x)\, dm(x) = e$ for all i.

(ii) $\int_{\mathbf{R}} \|\delta_i(x)\|\, dm(x) \le M_1$, $\int_{\mathbf{R}} \|\varepsilon_i(x)\|\, dm(x) \le M_2$, for all i for some $M_1, M_2 > 0$.

(iii) $\operatorname{supp} \delta_i \to 0$, $\operatorname{supp} \varepsilon_i \to 0$ as $i \to \infty$.

We first prove that $\int_{\mathbf{R}} \delta_i * \varepsilon_i(x)\, dm(x) = e$ for all i.

Let $\operatorname{supp} \delta_i \subset K_1$ for all i, $\operatorname{supp} \varepsilon_i \subset K_2$ for all i, $G_i = \operatorname{supp} \delta_{i*} \varepsilon_i$ so that $G_i \subset K_1 + K_2$.

$$\begin{aligned}\int_{\mathbf{R}} (\delta_i * \varepsilon_i)(x)\, dm(x) &= \int_{G_i} (\delta_i * \varepsilon_i)(x)\, dm(x) \\ &= \int_{G_i} \left(\int_{K_1} \delta_i(t) \varepsilon_i(x-t)\, dm(t) \right) dm(x) \\ &= \int_{K_1} \left(\int_{G_i} \delta_i(t) \varepsilon_i(x-t)\, dm(x) \right) dm(t) \\ &= e \quad \text{(by Fubini's theorem).}\end{aligned}$$

Further,

$$\begin{aligned}\int_{\mathbf{R}} \|\delta_{i*} \varepsilon_i(x)\|\, dm(x) &\le \int_{G_i} \|\delta_{i*} \varepsilon_i(x)\|\, dm(x) \\ &\le \int_{G_i} \int_{K_1} \|\delta_i(t)\|\, \|\varepsilon_i(x-t)\|\, dm(t)\, dm(x) \\ &\le \int_{K_1} \|\delta_i(t)\| \left(\int_{G_i} \|\varepsilon_i(x-t)\|\, dm(x) \right) dm(t) \\ &\le \int_{K_1} \|\delta_i(t)\| \left(\int_{K_2} \|\varepsilon_i(s)\|\, dm(s) \right) dm(t) \\ &\le M_1 M_2 \quad \forall\, i.\end{aligned}$$

Since $\operatorname{supp}(\delta_{i*} \varepsilon_i) \subseteq \operatorname{supp} \delta_i + \operatorname{supp} \varepsilon_i$, we get $\operatorname{supp}(\delta_{i*} \varepsilon_i) \to 0$ as $i \to \infty$, completing the proof of our theorem. $\square$

In view of Theorems 3.3 and 3.4, the family Δ can be called as a family of delta sequences in the sense of [**4**].

We now verify that the convergence in $\mathcal{L}^2(A)$ satisfies the following conditions.

Theorem 3.5. (i) *If* $\lim_{n\to\infty} f_n = f$ *in* $\mathcal{L}^2(A)$, *then for* $\delta \in S$, $\lim_{n\to\infty} f_n * \delta = f * \delta$.

(ii) *If* $\lim_{n\to\infty} f_n = f$ *in* $\mathcal{L}^2(A)$,*then for* $(\delta_n) \in \Delta$, $\lim_{n\to\infty} f_n * \delta_n = f$.

Proof. $\|f_n * \delta - f * \delta\|_2 = \|(f_n - f) * \delta\|_2 \le \|f_n - f\|_2 \|\delta\|_1$, by Theorem 2.8, which tends to zero as $n \to \infty$. Consider $\|f_n * \delta_n - f\|_2 \le \|f_n - f\|_2 \|\delta_n\|_1 + \|f * \delta_n - f\|_2$, the first term on the right side tends to zero as $n \to \infty$ by the property (ii) of delta sequences, and the second term also tends to zero as observed in the proof of Theorem 3.3.

In view of Theorems 3.3, 3.4 and 3.5, we can construct the Boehmian space in the canonical way using $\mathcal{L}^2(A)$ and Δ, see [**4**]. This space we denote by $\mathcal{B}(\mathcal{L}^2(A), \Delta)$. Convergence in this space is taken as δ-convergence. □

Theorem 3.6. *The mapping* $f \to [f * \delta_i/\delta_i]$ *where* $(\delta_i) \in \mathcal{D}(A)$ *is a continuous imbedding of* $\mathcal{L}^2(A)$ *into* $B(\mathcal{L}^2(A), \Delta)$.

