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HYPERNORMAL FORMS FOR EQUILIBRIA OF VECTOR FIELDS. CODIMENSION ONE LINEAR DEGENERACIES

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ABSTRACT. The Poincare-Dulac-Birkhoff normal form theorem determines how much a vector field can be simplified, depending uniquely on its linear part. Nevertheless, taking into account the nonlinear terms, it is possible to obtain further simplifications in the classical normal form. In this paper we define the hypernormal forms, which are the simplest that we can achieve using $\mathcal{C}^\infty$-conjugation.

In practice, the computation of a hypernormal form requires the solution of some nonlinear equations. For this reason, we define the pseudohypernormal form, which is not as general as the hypernormal form, but its computation involves only linear equations.

We characterize the hypernormal forms using the theory of transformations based on the Lie transforms. As examples, we work out the two cases of codimension one linear degeneracies: saddle-node and Hopf singularities, using the method previously presented. Finally, in both examples, we consider additional simplifications that can be obtained using $\mathcal{C}^\infty$-equivalence.

1. Introduction. The normal form theory is a powerful tool for the analysis of local bifurcation problems near a nonhyperbolic equilibrium point. The underlying idea in this theory is to use near-identity transformations to remove, in the analytic expression of the vector field, the terms that are inessential in the local dynamical behavior.

The normal form theorem determines how it is possible to simplify the analytic expression of a vector field, taking uniquely into account the linear part of the vector field. Our goal is to show how to obtain further simplifications on the classical normal form, considering the nonlinear terms of the vector field.

We start with a brief summary of the basic ideas of the normal form theory. Consider the system

$$\dot{x} = f(x), \quad x \in \mathbf{K}^n, \tag{1.1}$$

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where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and f is a $\mathcal{C}^\infty$ vector field. Suppose we have an equilibrium at $x = 0$. Usually, we deal with Taylor expansions of the vector field. For this reason, we will introduce the following notation: $\mathcal{J}^k f$ denotes the k-jet of f. Likewise, f_k stands for the homogeneous part of degree k of f, that is, $f_k = \mathcal{J}^k f - \mathcal{J}^{k-1} f$. We can formally write the system in the form

$$\dot{x} = Ax + \sum_{j\geq 2} f_j(x), \tag{1.2}$$

where $A = Df(0)$ is the Jacobian matrix at the origin and $f_j \in \mathcal{H}_j^n$, the space of polynomial homogeneous vector fields of degree j, for $j \geq 2$. Making the near-identity transformation

$$x = \phi(y) = y + P_k(y), \tag{1.3}$$

where $k \geq 2$ is fixed and P_k is a homogeneous polynomial of degree k, we obtain

$$\begin{aligned} \dot{y} &= [D\phi(y)]^{-1} f(\phi(y)) \\ &= [I + D_y P_k(y)]^{-1} A(y + P_k(y)) \\ &\quad + \sum_{j\geq 2} [I + D_y P_k(y)]^{-1} f_k(y + P_k(y)) \\ &= g(y) = Ay + \sum_{j\geq 2} g_j(y). \end{aligned} \tag{1.4}$$

It is easy to show, see Guckenheimer and Holmes [**14**], that we have

Proposition 1.1.

$$g_j(y) = f_j(y), \quad \text{for } j = 2, 3, \dots, k-1,$$

and

$$g_k(y) = f_k(y) - \{D_y P_k(y) Ay - AP_k(y)\}.$$

The expression of g_k suggests the definition of the following linear operator, called the *homological operator*:

$$\begin{aligned} L_k^A : \mathcal{H}_k^n &\longrightarrow \mathcal{H}_k^n \\ P &\longmapsto L_k^A(P), \end{aligned}$$

where $L_k^A P(x) = D_x P(x) Ax - AP(x)$.

We consider a complementary subspace $\mathrm{Cor}\, L_k^A$ to the range $\mathrm{Rang}\, L_k^A$ of the linear operator L_k^A, that is, $\mathcal{H}_k^n = \mathrm{Rang}\, L_k^A \oplus \mathrm{Cor}\, L_k^A$. Thus, we can write $f_k = f_k^r + f_k^c$, where $f_k^r \in \mathrm{Rang}\, L_k^A$ and $f_k^c \in \mathrm{Cor}\, L_k^A$. Then there exists $P_k \in \mathcal{H}_k^n$ such that

$$L_k^A P_k = f_k^r. \tag{1.5}$$

In this way we obtain

$$g_k = f_k - L_k^A P_k = f_k^c \in \mathrm{Cor}\, L_k^A. \tag{1.6}$$

Taking $k = 2, 3, \ldots$, we can achieve $g_j \in \mathrm{Cor}\, L_j^A$ for all $j \geq 2$. Now, using a version of Borel's theorem, the normal form theorem is obtained, see, e.g., Vandervauwhede [**18**].

Theorem 1.2. *There exists a $\mathcal{C}^\infty$-diffeomorphism ϕ verifying $\phi(0) = 0$, $D\phi(0) = I$, such that the change of variables $x = \phi(y)$ transforms* (1.2) *into* (1.4) *where $g_j \in \mathrm{Cor}\, L_j^A$, for all $j \geq 2$.*

The choice of the complementary subspace $\mathrm{Cor}\, L_k^A$ is not unique. In Elphick et al. [**10**], a method to obtain a possible complementary subspace is presented, by defining an adequate inner product in $\mathcal{H}_k^n$.

The normal form theorem determines classes of vector fields as simple as possible, depending upon the linear part, which characterizes the homological operator. The key of the problem of obtaining the normal form of order k is to solve the homological equation (1.5). This linear equation has, in general, a nonunique solution that will depend on an arbitrary additive term belonging to $\mathrm{Ker}\, L_k^A$. So, a number of arbitrary constants will appear in the expression of the solution and, consequently, in the normal form of order greater than k. These constants can be used, depending on the form of the nonlinear terms, to obtain further simplifications in the normal form, leading to the concept of hypernormal forms.

Note the difference between the procedures of obtaining simplifications in the vector field; whereas the linear part of f determines the simplifications expressed by the normal form theorem, in the hypernormal form we take into account the nonlinear terms of f in order to obtain further reductions in the classical normal form.

The utility of these hypernormal forms is evident when we analyze bifurcation problems of codimension greater than those expressed by the linear part; for instance, if we have some degeneracy in the nonlinear terms.

The near-identity transformations leading to normal forms were introduced by Poincare in the study of differential equations. Afterwards they were used by Dulac and Lyapunov, and developed by Birkhoff. At the end of the sixties, several authors, Hori, Garrido, Gröbner, Knapp, ..., presented an approach to the subject using Lie transforms, although mainly devoted to celestial mechanic and Hamiltonian systems, see Chow and Hale [**5**], Lichtenberg and Lieberman [**15**]. The improvements brought out by Deprit [**8**] provided a recursive procedure, suitable to algebraic computation. In Takens [**16**], Ushiki [**17**], Gamero et al. [**11, 12, 13**], the methods of Lie transforms were used to obtain hypernormal forms for linear degeneracies of codimension less than three. In the first two papers, the authors do not use the recursive formulation of Deprit, which is used by the last ones to obtain algorithms specifically adapted to the symbolic computation.

The advantage of the recursive procedure based on the Lie transforms is that we have a precise management of the effect of the change (1.3) in the terms of order greater than k. Moreover, the method of the Lie transforms is not only useful to perform changes of variables, it will also be of great interest in the theoretical analysis of hypernormal forms.

Ushiki [**17**] was the first author to obtain explicitly hypernormal forms. Nevertheless, the method used in the quoted work (revised with more details in Chua and Kokubu [**6**]) requires the resolution of a number of linear ordinary differential equations and does not seem to be computationally effective. In fact, the hypernormal forms were obtained only up to lower degree. Using a different approach, Baider and Sanders [**4**] have obtained hypernormal forms up to an arbitrary order for the Takens-Bogdanov bifurcation in some particular cases.

Using the procedure presented in the present paper, we have obtained hypernormal forms up to an arbitrary order in the cases of saddle-node and Hopf singularities, considered in Sections 3 and 4, and also for Takens-Bogdanov and Hopf-zero singularities, see [**1, 2**].

Now we review the basic ideas of the transformation theory based upon Lie transforms. Let us consider the system (1.1) and a vector field $U(x)$ such that $f, U \in \mathcal{C}^\infty$ in a neighborhood of the origin and $f(0) = U(0) = 0$. We will perform the change of variables $x = u(y, \varepsilon)$, where u is the unique solution of the problem

$$\frac{\partial}{\partial \varepsilon} u(y, \varepsilon) = U(u(y, \varepsilon)), \qquad u(y, 0) = y, \tag{1.7}$$

that is, u is the flow of the autonomous system generated by U.

The transformed vector field of f depends on the generator U in the following form, see Chow and Hale [**5**],

$$g(y, \varepsilon) = f(y) + \sum_{n \geq 1} T_U^n(f)(y) \frac{\varepsilon^n}{n!}, \tag{1.8}$$

where $T_U(f) = [f, U] = Df \cdot U - DU \cdot f$ is the Lie product of f and U, and $T_U^n(f) = T_U \circ \overset{n}{\cdots} \circ T_U(f)$. This vector field verifies

$$\begin{gathered} \frac{\partial g}{\partial \varepsilon}(y, \varepsilon) = [g(y, \varepsilon), U(y)] = T_U(g(y, \varepsilon)), \\ g(y, 0) = f(y). \end{gathered} \tag{1.9}$$

This characterization of the transformed vector field is the one used by Ushiki [**17**]. Here we will use (1.8) in our analysis, taking $\varepsilon = 1$. The transformed vector field corresponding to this value of ε is denoted by $g = U ** f$. This procedure of doing changes of variables is general, i.e., given $k \in \mathbf{N}$, $k \geq 1$, and a local diffeomorphism ϕ such that $\phi(0) = 0$, there exists a generator U such that $\mathcal{J}^k g = \mathcal{J}^k(U ** f)$, where $g = \phi * f$ denotes the transformed vector field of f by ϕ. In the orientation preserving case, this statement is contained in Lemma 4.2 of Ashkenazi and Chow [**3**]. In the orientation reversing case, it is easily obtained in a similar way but takes into account the complex domain.

This paper is organized as follows. In Section 2 we define the hypernormal form of order k for a given vector field by induction. To obtain a hypernormal form, it is necessary to solve some nonlinear equations. For this reason, we introduce the concept of pseudohypernormal form, which does not contain all the possible simplifications but is applicable in practice because it only involves linear equations.

In Theorem 2.3 we characterize the pseudohypernormal form of order k, by defining an adequate subspace on $\operatorname{Cor} L_k^A$.

We also consider in Section 2 the possibilities of obtaining further simplifications in the pseudohypernormal form, which is done making a linear transformation that commutes with the matrix of the linear part of the vector field. We finish this section with Proposition 2.4, useful to remove the terms of order greater than k that remain in the vector field, when we achieve the hypernormal form of order k, e.g., the flat terms.

In Section 3 we analyze, as an application, the saddle-node singularity. We obtain the same result as Takens [**16**] using $\mathcal{C}^\infty$-conjugation, and we show the improvements achieved by means of the use of $\mathcal{C}^\infty$-equivalence, that is, using, besides the near-identity transformation, a reparametrization of the time depending on the state variables.

Finally, in Section 4, we present the simplest normal form, under $\mathcal{C}^\infty$-conjugacy and $\mathcal{C}^\infty$-equivalence, for a germ whose linear part has a pair of pure imaginary eigenvalues $\pm i$. The hypernormal forms are determined by the first nonzero coefficient in the radial and azimuthal components. Several particular cases are considered, where we also give the expressions of the hypernormal form coefficients.

2. Hypernormal form theory. As mentioned before, to give the formal definition of hypernormal form up to a given order k for the system (1.2), we will proceed by induction in k. We say that (1.2) is a *hypernormal form* up to first order if A is put in Jordan normal form.

For $k \geq 2$, consider the set

$$\mathcal{X}_k = \left\{ U \in \bigoplus_{j=1}^{k} \mathcal{H}_j^n : \mathcal{J}^1(U * * f) = A, (U * * f)_j \in \operatorname{Cor} L_j^A, \right. \\ \left. j = 2, \dots, k \right\}, \tag{2.1}$$

that corresponds to the generators of the changes of variables that transforms (1.2) in normal form up to order k.

Let $\mathcal{B}_j = \{u_1^j, \dots, u_{l_j}^j\}$ be a basis of $\operatorname{Cor} L_j^A$ for $2 \leq j \leq k$. Then, for each $U \in \mathcal{X}_k$, we can write the k-jet of the vector field transformed of

f by U in the form

$$\mathcal{J}^k(U**f)(x) = Ax + \sum_{j=2}^{k}\sum_{i=1}^{l_j} h_{ji}(U)u_i^j, \tag{2.2}$$

where $h_{ji} : \mathcal{X}_k \mapsto \mathbf{K}$, $2 \le j \le k$, $1 \le i \le l_j$.

Define $\mathcal{M}_k : \mathcal{X}_k \to \mathbf{N}$ by

$$\mathcal{M}_k(U) = \operatorname{card}\{(j,i) \in \mathbf{N}^2 : h_{ji}(U) = 0, 2 \le j \le k, 1 \le i \le l_j\}, \tag{2.3}$$

that is, $\mathcal{M}_k(U)$ is the number of vanishing coefficients in the k-degree normal form corresponding to U.

As the range of the map $\mathcal{M}_k$ is a finite set, there exists $\tilde{U} \in \mathcal{X}_k$ such that

$$\mathcal{M}_k(\tilde{U}) = \max_{U\in\mathcal{X}_k} \mathcal{M}_k(U). \tag{2.4}$$

If $\tilde{U}$ verifies (2.4), we say that $\mathcal{J}^k(\tilde{U}**f)$ is a *hypernormal form* up to order k. In other words, a hypernormal form is a normal form having the maximum number of vanishing coefficients.

Notice that, if $\mathcal{J}^k(\tilde{U}**f)$ is a hypernormal form up to order k, it is not true that, for $2 \le j < k$, $\mathcal{J}^j(\tilde{U}**f)$ is a hypernormal form up to order j, because $\mathcal{M}_j(\mathcal{J}^j(\tilde{U})) \ne \max_{U\in\mathcal{X}_j} \mathcal{M}_j(U)$ in general. For instance, in the normal form in cylindrical coordinates for the Hopf-zero singularity, see [**2**], the simplifications in the higher order terms are more convenient that those achieved in the azimuthal component. Thus, in this particular case, a truncated hypernormal form cannot contain the maximum number of simplifications. A similar comment holds in the case of the Hopf-Hopf singularity.

In the following, we present a recursive procedure to obtain hypernormal forms, with the following feature: if we truncate a hypernormal form of order k to an order $j < k$, we will obtain a hypernormal form up to order j.

Assume that the system (1.2) has been put in hypernormal form up to order $k-1$. Using (1.8), we can write the transformed vector field of f by means of U in the form

$$\begin{aligned} f^*(U) = U**f &= f + \sum_{n\ge1} \frac{(T_U^n(f))}{n!} \\ &= f + [f,U] + \frac{1}{2!}[[f,U],U] + \frac{1}{3!}[[[f,U],U],U] + \cdots . \end{aligned} \tag{2.5}$$

In order to leave unaltered the hypernormal form up to order $k-1$, we require $\mathcal{J}^{k-1}[f,U]=0$. This assumption has the following consequences:

Lemma 2.1. *Assume that the vector field f and the generator U satisfy $\mathcal{J}^{k-1}[f,U]=0$, and denote by $U_1=\mathcal{J}^1U$ the linear part of U. Then*

(2.6)
$$\mathcal{J}^{k-1}f^*(U)=\mathcal{J}^{k-1}f,$$

(2.7)
$$(f^*(U))_k=f_k+[f,U]_k+\sum_{n\geq 1}\frac{(T^n_{U_1}([f,U]_k))}{(n+1)!},\quad \textit{if } U_1\not\equiv 0,$$

(2.8)
$$(f^*(U))_k=f_k+[f,U]_k,\quad \textit{if } U_1\equiv 0.$$

Remark 1. Equality (2.6) corresponds to Lemma 4.6 of Chua and Kokubu [**6**].

Remark 2. In our approach the keys are (2.7) and (2.8) which permit us to manage algebraically the k-degree terms. In Ushiki [**17**] and Chua and Kokubu [**6**], these terms are characterized by means of linear ordinary differential equations, see (1.9). Equation (2.8) shows that, taking U with zero linear part, the k-degree terms of the transformed vector field can be easily obtained, because we avoid the infinite series in (2.5).