Proof. The mapping is one-to-one since $[f * \delta_i/\delta_i] = [g * \delta_i/\delta_i]$ implies $(f * \delta_i) * \delta_j = (g * \delta_i) * \delta_j$ for all i, j and, in particular, writing $\delta_i * \delta_i = \delta_i^2$ we have $f * \delta_i^2 = g * \delta_i^2$. By applying Theorem 3.1, Theorem 3.4 and Theorem 3.3, successively, we get $f = g$. We now show that this map is continuous.

Let $f_n \to 0$ in $\mathcal{L}^2(A)$. We claim that $x_n = [f_n * \delta_i/\delta_i] \xrightarrow{\delta} 0$ in $\mathcal{B}(\mathcal{L}^2(A), \Delta)$. We need only observe that $x_n * \delta_i = f_n * \delta_i \to 0$ in $\mathcal{L}^2(A)$ for each i by Theorem 3.5. □

Lemma 3.7. *If* $f \in \mathcal{L}^2(A)$, $g \in \mathcal{D}(A)$, *then* $(f * g)^\wedge = \hat{f}\hat{g}$.

Proof. Let $f_n \to f$ in $\mathcal{L}^2(A)$ where $f_n \in \mathcal{L}^1(A) \cap \mathcal{L}^2(A)$. Such a sequence exists as we saw in the proof of Theorem 2.20. By Theorem 3.5, $f_n * g \to f * g$ in $\mathcal{L}^2(A)$. So by Plancherel theorem (Theorem 2.20)

$(f_n * g) \to (f * g)^\wedge$ in $\mathcal{L}^2(A)$. Thus, by Lemma 2.14,

$$\hat{f}_n \hat{g} \longrightarrow (f * g)^\wedge \quad \text{in} \quad \mathcal{L}^2(A). \tag{12}$$

On the other hand, $f_n \to f$ implies $\hat{f}_n \to \hat{f}$ in $\mathcal{L}^2(A)$. Now, as in the classical case, $\hat{g}$ as a function of t is bounded. Hence,

$$\hat{f}_n \hat{g} \longrightarrow \hat{f} \hat{g} \quad \text{in} \quad \mathcal{L}^2(A). \tag{13}$$

Now (12) and (13) imply $(f * g)^\wedge = \hat{f}\hat{g}$.

We shall now describe yet another Boehmian space which contains $\mathcal{L}^2(A)$. Let $G = \mathcal{L}^2(A)$ and $S_1 = \hat{S} = \{\hat{\delta}/\delta \in S\}$ where $S = \mathcal{D}(A)$. For $f \in G$, $\hat{\delta} \in S_1$, we define $(f\hat{\delta})(x) = f(x)\hat{\delta}(x)$ for all $x \in \mathbf{R}$. □

Lemma 3.8. *If $f \in G$ and $\hat{\delta} \in S_1$, then $f\hat{\delta} \in G$.*

Proof. We observe that $f\hat{\delta}$ is Borel measurable and

$$\begin{aligned}
\int_{\mathbf{R}} \|f\hat{\delta}\|_A^2 \, dm(t) &= \int_{\mathbf{R}} \|f(t)\|_A^2 \|\hat{\delta}(t)\|_A^2 \, dm(t) \\
&\leq \int_{\mathbf{R}} \|f(t)\|_A^2 \|\delta\|_1^2 \, dm(t), \\
&\quad \text{since} \quad \forall\, t \ \|\hat{\delta}(t)\|_A \leq \|\delta\|_1 \\
&= \|f\|_2^2 \|\delta\|_1^2 \\
&< \infty.
\end{aligned}$$

Hence $f\hat{\delta} \in G$. □

Lemma 3.9. *The mapping $(f, \hat{\delta}) \to f\hat{\delta}$ from $G \times S \to G$ satisfies the following properties.*

(i) *If $\hat{\delta}_1, \hat{\delta}_2 \in S_1$, then $\hat{\delta}_1\hat{\delta}_2 \in S_1$.*

(ii) *If $f, g \in G$ and $\hat{\delta} \in S_1$, then $(f+g)\hat{\delta} = f\hat{\delta} + g\hat{\delta}$.*

(iii) $\hat{\delta}_1\hat{\delta}_2 = \hat{\delta}_2\hat{\delta}_1$ for $\hat{\delta}_1, \hat{\delta}_2 \in S$.

(iv) *If $f \in G$ and $\hat{\delta}, \hat{\varepsilon} \in S_1$, then $(f\hat{\delta})\hat{\varepsilon} = f(\hat{\delta}\hat{\varepsilon})$.*

Proof. Since A is a commutative Banach algebra, the required results are immediate. $\square$

Definition 3.10. The set of all sequences $(\hat{\delta}_i)$ such that $(\delta_i) \in \Delta$ is denoted by $\hat{\Delta}$.

Lemma 3.11. *Let $f, g \in G$ and $(\hat{\delta}_i) \in \hat{\Delta}$ such that $f\hat{\delta}_i = g\hat{\delta}_i$ for all i. Then $f = g$ in $\mathcal{L}^2(A)$.*

Proof. Since $f \in \mathcal{L}^2(A)$, $\tilde{\hat{f}}$ also belongs to $\mathcal{L}^2(A)$. As in the proof of Theorem 3.3., $\tilde{\hat{f}} * \delta_i \to \tilde{\hat{f}}$ in $\mathcal{L}^2(A)$ as $i \to \infty$. Since the Plancherel transform is continuous on $\mathcal{L}^2(A)$, we get $\hat{\tilde{\hat{f}}}\hat{\delta}_i \to \hat{\tilde{\hat{f}}}$. Equivalently, we get $f\hat{\delta}_i \to f$. In a similar way, we get $g\hat{\delta}_i \to g$. The lemma now follows. $\square$

We note that, if $(\hat{\delta}_i), (\hat{e}_i)$ are two delta sequences in $\hat{\Delta}$ then, by definition, $(\delta_i), (e_i)$ are delta sequences in Δ. So, by Theorem 3.4, $(\delta_i * e_i) \in \Delta$. Thus, by Lemma 3.7, $(\hat{\delta}_i \hat{e}_i) \in \hat{\Delta}$. In view of the above lemmas, the elements of $\hat{\Delta}$ can be called as delta sequences (in the sense of [**4**]).

Now we shall verify that the convergence in G satisfies the following conditions.

Lemma 3.12. (i) *If $f_n \to f$ in $\mathcal{L}^2(A)$ and $\hat{\delta} \in S_1$, then $f_n\hat{\delta} \to f\hat{\delta}$ in $\mathcal{L}^2(A)$.*

(ii) *If $f_n \to f$ in $\mathcal{L}^2(A)$ and $(\hat{\delta}_n) \in \hat{\Delta}$, then $f_n\hat{\delta}_n \to f$ in $\mathcal{L}^2(A)$.*

Proof. (i) Since $\hat{\delta}(t)$ as a function of t is bounded, we get the result.

(ii) follows by Plancherel Theorem 2.20 and Theorem 3.5(ii). $\square$

Lemma 3.13. *The mapping*

$$\iota : f \longrightarrow \left[\frac{f\hat{\delta}_i}{\hat{\delta}_i}\right], \quad (\hat{\delta}_i) \in \hat{\Delta} \tag{14}$$

is a continuous imbedding of $\mathcal{L}^2(A)$ *into* $\mathcal{B}(\mathcal{L}^2(A), \hat{\Delta})$.

Proof. If $f \in \mathcal{L}^2(A)$, we note that $[f\hat{\delta}_n/\hat{\delta}_n]$ is a quotient sequence in the sense of [**4**] as $f\hat{\delta}_n\hat{\delta}_m = f\hat{\delta}_m\hat{\delta}_n$. So $[f\hat{\delta}_n/\hat{\delta}_n]$ belongs to $\mathcal{B}(\mathcal{L}^2(A), \hat{\Delta})$. This map ι is one-to-one since

$$\left[\frac{f\hat{\delta}_n}{\hat{\delta}_n}\right] = \left[\frac{g\hat{\delta}_n}{\hat{\delta}_n}\right]$$

implies $f\hat{\delta}_n\hat{\delta}_m = g\hat{\delta}_m\hat{\delta}_n$, for all n, m. In particular, $f\delta_n^2 = g\delta_n^2$ for all n. As usual, letting $n \to \infty$, we get $f = g$ in $\mathcal{L}^2(A)$.