Proof. Equation (2.6) is easily obtained from (2.5) using that, if $P\in\mathcal{H}^n_k$ and $Q\in\mathcal{H}^n_j$, then $[P,Q]\in\mathcal{H}^n_{k+j-1}$. To prove (2.7) and (2.8), we observe that the homogeneous terms of order k in $f^*(U)$ are:

$$\begin{aligned}(f^*(U))_k&=f_k+[f,U]_k+\frac{1}{2!}[[f,U],U]_k+\frac{1}{3!}[[[f,U],U],U]_k+\cdots\\&=f_k+[f,U]_k+\frac{1}{2!}[[f,U]_k,U_1]+\frac{1}{3!}[[[f,U]_k,U_1],U_1]+\cdots.\end{aligned}$$

In fact,

(2.9)
$$(f^*(U))_k = f_k + \sum_{n\geq 1} \frac{(T_U^n(f))_k}{n!} = f_k + [f,U]_k + \sum_{n\geq 1} \frac{(T_{U_1}^n([f,U]_k))}{(n+1)!}.$$

In the above expression, the generator U appears nonlinearly in the terms of the last sum. Nevertheless, if we take $U_1 = 0$, that is, taking U with zero linear part, we have $T_{U_1} \equiv 0$ and the quoted sum would not appear. In this case, the effect of U in $(f^*(U))_k$ is linear as shown in (2.8). □

Due to (2.8), we will perform a change of variables corresponding to the generator $U = V + U_k$, where

$$V \in \mathcal{W}_k = \left\{ V \in \bigoplus_{i=1}^{k-1} \mathcal{H}_i^n : \mathcal{J}^{k-1}[f,V] = 0 \right\} \quad \text{and} \quad U_k \in \mathcal{H}_k^n. \tag{2.10}$$

Observe that $\mathcal{W}_k$ is a subspace of $\oplus_{i=1}^{k-1}\mathcal{H}_i^n$.

We have $\mathcal{J}^{k-1}[f, V + U_k] = 0$, and so the change leaves the terms of order less than $k-1$ unaltered. Decompose $\mathcal{H}_k^n = \text{Rang } L_k^A \oplus \text{Cor } L_k^A$, and consider

$$\mathcal{B}_1 = \{v_1, \dots, v_{r_k}\} \quad \text{and} \quad \mathcal{B}_2 = \{u_1, \dots, u_{l_k}\}, \tag{2.11}$$

basis of Rang L_k^A and Cor L_k^A, respectively. Then we can write

$$(f^*(U))_k = k_1 v_1 + \cdots + k_{r_k} v_{r_k} + h_1 u_1 + \cdots + h_{l_k} u_{l_k}, \tag{2.12}$$

where the coefficients depend on U and consequently depend on V and U_k.

However, the coefficients h_j, $j = 1, \dots, l_k$, do not depend on U_k.

Lemma 2.2. *The projection of $f^*(U)$ onto* Cor L_k^A *is independent of U_k.*

Proof. Using that $U = V + U_k$, where $V \in \mathcal{W}_k$, we can write (2.7) in the form

$$\begin{aligned} (f^*(U))_k &= f_k + [f,V]_k + [f_1, U_k] \\ &\quad + \sum_{n\geq 1} \frac{(T_{U_1}^n([f,V]_k) + T_{U_1}^n([f_1,U_k]))}{(n+1)!}, \end{aligned} \tag{2.13}$$

where $U_1 = \mathcal{J}^1(U) = \mathcal{J}^1(V) = V_1$ is the linear part of the generator. It is enough to show that all the terms containing U_k belong to Rang L_k^A. We have $[f_1, U_k] = -L_k^A(U_k) \in$ Rang L_k^A. Using $U_1 = V_1$, we obtain $T_{U_1}(f_1) = [f_1, V_1] = 0$ because $V \in \mathcal{W}_k$. Using Jacobi's identity, we deduce

$$\begin{aligned} T_{U_1}([f_1, U_k]) &= [[f_1, U_k], U_1] = -[[U_k, U_1], f_1] - [[U_1, f_1], U_k] \\ &= -[[U_k, U_1], f_1] \in \text{Rang}\, L_k^A. \end{aligned}$$

Analogously, it can be proved that $T_{U_1}^n([f_1, U_k]) \in \text{Rang}\, L_k^A$ for $n \geq 2$. □

Using this lemma, we can write

(2.14)
$$(f^*(U))_k = k_1(U)v_1 + \cdots + k_{r_k}(U)v_{r_k} + h_1(V)u_1 + \cdots + h_{l_k}(V)u_{l_k},$$

where

$$\begin{aligned} k_i &: \mathcal{W}_k \oplus \mathcal{H}_k^n \longmapsto \mathbf{K}, \quad 1 \leq i \leq r_k, \\ h_j &: \mathcal{W}_k \longmapsto \mathbf{K}, \qquad\quad\ \ 1 \leq j \leq l_k. \end{aligned}$$

At this point, we make a near-identity transformation in order to annihilate the part of $f^*(U)$ belonging to Rang L_k^A. This is done taking $V_k \in \mathcal{H}_k^n$, depending on $U \in \mathcal{W}_k \oplus \mathcal{H}_k^n$, adequately. In this way, we obtain a transformed vector field $g = V_k * *(f^*(U))$ such that

$$g_k = g_k(V) = h_1(V)u_1 + \cdots + h_{l_k}(V)u_{l_k}, \tag{2.15}$$

with $V \in \mathcal{W}_k$.

We define

$$\begin{gathered} \mathcal{N}_k : \mathcal{W}_k \longmapsto \mathbf{N} \\ \mathcal{N}_k(V) = \text{card}\, \{i \in \mathbf{N} : h_i(V) = 0, 1 \leq i \leq l_k\}. \end{gathered}$$

Using that the range of $\mathcal{N}_k$ is finite, we can conclude that there exists $\tilde{V} \in \mathcal{W}_k$ such that

$$\mathcal{N}_k(\tilde{V}) = \max_{V \in \mathcal{W}_k} \mathcal{N}_k(V). \tag{2.16}$$

If $\tilde{V} \in \mathcal{W}_k$ verifies (2.16), $g(\tilde{V})$ is a hypernormal form of f up to order k.

Note that $h_j(V)$, $1 \leq j \leq l_k$, depend nonlinearly on $V \in \mathcal{W}_k$, but if we take V with zero linear part, that is, $V \in \mathcal{W}_k \cap \oplus_{i=2}^{k-1} \mathcal{H}_i^n$, the above functions depend linearly on V, for $1 \leq j \leq l_k$, see (2.8).

In practice, it is difficult to obtain the expressions for the above scalar functions. Moreover, even in the case that we could obtain these expressions, it is complicated to compute $\tilde{V}$ that verifies (2.16), due to the nonlinearity of these expressions.

For this reason, our approach in the applications will not be as general as the one previously presented. In fact, we will only consider the linear effect of the generator, and the obtained simplified normal form will be called *pseudohypernormal* form. We proceed, as above, by induction. The pseudohypernormal form of first order agrees with the hypernormal form of this order.

Consider $k \geq 2$, and suppose that (1.2) has been put in pseudohypernormal form of order $k-1$. Define the set $\mathcal{V}_k$ by

$$\mathcal{V}_2 = \{0\}$$

and

$$\mathcal{V}_k = \{\tilde{U} \in \mathcal{H}_2^n \oplus \cdots \oplus \mathcal{H}_{k-1}^n : \mathcal{J}^{k-1}[f, \tilde{U}] = 0\} \quad \text{for } k \geq 3. \tag{2.17}$$

Obviously, $\mathcal{V}_k$ is a subspace of $\oplus_{i=2}^{k-1} \mathcal{H}_i^n$, and it is only determined by $\mathcal{J}^{k-1} f$.

In addition, define the linear operator

$$\begin{aligned} &\mathcal{L}_k : \mathcal{V}_k \longmapsto \mathcal{H}_k^n, \\ &\mathcal{L}_k(\tilde{U}) = [f, \tilde{U}]_k. \end{aligned}$$

The set

$$\begin{aligned} \mathcal{C}_k = \{h \in \operatorname{Cor} L_k^A :\ &\text{there exists } \tilde{U} \in \mathcal{V}_k \\ &\text{such that } \mathcal{L}_k(\tilde{U}) + h \in \operatorname{Rang} L_k^A\}, \end{aligned} \tag{2.18}$$

is a subspace of $\operatorname{Cor} L_k^A$. Consequently, $\operatorname{Cor} L_k^A$ may be decomposed as

$$\operatorname{Cor} L_k^A = \mathcal{C}_k \oplus \mathcal{D}_k, \tag{2.19}$$

where $\mathcal{D}_k$ is a complementary subspace of $\mathcal{C}_k$ in $\operatorname{Cor} L_k^A$. Then we can write

$$\mathcal{H}_k^n = \operatorname{Rang} L_k^A \oplus \mathcal{C}_k \oplus \mathcal{D}_k. \tag{2.20}$$

We say that (1.2) is put in pseudohypernormal form of order k if $f_k \in \mathcal{D}_k$. In the next theorem, we show that it is always possible to reduce the vector field (1.2) to a pseudohypernormal form of order k:

Theorem 2.3. *There exists a near-identity transformation that carries over the vector field* (1.2) *to* (1.4)*, where*

$$g_i = f_i, \quad \textit{for } 1 \leq i \leq k-1,$$
$$g_k \in \mathcal{D}_k.$$

Proof. We consider the generator $U = V + U_k$, where $V \in \mathcal{V}_k$ and $U_k \in \mathcal{H}_k^n$, and denote the transformed vector field of f by $g = U ** f$.

From Lemma 2.1, using that $\mathcal{J}^{k-1}([f,U]) = 0$ and $\mathcal{J}^1 U = 0$, we obtain that $g_j = f_j$ for $1 \leq j \leq k-1$, and

$$g_k = f_k + [f,U]_k = f_k + [f_1, U_k] + [\mathcal{J}^{k-1} f, V]_k. \tag{2.21}$$

On the other hand, we can write

$$f_k = f_k^1 + f_k^2 + f_k^3, \tag{2.22}$$

where $f_k^1 \in \operatorname{Rang} L_k^A$, $f_k^2 \in \mathcal{C}_k$ and $f_k^3 \in \mathcal{D}_k$, see (2.20).

As $f_k^2 \in \mathcal{C}_k$, there exists $V \in \mathcal{V}_k$ such that $\mathcal{L}_k(V) + f_k^2 = \tilde{f}_k$, with $\tilde{f}_k \in \operatorname{Rang} L_k^A$. Using that $f_k^1 + \tilde{f}_k \in \operatorname{Rang} L_k^A$, we can take $U_k \in \mathcal{H}_k^n$ such that $L_k^A(U_k) = f_k^1 + \tilde{f}_k$.

In such a manner, we obtain $g_k = f_k^3 \in \mathcal{D}_k$. $\square$

Remark 1. The $(k-1)$-set of f determines the structure of $\mathcal{D}_k$ and, therefore, characterizes the pseudohypernormal form of order k.

Remark 2. Theorem 1.2 of the normal form affirms that if two vector fields f, g verify $f(0) = g(0) = 0$ and $\mathcal{J}^1 f = \mathcal{J}^1 g = A$, that is, they have the same linear part, then the vector fields can be transformed into others f^*, g^* such that $f_k^*, g_k^* \in \operatorname{Cor} L_k^A$. On the other hand, Theorem 2.3 of the pseudohypernormal form affirms that if two vector fields f, g verify $f(0) = g(0) = 0$ and $\mathcal{J}^{k-1} f = \mathcal{J}^{k-1} g$, then they can be transformed into others f^*, g^* such that $f_k^*, g_k^* \in \mathcal{D}_k$.

The proof of Theorem 2.3 provides a method to obtain a pseudohypernormal form for a given vector field that requires only the resolution of linear equations. In general, the pseudohypernormal form is not the simplest one. Nevertheless, in some cases, we will be able to obtain further simplifications by considering the nonlinear effect of a special kind of generator that is easily handled. For that, we will use the k-jet of the vector field f, instead of the $(k-1)$-jet, to obtain new simplifications in the k-degree terms of the pseudohypernormal form.

Consider a generator $U_1(x) \in \mathcal{H}_1^n$ such that $\mathcal{J}^{k-1}[f, U_1] = 0$. In this way, the terms of order less than $k-1$ remain unaltered. In other words, if we denote the transformed vector field by $h = U_1 ** f = h_1 + \cdots + h_k + \cdots$, then $h_i = f_i$ for $1 \le i \le k-1$.

It is easy to check that, if $U_1(x) = Bx$, then the change of variables is given by $x = e^B y$, and so $h_k(y) = e^{-B} f_k(e^B y)$.

For our convenience, we assume that the basis $\mathcal{B}_2$ of $\operatorname{Cor} L_k^A$ given in (2.11) has been chosen such that it contains a basis of $\mathcal{D}_k$: $\{u_1, \ldots, u_{d_k}\}$.

Applying Theorem 2.3, we obtain a pseudohypernormal form g of order k. So, $g_i = h_i = f_i$ for $1 \le i \le k-1$. Moreover, g_k is the projection of h_k onto $\mathcal{D}_k$ and can be expressed as:

$$g_k = g_k(U_1) = l_1(U_1) u_1 + \cdots + l_{d_k}(U_1) u_{d_k} \in \mathcal{D}_k. \tag{2.23}$$

Consider the set $\mathcal{U}_k = \{U_1 \in \mathcal{H}_1^n : \mathcal{J}^{k-1}[f, U_1] = 0\}$ and the transformation $\tilde{\mathcal{N}} : \mathcal{U}_k \mapsto \mathbf{N}$, defined by

$$\tilde{\mathcal{N}}(U_1) = \operatorname{card}\{i \in \mathbf{N} : l_i(U_1) = 0, 1 \le i \le d_k\}. \tag{2.24}$$

Obviously, there exists $\tilde{U}_1 \in \mathcal{U}_k$ such that

$$\tilde{\mathcal{N}}_k(\tilde{U}_1) = \max_{U_1 \in \mathcal{U}_k} \tilde{\mathcal{N}}_k(U_1). \tag{2.25}$$

The transformed vector field corresponding to the generator $\tilde{U}_1$ is called a *reduced pseudohypernormal form.*

Although several kinds of normal forms defined before (hypernormal, pseudohypernormal and reduced pseudohypernormal) do not agree in general, in some specific cases that we have analyzed, corresponding to saddle-node, Hopf, Takens-Bogdanov, Hopf-zero, the three ones yield the maximum simplification. The most cumbersome part of the analysis in these cases is to check that the local expression reached of a vector field is the simplest one.

In other cases, e.g., the triple-zero linear degeneracy, the reduced pseudohypernormal form is simpler than the pseudohypernormal form. The reason is that, in this case, we can use a linear generator that leaves unaltered the linear part and allow us to simplify some higher order terms.

So far, we have been concerned about formal hypernormal forms. We end this section with a proposition, where we obtain a sufficient condition which enables us to remove those higher order terms that are not essential in the local behavior. In particular, it will be useful to annihilate the ∞-flat terms, to be sure that the hypernormal form obtained is convergent in some neighborhood of the origin.

Proposition 2.4. *Consider the system*

$$\dot{x} = f(x, \varepsilon) = g(x) + \varepsilon h(x), \quad x \in \mathbf{K}^n,\, \varepsilon \in \mathbf{K}, \tag{2.26}$$

with $g, h \in \mathcal{C}^\infty$ *in a neighborhood of the origin* $\mathcal{V}$. *Assume that there exists a vector field* $U \in \mathcal{C}^\infty$ *in* $\mathcal{V} \times \mathbf{K}$ *such that*

$$h(x) + [f(x, \varepsilon), U(x, \varepsilon)] = 0, \quad x \in \mathcal{V},\, \varepsilon \in \mathbf{K}. \tag{2.27}$$

Then the system (2.26) *is* $\mathcal{C}^\infty$*-conjugate to*

$$\dot{x} = g(x). \tag{2.28}$$

Proof. We consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = F(x, \xi) = \begin{pmatrix} f(x, \xi) \\ 0 \end{pmatrix}. \tag{2.29}$$

If we denote the transformation of coordinates corresponding to the generator $V(x,\xi) = \binom{U(x,\xi)}{1}$, where $U(x,\xi)$ verifies (2.27), by

$$x = u_1(y,\zeta,\varepsilon), \qquad \xi = u_2(y,\zeta,\varepsilon), \tag{2.30}$$

then we have

$$\frac{\partial u_1}{\partial \varepsilon}(y,\zeta,\varepsilon) = U(u_1(y,\zeta,\varepsilon), u_2(y,\zeta,\varepsilon)), \qquad \frac{\partial u_2}{\partial \varepsilon}(y,\zeta,\varepsilon) = 1,$$
$$u_1(y,\zeta,0) = y, \qquad u_2(y,\zeta,0) = \zeta.$$

Therefore, $\xi = u_2(y,\zeta,\varepsilon) = \zeta+\varepsilon$, and $(\partial u_1/\partial\varepsilon)(y,\zeta,\varepsilon) = U(u_1(y,\zeta,\varepsilon), \zeta+\varepsilon)), u_1(y,\zeta,0) = y$.