We now claim that the map ι is continuous. Let $f_n \to 0$ in $\mathcal{L}^2(A)$ as $n \to \infty$. We claim that

$$x_n = \left[\frac{f_n\hat{\delta}_i}{\hat{\delta}_i}\right] \xrightarrow{\delta} 0.$$

By Theorem 3.12, $f_n\hat{\delta}_i \to 0$ in $L^2(A)$ as $n \to \infty$. Thus $x_n \xrightarrow{\delta} 0$.

In view of the above lemmas, we see that $\mathcal{B}(\mathcal{L}^2(A), \hat{\Delta})$ can be regarded as a Boehmian space. We shall equip this with its usual δ-convergence. □

4. Fourier transform.

Definition 4.1. Let $x = [f_n/\phi_n] \in \mathcal{B}(\mathcal{L}^2(A), \Delta)$. We define the *Fourier transform* of x as $[\hat{f}_n/\hat{\phi}_n] \in \mathcal{B}(\mathcal{L}^2(A), \hat{\Delta})$ and denote it by $\hat{x}$.

Remark 4.2. The Fourier transform is well defined. If x has two representations, $x = [f_n/\phi_n] = [g_n/\xi_n]$, where $f_n, g_n \in \mathcal{L}^2(A)$ and $(\phi_n), (\xi_n) \in \Delta$, then $f_n * \xi_m = g_m * \phi_n$. Taking the Plancherel transform on both sides and using Lemma 3.7, we get $\hat{f}_n\hat{\xi}_m = \hat{g}_m\hat{\phi}_n$. So $[\hat{f}_n/\hat{\phi}_n] = [\hat{g}_n/\hat{\xi}_n] \in \mathcal{B}(\mathcal{L}^2(A), \hat{\Delta})$.

Theorem 4.3. *Let* $F : \mathcal{B}(\mathcal{L}^2(A), \Delta) \to \mathcal{B}(\mathcal{L}^2(A), \hat{\Delta})$ *be defined by* $F(x) = \hat{x}$. *Then* F *is a continuous one-to-one map from* $\mathcal{B}(\mathcal{L}^2(A), \Delta)$ *onto* $\mathcal{B}(\mathcal{L}^2(A), \hat{\Delta})$.

Proof. Let $(x_n) \xrightarrow{\delta} 0$ in $\mathcal{B}(\mathcal{L}^2(A),\Delta)$, say $x_n = [f_{n,i}/\phi_i]$. (We can take a common δ-sequence for the denominators of all x_n's, see, for example, [**4**]).

We claim that $\hat{x}_n = [\hat{f}_{n,i}/\hat{\phi}_i] \xrightarrow{\delta} 0$ in $\mathcal{B}(\mathcal{L}^2(A),\hat{\Delta})$. By hypothesis for each fixed i as $n \to \infty$, $(f_{n,i}) \to 0$ in $\mathcal{L}^2(A)$ with respect to $\| \|_2$. So for each fixed i as $n \to \infty$, $(f_{n,i}) \to 0$ in $\mathcal{L}^2(A)$ with respect to $\| \|_H$. By Theorem 2.20, for each fixed i as $n \to \infty$, $(\hat{f}_{n,i}) \to 0$ in $\mathcal{L}^2(A)$ with respect to $\| \|_H$ and so, for each fixed i as $n \to \infty$, $(\hat{f}_{n,i}) \to 0$ in $\mathcal{L}^2(A)$ with respect to $\| \|_2$. Thus $\hat{x}_n \xrightarrow{\delta} 0$ in $\mathcal{B}(\mathcal{L}^2(A),\hat{\Delta})$.