The transformed vector field of (2.29) by the change of variables (2.30), G, satisfies the following Cauchy problem

$$\frac{\partial G}{\partial \varepsilon}(y,\zeta,\varepsilon) = [G(y,\zeta,\varepsilon), V(y,\zeta)],$$
$$G(y,\zeta,0) = F(y,\zeta).$$

By hypothesis, U verifies (2.27), hence the vector field $F(y,\zeta)$ also satisfies the previous Cauchy problem. Then $G(y,\zeta,\varepsilon) = F(y,\zeta)$.

Therefore, the above change of variables carries the vector field (2.29) into itself. Consequently, we obtain

$$(D_y u_1(y,\zeta,\varepsilon))^{-1} f(u_1(y,\zeta,\varepsilon), \zeta+\varepsilon) = f(y,\zeta) \tag{2.31}$$

for any y and ζ in a neighborhood of the origin in $\mathbf{K}^n$ and $\mathbf{K}$, respectively, and $\varepsilon \in \mathbf{K}$.

In particular, the transformation $x = u_1(y,0,\varepsilon)$ transforms the system (2.26) into

$$\dot{y} = (D_y u_1(y,0,\varepsilon))^{-1} f(u_1(y,0,\varepsilon),\varepsilon) = g(y). \qquad \square \tag{2.32}$$

3. Saddle-node singularity. In this section we apply the above ideas and results to the easiest case of nonhyperbolicity: the saddle-node bifurcation. Theorem 3.1 of the hypernormal form was previously obtained by Takens [**16**]. In Theorem 3.2 we will also consider the effect of the reparametrization of the time, $\mathcal{C}^\infty$-equivalence, which will provide further simplifications.

We begin with the scalar field

$$\dot{x} = f(x), \tag{3.1}$$

where $f \in \mathcal{C}^\infty$ in a neighborhood of the origin in $\mathbf{R}$. We assume $f(0) = f'(0) = \cdots = f^{(r-1)}(0) = 0$, and $f^{(r)}(0) \neq 0$. Consequently, for any $N \in \mathbf{N}$, $N > r$, the system may be written in the form

$$\dot{x} = \alpha_r x^r + \alpha_{r+1} x^{r+1} + \cdots + \alpha_N x^N + \mathcal{O}(|x|^{N+1}), \tag{3.2}$$

with $\alpha_r \neq 0$, or equivalently,

$$\dot{x} = \sum_{j=r}^{N} f_j(x) + \mathcal{O}(N+1), \tag{3.3}$$

where $f_j(x) = \alpha_j x^j$ for $j \geq r$.

Theorem 3.1. *The system* (3.2), *with* $\alpha_r \neq 0$, *is* $\mathcal{C}^\infty$*-conjugate to*

$$\dot{x} = \alpha_r^* x^r + \alpha_{2r-1}^* x^{2r-1}, \tag{3.4}$$

where $\alpha_r^* = \operatorname{sig}(\alpha_r)$.

Proof. Firstly, we will show that

$$\dot{x} = \alpha_r^* x^r + \alpha_{2r-1}^* x^{2r-1} + h(x), \tag{3.5}$$

where $\alpha_r^* = \operatorname{sig}(\alpha_r)$ and $h(x) = \mathcal{O}(|x|^{N+1})$, is a hypernormal form for the system (3.2) up to order N.

For that, we consider the generator $U(x) = ax^k$ with $k \geq 2$, which has null linear part. Since $\mathcal{J}^{r+k-1}[f, U] = 0$, using Lemma 2.1 we deduce that the transformed scalar field $f^* = U **f$ verifies $f_j^* = f_j$ for $j \leq r + k - 2$, and

$$f_{r+k-1}^* = f_{r+k-1} + [f, U]_{r+k-1} = (\alpha_{r+k-1} + (r-k)\alpha_r a) x^{r+k-1}. \tag{3.6}$$

Therefore, in the case $k \neq r$, we can choose a such that $f^*_{r+k-1} = 0$. If we proceed with the change for f with $k = 2$, subsequently on the transformed field with the change for $k = 3$, and so on, until $k = N - r + 1$, except for $k = r$, we get a field $\mathcal{C}^\infty$-conjugate to the first one, given by

$$f^*(x) = \alpha_r x^r + \tilde{\alpha}_{2r-1} x^{2r-1} + \mathcal{O}(|x|^{N+1}). \tag{3.7}$$

Finally, taking a generator $U(x) = ax$ and using (2.5), we obtain

$$\begin{aligned} f^*(x) &= \alpha_r e^{a(r-1)} x^r + \tilde{\alpha}_{2r-1} e^{2a(r-1)} x^{2r-1} + \mathcal{O}(|x|^{N+1}) \\ &= \operatorname{sig}(\alpha_r) x^r + \alpha^*_{2r-1} x^{2r-1} + \mathcal{O}(|x|^{N+1}), \end{aligned} \tag{3.8}$$

choosing a adequately.

To finish the proof, we apply Proposition 2.4 to annihilate the term $h(x)$ that appears in (3.5). Note that $h \in \mathcal{C}^\infty$ in a neighborhood of the origin and $\mathcal{J}^N h(x) = 0$, with N arbitrarily large. We define $f(x,\varepsilon) = \alpha^*_r x^r + \alpha^*_{2r-1} x^{2r-1} + \varepsilon h(x)$. We need to prove that the equation (2.27) has some solution. This equation becomes

$$\begin{aligned} &\left[\alpha^*_r + \alpha^*_{2r-1} x^{r-1} + \varepsilon \frac{h(x)}{x^r}\right] \frac{\partial}{\partial x} V(x,\varepsilon) \\ &\quad - \left[(r-1)\alpha^*_{2r-1} x^{r-2} - \varepsilon r \frac{h(x)}{x^{r+1}} + \varepsilon \frac{h'(x)}{x^r}\right] V(x,\varepsilon) - \frac{h(x)}{x^{2r}} = 0, \end{aligned} \tag{3.9}$$

where $U(x,\varepsilon) = x^r V(x,\varepsilon)$. We note that (3.9) agrees with the first equation of [**16**, p. 177]. So, taking $N \geq 2r$ and ε in a compact subset K, we can be sure that there exists a unique solution $V(x,\varepsilon)$, defined for x in a neighborhood of the origin and $\varepsilon \in K$, of this equation such that $V(0,\varepsilon) = 0$. □

The hypernormal form under $\mathcal{C}^\infty$-equivalence, used by Dumortier et al. [**9**], is the following:

Theorem 3.2. *The scalar field* (3.2)*, with $\alpha_r \neq 0$, is $\mathcal{C}^\infty$-equivalent to*

$$x' = \operatorname{sig}(\alpha_r) x^r. \tag{3.10}$$

Proof. Firstly, we use Theorem 3.1 to transform (3.2) into (3.4). Next, we reparametrize the time by $(dT/dt) = (1/P(x))$, where $P(x) = 1 - (\alpha^*_{2r-1}/\mathrm{sig}\,(\alpha_r))x^{r-1}$ (note that $P(x)$ is locally positive). In this way, we obtain the system

$$x' = \frac{dx}{dT} = g(x) = \mathrm{sig}\,(\alpha_r)x^r + b\alpha_{2r-1}x^{3r-2} + \mathcal{O}(|x|^{N+1}). \tag{3.11}$$

Finally, using the procedure carried out in the proof of Theorem 3.1, we obtain the field

$$g^*(x) = \mathrm{sig}\,(\alpha_r)x^r. \qquad \square \tag{3.12}$$

4. Hopf singularity. In this last section we will use the formulation of the changes of variables in terms of the Lie transforms to obtain theoretical results about the hypernormal form for the Hopf bifurcation, which is determined by the knowledge of the first nonvanishing coefficients of the normal form (see Theorem 4.3). Also, in subsection 4.2, we analyze further simplifications that can be achieved not only by coordinate transformations, but by using additionally a reparametrization of the time depending on the state variables, that is, by means of the use of $\mathcal{C}^\infty$-equivalence.

We start this section with a review of the basic ideas of the normal form for the Hopf bifurcation. Let us consider a system

$$\begin{aligned} \dot{x} &= F(x,y),\\ \dot{y} &= G(x,y), \end{aligned} \tag{4.1}$$

where $F, G \in \mathcal{C}^\infty(\mathcal{V}, \mathbf{R})$, $\mathcal{V}$ is a neighborhood of the origin in $\mathbf{R}^2$, with an equilibrium point at the origin ($F(0,0) = G(0,0) = 0$) with eigenvalues $\pm i$. It is usual to introduce complex variables by

$$z = x + iy, \qquad \bar{z} = x - iy. \tag{4.2}$$

The system (4.1) becomes

$$\begin{aligned} \dot{z} &= P(z,\bar{z}),\\ \dot{\bar{z}} &= \overline{P(z,\bar{z})}, \end{aligned} \tag{4.3}$$

where $P(z,\bar z) = F(((z+\bar z)/2),((z-\bar z)/2i)) + iG(((z+\bar z)/2),((z-\bar z)/2i)) \in \mathcal{C}^\infty(\mathcal{V}^*;\mathbf{C})$ in some neighborhood $\mathcal{V}^*$ of the origin in $\mathbf{C}^2$. It is easy to show that $P(z,\bar z) = iz + \mathcal{O}(|z,\bar z|^2)$.

We consider the vector spaces over $\mathbf{R}$ (not over $\mathbf{C}$) given by

$$\begin{aligned} \overline{\mathcal{H}}_k &= \left\{ \begin{pmatrix} P(z,\bar z) \\ \overline{P(z,\bar z)} \end{pmatrix} : P \in \mathcal{H}^1_k(\mathbf{C}^2) \right\} \\ &= \left\{ \sum_{p,q\in\mathbf{N}:p+q=k} A_{pq} \begin{pmatrix} z^p\bar z^q \\ 0 \end{pmatrix} + \overline{A_{pq}} \begin{pmatrix} 0 \\ \bar z^p z^q \end{pmatrix} : A_{pq} \in \mathbf{C} \right\}, \end{aligned} \tag{4.4}$$

where $\mathcal{H}^1_k(\mathbf{C}^2)$ is the space of homogeneous polynomials of degree k in the variables $z, \bar z$. The Lie product in complex coordinates verifies the following property:

Lemma 4.1. *Assume that* $\begin{pmatrix} f \\ \bar f \end{pmatrix} \in \overline{\mathcal{H}}_j$, $\begin{pmatrix} U \\ \overline{U} \end{pmatrix} \in \overline{\mathcal{H}}_k$. *Then* $\left[\begin{pmatrix} f \\ \bar f \end{pmatrix}, \begin{pmatrix} U \\ \overline{U} \end{pmatrix}\right] \in \overline{\mathcal{H}}_{j+k-1}$.

Proof. It is enough to note that

$$\begin{aligned} \left[\begin{pmatrix} f \\ \bar f \end{pmatrix}, \begin{pmatrix} U \\ \overline{U} \end{pmatrix}\right] &= \begin{pmatrix} f_z U + f_{\bar z}\overline{U} - U_z f - U_{\bar z}\bar f \\ \bar f_z U + \bar f_{\bar z}\overline{U} - \overline{U}_z f - \overline{U}_z \bar f \end{pmatrix} \\ &= \begin{pmatrix} Q \\ \overline{Q} \end{pmatrix} \in \overline{\mathcal{H}}_{j+k-1}. \qquad \square \end{aligned} \tag{4.5}$$

The structure of the vector fields, vector spaces and Lie product suggest to define the ones corresponding to the first component. So we deal with

$$\dot z = P(z,\bar z), \tag{4.6}$$

instead of (4.3), having in mind that the second component is the complex conjugate of the first one. We define the spaces

$$\mathcal{H}_k = \left\{ \sum_{p+q=k} A_{pq} z^p \bar z^q : A_{pq} \in \mathbf{C} \right\}, \tag{4.7}$$

with the following basis:

$$\{u_{pq} = z^p \bar{z}^q, v_{pq} = iz^p \bar{z}^q : p + q = k\}. \tag{4.8}$$

Finally, the Lie product is given by

$$[f, U] = f_z U + f_{\bar{z}} \overline{U} - U_z f - U_{\bar{z}} \bar{f}, \tag{4.9}$$

and so, if $f \in \mathcal{H}_j$, $U \in \mathcal{H}_k$, then $[f, U] \in \mathcal{H}_{j+k-1}$. The homological operator, which determines the normal form, is

$$L_k : H_k \longmapsto \mathcal{H}_k, \qquad L_k(U) = [U, iz]. \tag{4.10}$$

It is easy to check that

$$L_k(u_{pq}) = (p - 1 - q)v_{pq}, \qquad L_k(v_{pq}) = (q + 1 - p)u_{pq}, \tag{4.11}$$

for all $p, q \in \mathbf{N}$, $p + q = k$. From this, it is obtained that $\{u_{pq}, v_{pq} : p - 1 - q \neq 0, p + q = k\}$ is a basis of $\operatorname{Rang} L_k$, the range of the linear operator L_k. Also, $G_k = \{u_{pq}, v_{pq} : p - 1 - q = 0, p + q = k\}$ is a complementary subspace of $\operatorname{Rang} L_k$ in $\mathcal{H}_k$. Note that $G_k = \operatorname{Ker} L_k$ is the orthogonal complementary subspace defined in Elphick et al. [**10**], and $G_{2m} = \{0\}$, $G_{2m+1} = \{\alpha z(z\bar{z})^m + \beta iz(z\bar{z})^m : \alpha, \beta \in \mathbf{R}\}$.

Hence it is easily obtained that a normal form for (4.6) is

$$\dot{z} = \sum_{j=0}^{\infty} f_{2j+1}(z, \bar{z}) = (0, 1)G_1 + \sum_{j=1}^{\infty} (\alpha_{2j+1}, \beta_{2j+1})_{G_{2j+1}}, \tag{4.12}$$

where we have denoted

$$(\alpha, \beta)_{G_{2m+1}} = \alpha u_{m+1,m} + \beta v_{m+1,m} = \alpha z(z\bar{z})^m + \beta iz(z\bar{z})^m \in G_{2m+1}, \tag{4.13}$$

for $\alpha, \beta \in \mathbf{R}$.

It is usual to express the above normal form in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$:

$$\dot{r} = \sum_{j=1}^{\infty} \alpha_{2j+1} r^{2j+1}, \qquad \dot{\theta} = 1 + \sum_{j=1}^{\infty} \beta_{2j+1} r^{2j}. \tag{4.14}$$

Now we present some previous definitions and results. The following operators are useful to obtain the coefficients of the normal form:

$$\prod\nolimits_1^{2m+1} : G_{2m+1} \longmapsto \mathbf{R}, \text{ defined by } \prod\nolimits_1^{2m+1}(\alpha,\beta)_{G_{2m+1}} = \alpha,$$
$$\prod\nolimits_2^{2m+1} : G_{2m+1} \longmapsto \mathbf{R}, \text{ defined by } \prod\nolimits_2^{2m+1}(\alpha,\beta)G_{2m+1} = \beta.$$

In the next lemma we summarize some properties of the Lie product and the vector subspaces associated with the homological operator.

Lemma 4.2. *Let us consider* $k, l, m \in \mathbf{N}$. *Then*

(a) $[\operatorname{Rang} L_k, G_{2m+1}] \subseteq \operatorname{Rang} L_{2m+k}$.

(b) $[G_{2l+1}, G_{2m+1}] \subseteq G_{2l+2m+1}$.

(c) *If* $f \in G_{2l+1}$ *and* $U \in G_{2m+1}$ *verifies* $\prod_1^{2k+1}(U) = 0$, *then* $\prod_1^{2l+2m+1}([f, U]) = 0$.

Proof. To prove (a), we consider $g_{2m+1} \in G_{2m+1} = \operatorname{Ker} L_{2m+1}$, $h_k \in \operatorname{Rang} L_k$. Then $h_k = [U_k, f_1]$ where $U_k \in \mathcal{H}_k$. Using Jacobi's identity, we obtain

$$\begin{aligned}[g_{2m+1}, h_k] &= -[U_k, [f_1, g_{2m+1}]] - [f_1, [g_{2m+1}, U_k]] \\ &= -[f_1, [g_{2m+1}, U_k]] \in \operatorname{Rang} L_{2m+k}.\end{aligned}$$

The rest of the proof is based on the following equality:

$$\begin{aligned}[(\alpha,\beta)_{G_{2l+1}}, (A,B)_{G_{2m+1}}] &= [\alpha u_{l+1,l} + \beta v_{l+1,l}, A u_{m+1,m} + B v_{m+1,m}] \\ &= 2(l-m)A\alpha u_{l+m+1,l+m} \\ &\quad + (2lA\beta - 2mB\alpha) v_{l+m+1,l+m}.\end{aligned} \tag{4.15}$$

From this, we can prove (b) easily, and taking $A = 0$, we deduce (c). □

4.1. Hypernormal form under $\mathcal{C}^\infty$-conjugation. Let us consider $N \in \mathbf{N}$ arbitrary. In our analysis we assume that the system (4.6) has been put in normal form up to order $2N + 1$. So this system can be written as

$$
\begin{aligned}
(4.16)\qquad \dot z = f(z,\bar z) &= \sum_{j=0}^{N} f_{2j+1} + \mathcal{O}(2N+3)\\
&= (0,1)_{G_1} + \sum_{j=1}^{N}(\alpha_{2j+1},\beta_{2j+1})_{G_{2j+1}} + \mathcal{O}(2N+3).
\end{aligned}
$$

The principal result of this subsection is the following theorem, which characterizes a hypernormal form up to order $2N+1$, using $\mathcal{C}^\infty$-conjugation, for the system (4.16) (and, therefore, for the system (4.1)).