We now prove that the map F is one-to-one. Let $\hat{x}_1 = \hat{x}_2$. Then $[\hat{f}_n/\hat{\phi}_n] = [\hat{g}_n/\hat{\xi}_n]$. So $\hat{f}_n\hat{\xi}_m = \hat{g}_m\hat{\phi}_n$. By Lemma 3.7, we get $(f_n * \xi_m)^\wedge = (g_m * \phi_n)^\wedge$. Since Plancherel transform is one-to-one, it follows that $f_n * \xi_m = g_m * \phi_n$. So we get $x_1 = x_2$. We claim that the map F is onto. Since Plancherel transform is onto by Theorem 2.20 given $y = [g_n/\xi_n]$ in $\mathcal{B}(\mathcal{L}^2(A),\hat{\Delta})$, we take $x = [f_n/\phi_n]$ where $\hat{f}_n = g_n$ and $\hat{\phi}_n = \xi_n$. It can be easily verified that $x \in \mathcal{B}(\mathcal{L}^2(A),\Delta)$ and $\hat{x} = y$. □

Lemma 4.4. *If* $x_1, x_2 \in \mathcal{B}(\mathcal{L}^2(A),\Delta)$, *then*

(i) $(\hat{x}_1 + \hat{x}_2) = \hat{x}_1 + \hat{x}_2$.

(ii) $(\lambda x)^\wedge = \lambda\hat{x}$, $\lambda \in \mathbf{C}$, *where addition and multiplication are defined as usual for Boehmians.*

Proof. Results are immediate from the definitions. □

From the above theorems we see that the Plancherel transform is a one-to-one continuous linear mapping from $\mathcal{B}(\mathcal{L}^2(A),\Delta)$ onto $\mathcal{B}(\mathcal{L}^2(A),\hat{\Delta})$.

Remark 4.5. (i) $f \in \mathcal{L}^2(A)$ can be identified with the element $x = [f * \delta_i/\delta_i] \in \mathcal{B}(\mathcal{L}^2(A),\Delta)$ where (δ_i) is any delta sequence in Δ. Its Plancherel transform as a Boehmian is $[(f * \delta_i)^\wedge/\hat{\delta}_i] = [\hat{f}\hat{\delta}_i/\hat{\delta}_i]$. This latter Boehmian is nothing but the identification of $\hat{f}$ in $\mathcal{B}(\mathcal{L}^2(A),\hat{\Delta})$. So the Plancherel transform on $\mathcal{B}(\mathcal{L}^2(A),\Delta)$ is an extension of the Plancherel transform on $\mathcal{L}^2(A)$.

(ii) If $x = [f_n/\phi_n] \in \mathcal{B}(\mathcal{L}^2(A), \Delta)$ and $y = [g_n/\xi_n]$, $g_n \in \mathcal{D}(A)$, $\xi_n \in \Delta$, we can define $x*y = [f_n*g_n/\phi_n*\xi_n]$. In this case $(x*y)^\wedge = \hat{x}\hat{y}$ holds by Lemma 3.7.

5. As already observed, the Plancherel transform theory on $\mathcal{L}^2(A)$ developed here is an extension of the classical Plancherel theorem. In the literature there are three types of Boehmian spaces, on which the theory of Fourier transform is developed, viz., $\mathcal{L}^1$ Boehmians [**6**], tempered Boehmians [**5**] and more general tempered Boehmians [**8**]. In all these cases the Fourier transform was defined as a classical distribution. Since, in an arbitrary separable Banach algebra, division does not make sense, we have chosen the natural approach to define our Plancherel transform. However, the space $\mathcal{L}^2(A)$, the space of compactly supported $[A : A]$ valued distributions belonging to $[\mathcal{D}(A) : A]$, are all subspaces of $\mathcal{B}(\mathcal{L}^2(A), \Delta)$. Thus the space $\mathcal{B}(\mathcal{L}^2(A), \Delta)$ is larger than $\mathcal{L}^2(A)$. Moreover, in the classical case where A is replaced by $\mathbf{C}$, one can identify each element of $\mathcal{B}(\mathcal{L}^2(A), \Delta)$ as an element of $\mathcal{B}_{\mathcal{T}}$ defined in [**5**] and also verify that the original definition of $\hat{x}$ for $x \in \mathcal{B}_{\mathcal{T}}$ coincides with our definition.

However these two definitions are qualitatively different (since there is no natural way in which $\mathcal{D}'$ can be identified as a subspace of $\mathcal{B}(\mathcal{L}^2(A), \Delta)$). Further, in the case of vector valued functions, the classical technique in which the Fourier transform can be identified as a distribution, no longer works.

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