Theorem 4.3. *Let us consider the normal form* (4.16) *for the system* (4.6), *and define* $r, s \in \mathbf{N}$ *by*

$$
\begin{aligned}
(4.17)\qquad r &= \min\left\{ j \in \{1,\dots,N\} : \prod\nolimits_1^{2j+1}(f_{2j+1}) = \alpha_{2j+1} \neq 0\right\},\\
s &= \min\left\{ j \in \{1,\dots,N\} : \prod\nolimits_2^{2j+1}(f_{2j+1}) = \beta_{2j+1} \neq 0\right\},
\end{aligned}
$$

that is, α_r *and* β_s *are the first nonvanishing coefficients of the normal form* (4.16). *Then* f *is* $\mathcal{C}^\infty$*-conjugate to*

$$
(4.18)\qquad f^* = (0,1)_{G_1} + \sum_{j=1}^{N}(\alpha^*_{2j+1},\beta^*_{2j+1})_{G_{2j+1}} + \mathcal{O}(2N+3),
$$

where we have the following possibilities:

(i) *If* $s > r$, *then*

(a) $\alpha^*_{2j+1} = 0$ *for* $j = 1,\dots,N$, $j \neq r, 2r$,

(b) $\alpha^*_{2r+1} = \operatorname{sig}(\alpha_{2r+1})$,

(c) $\beta^*_{2j+1} = 0$ *for* $j = 1,\dots,N$.

(ii) *If* $s = r$, *then*

(a) $\alpha^*_{2j+1} = 0$ *for* $j = 1,\dots,N$, $j \neq r, 2r$,

(b) $\alpha^*_{2r+1} = \operatorname{sig}(\alpha_{2r+1})$,

(c) $\beta^*_{2j+1} = 0$ *for* $j = 1, \dots, N$, $j \neq r$.

(iii) *If* $s < r$, *then*

(a) $\alpha^*_{2j+1} = 0$ *for* $j = 1, \dots, r-1$,

(b) $\beta^*_{2j+1} = 0$ *for* $j = 1, \dots, s-1$,

(c) $\alpha^*_{2r+2j+1} = 0$ *or* $\beta^*_{2s+2j+1} = 0$ (*only one of them vanish*), *for* $j = 1, \dots, r-s$,

(d) $\alpha^*_{2r+1} = \operatorname{sig}(\alpha_{2r+1})$,

(e) $\alpha^*_{2j+1} = 0$ for $j = 2r - s + 1, \dots, N$, $j \neq 2r$,

(f) $\beta^*_{2j+1} = 0$ for $j = r+1, \dots, N$.

Moreover, the above are the maximum number of simplifications that can be achieved using $\mathcal{C}^\infty$*-conjugation.*

Proof. First we consider the case $s \geq r$. In this case the vector field is

$$f = f_1 + f_{2r+1} + \cdots + f_{2N+1} + \mathcal{O}(2N+3), \tag{4.19}$$

where $\alpha_{2r+1} = \prod_1^{2r+1}(f_{2r+1}) \neq 0$. Applying item (b) of Lemma 4.2, we obtain that the transformed vector field of (4.16) by the generator

$$U = (A, B)_{G_{2m+1}} \in G_{2m+1}, \quad m \geq 1, \tag{4.20}$$

is also in normal form up to order $2N+1$, that is, $(f^*)_{2j+1} \in G_{2j+1}$ for all $j = 1, \dots, N$.

As $U \in G_{2m+1} = \operatorname{Ker} L_{2m+1}$, we have $[f_1, U] = 0$. Therefore, $[f, U] = [f_{2r+1} + \cdots, U]$, and so $\mathcal{J}^{2r+2m}[f, U] = 0$. Using that $\mathcal{J}^1 U = 0$, from (2.6) and (2.8), we obtain

$$\begin{aligned} &\mathcal{J}^{2r+2m}(f^*) = \mathcal{J}^{2r+2m} f, \\ &\quad f^*_{2r+2m+1} = (f + [f, U])_{2r+2m+1} = f_{2r+2m+1} + [f_{2r+1}, U]. \end{aligned} \tag{4.21}$$

By (4.15), we get

$$\begin{aligned} &\prod\nolimits_1^{2r+2m+1}(f^*) = \alpha_{2r+2m+1} + 2(r-m)A\alpha_{2r+1}, \\ &\prod\nolimits_2^{2r+2m+1}(f^*) = \beta_{2r+2m+1} + 2rA\beta_{2r+1} - 2mB\alpha_{2r+1}. \end{aligned} \tag{4.22}$$

So, for $m \neq r$, it is possible to choose A, B in order to annihilate the terms of order $2r+2m+1$ in f^*. In the case $m = r$, we can select B such that $\prod_2^{2r+2m+1}(f^*) = 0$.

In summary, using the generator (4.20), we leave the normal form up to order $2r+2m$ unaltered, and we annihilate the coefficients of order $2r+2m+1$. The above procedure holds for all values of $m \neq r$. For $m = r$ we can only annihilate $\prod_2^{2r+2m+1}(f^*)$. In this way, making successive changes of variables over (4.16), we obtain a system (4.18), whose coefficients verify the conditions expressed in the theorem for the first two cases.

Finally, to achieve $\alpha^*_{2r+1} = \operatorname{sig}(\alpha_{2r+1})$, it is enough to consider a generator $U = (A, 0)_{G_1} \in G_1$. In this case, from (2.7), the transformed vector field (2.5) is

$$
\begin{aligned}
f^* &= f_1 + \sum_{j=r}^{N} \left[\sum_{n=0}^{\infty} \frac{(2jA)^n}{n!} f_{2j+1} \right] + \mathcal{O}(2N+3) \\
&= f_1 + \sum_{j=r}^{N} e^{2jA} f_{2j+1} + \mathcal{O}(2N+3),
\end{aligned}
\tag{4.23}
$$

and, selecting A adequately, we obtain $\prod_1^{2r+1}(f^*) = \operatorname{sig}(\alpha_{2r+1})$.

Now we consider the case $s < r$. The system (4.16) is

$$
f = f_1 + f_{2s+1} + \cdots + f_{2r+1} + \cdots + f_{2N+1} + \mathcal{O}(2N+3), \tag{4.24}
$$

where

$$
\begin{aligned}
&\prod\nolimits_1^{2j+1}(f_{2j+1}) = \alpha_{2j+1} = 0, \quad \text{for } j = s, \dots, r-1 \\
&\prod\nolimits_1^{2r+1}(f_{2r+1}) = \alpha_{2r+1} \neq 0, \\
&\prod\nolimits_2^{2s+1}(f_{2s+1}) = \beta_{2s+1} \neq 0.
\end{aligned}
\tag{4.25}
$$

For our convenience, the procedure will be carried out in two steps. First we consider a generator

$$
U = (A, 0)_{G_{2m+1}} \in G_{2m+1}. \tag{4.26}
$$

Using the same ideas as in the proof of the first two items, but now taking $B = 0$ and s instead of r, we deduce that the transformed vector field f^* remains unalterated up to order $2s+2m+1$, that is, $\mathcal{J}^{2s+2m}(f^*) = \mathcal{J}^{2s+2m} f$. Moreover, from the expressions for the terms of order $2s+2m+1$: $f^*_{2s+2m+1} = f_{2s+2m+1} + [f_{2s+1}, U]$; and using that $\alpha_{2s+1} = 0$, we obtain

$$\begin{aligned} &\prod\nolimits_1^{2s+2m+1}(f^*_{2s+2m+1}) = \alpha_{2s+2m+1}, \\ &\prod\nolimits_2^{2s+2m+1}(f^*_{2s+2m+1}) = \beta_{2s+2m+1} + 2sA\beta_{2s+1}. \end{aligned} \tag{4.27}$$

With respect to the terms of order greater than $2s+2m+1$, we note that $\prod_1^{2k-1}(f) = \alpha_{2k-1} = 0$ for $k \le r$. Applying item (c) of Lemma 4.2, we obtain $\prod_1^{2k-1}(T_U(f)) = 0$ for $k = 1, \dots, r+m$. Analogously, it is obtained that $\prod_1^{2k-1}(T_U^n(f)) = 0$ for any $n > 1$. From (2.5), we get

$$\prod\nolimits_1^{2k-1}(f^*) = \alpha_{2k-1}, \quad \text{for } k = 1, \dots, r+m. \tag{4.28}$$

Using (4.15), we obtain the expressions for the terms of order $2r+2m+1$:

$$\prod\nolimits_1^{2r+2m+1}(f^*) = \alpha_{2r+2m+1} + 2(r-m)A\alpha_{2r+1}. \tag{4.29}$$

As $\alpha_{2r+1} \neq 0$, $\beta_{2s+1} \neq 0$, we deduce that, for $m \neq r$, it is possible to achieve $\prod_2^{2s+2m+1}(f^*) = 0$ or $\prod_1^{2r+2m+1}(f^*) = 0$ by selecting A adequately. In the case $m = r$, there is only one possibility: to select A such that $\prod_2^{2s+2m+1}(f^*) = 0$.

In this way we make a change corresponding to a generator (4.26) with $m = 1$, and we can annihilate α^*_{2r+3} or β^*_{2s+3}. Next we make another change corresponding to $m = 2$, and we get $\alpha^*_{2r+5} = 0$ or $\beta^*_{2s+5} = 0$. This last change does not modify the terms annihilated before. We proceed successively increasing m up to $m = N - r$, except for $m = r$, where only we can annihilate $\beta^*_{2s+2r+1}$.

In the second step, we will use a generator

$$U = (0, B)_{G_{2m+1}} \in G_{2m+1}. \tag{4.30}$$

In this case, we have $\prod_1^{2k-1}(U) = 0$ for $k = 1, \dots, m+1$. Applying item (c) of Lemma 4.2 we obtain $\prod_1^{2k-1}(T_U(f)) = 0$ for $k = 1, \dots, r+m$.

Also we have $\prod_1^{2k-1}(f) = 0$ for $k = 1, \dots, r$. Using (4.15) with $\alpha = A = 0$, we deduce that $\prod_2^{2k-1}([f, U]) = 0$ for $k = 1, \dots, r+m$.

In summary, $\mathcal{J}^{2r+2m-1}[f, U] = 0$. Using (2.6) we obtain that the transformed vector field remains unalterated up to order $2r+2m-1$, i.e., $\mathcal{J}^{2r+2m-1}(f^*) = \mathcal{J}^{2r+2m-1}f$. With respect to the $(2r+2m+1)$-order terms, using (4.15) with $A = 0$, we obtain $\prod_1^{2r+2m+1}(f^*) = \alpha_{2r+2m+1}$.

Moreover, for $k = 1, \dots, N+1$, we have $\prod_1^{2k-1}(U) = 0$. Using item (c) of Lemma 4.2, we conclude that $\prod_1^{2k-1}([f, U]) = 0$, and also $\prod_1^{2k-1}(T_U^n(f)) = 0$ for all $n \geq 1$. So, from (2.5), we deduce $\prod_1^{2k-1}(f^*) = \alpha_{2k-1}$ for $k = 1, \dots, N+1$.

On the other hand, using (4.15) with $A = 0$, we obtain

$$\prod\nolimits_2^{2r+2m+1}(f^*) = \beta_{2r+2m+1} - 2mB\alpha_{2r+1}, \tag{4.31}$$

and it is possible to select B such that $\prod_2^{2r+2m+1}(f^*) = 0$. In other words, this change does not affect the terms α^*_{2k-1}, nor the terms β_{2k-1} with $k \leq r+m$, that we have previously annihilated. Moreover, we annihilate $\beta^*_{2r+2m+1}$. Taking successively $m = 1, 2, \dots$, we complete the proof of item (c) of the theorem.

Finally we will show that we have achieved the highest simplification. For this, it suffices to prove that the elements not used in the generator U do not produce additional simplifications. The quoted elements are of three kinds:

(a) Elements belonging to the range of the homological operator. These elements do not affect the complementary subspaces G_{2m+1}, see item (a) of Lemma 4.2.

(b) The element $(0, B)_{G_1}$, which does not affect G_{2m+1}, see (4.15) with $A = 0$, $m = 0$.

(c) The element $(A, 0)_{G_{2r+1}}$, which appears multiplied by zero, see (4.22) or (4.29). □

We note that the procedure carried out in the proof involves only the resolution of linear equations to determine the coefficients of the normal form that can be annihilated. A priori, these hypernormal forms are not necessarily the simplest ones (this kind of simplified normal form was called pseudohypernormal form). Nevertheless, we have shown that no further simplifications can be achieved by coordinate transformations. Therefore, we can assert that we have obtained a hypernormal form (the simplest that can be obtained via $\mathcal{C}^\infty$-conjugacy).

From Theorem 4.3, it is easy to obtain the following corollary, where we present a hypernormal form, based uniquely on the knowledge of the first nonvanishing coefficient α_{2r+1}.

Corollary 4.4. *Let us consider the system* (4.16)*, and let* $r = \min\{j \in \mathbf{N} : \alpha_{2j+1} \neq 0\}$*. Then* (4.16) *is* $\mathcal{C}^\infty$*-conjugate to*

$$(0,1)_{G_1} + (\alpha^*_{2r+1}, \beta^*_{2r+1})_{G_{2r+1}} + \sum_{j=1}^{r-1} (0, \beta^*_{2j-1})_{G_{2j+1}} + (\alpha^*_{4r+1}, 0)_{G_{4r+1}} + \mathcal{O}(2N+3), \tag{4.32}$$

where $\alpha^*_{2r+1} = \pm 1$*.*

4.2. Hypernormal form under $\mathcal{C}^\infty$-equivalence. The use of $\mathcal{C}^\infty$-equivalence allows further simplifications. In this case the hypernormal form is characterized by the first nonvanishing coefficient α_{2r+1} in the normal form (4.16).

Theorem 4.5. *Let us consider the system* (4.16)*, and let* $r = \min\{j \in \mathbf{N} : \alpha_{2j+1} \neq 0\}$*. Then* (4.16) *is* $\mathcal{C}^\infty$*-equivalent to*

$$(0,1)_{G_1} + (\alpha^*_{2r+1}, \beta^*_{2r+1})_{G_{2r+1}} + (\alpha^*_{4r+1}, 0)_{G_{4r+1}} + \mathcal{O}(2N+3), \tag{4.33}$$

where $\alpha^*_{2r+1} = \pm 1$ *and* $\alpha^*_{4r+1} = 0$ *or* $\beta^*_{2r+1} = 0$ (*only one of them vanishes*)*.*

Proof. We assume, without loss of generality, that we have applied Theorem 4.3, and so, system (4.16) has been put in hypernormal form. We consider two possibilities:

(a) We suppose that we deal with items (i) or (ii) of Theorem 4.3. System (4.16) can be written as

$$\dot{z} = f(z, \bar{z}) = (0,1)_{G_1} + (\alpha_{2r+1}, \beta_{2r+1})_{G_{2r+1}} + (\alpha_{4r+1}, 0)_{G_{4r+1}} + \mathcal{O}(2N+3), \tag{4.34}$$

where $\alpha_{2r+1} = \pm 1$ and $\beta_{2r+1} = 0$ in item (i) or $\beta_{2r+1} \neq 0$ in item (ii).

Multiplying the vector field by $P(z, \bar{z}) = 1 + A(z\bar{z})^r$, we get

$$(0,1)_{G_1} + (\alpha_{2r+1}, \beta_{2r+1} + A)_{G_{2r+1}} + (\alpha_{4r+1} + A_{\alpha_{2r+1}}, A\beta_{2r+1})_{G_{4r+1}} + (A\alpha_{4r+1}, 0)_{G_{6r+1}} + \mathcal{O}(2N+3). \tag{4.35}$$

As $\alpha_{2r+1} \neq 0$ and $\beta_{2j+1} = 0$ for $j = 1, \ldots, r-1$, using Theorem 4.3 we can annihilate α^*_{6r+1} and β^*_{4r+1}. Moreover, the vector field obtained is

$$(0,1)_{G_1} + (\alpha_{2r+1}, \beta_{2r+1} + A)_{G_{2r+1}} + (\alpha_{4r+1} + A\alpha_{2r+1}, 0)_{G_{4r+1}} + \mathcal{O}(2N+3). \tag{4.36}$$

Choosing A adequately, it is possible to achieve $\alpha^*_{4r+1} = 0$ or $\beta^*_{2r+1} = 0$.

(b) In case (iii) of Theorem 4.3, the vector field is given by

$$(0,1)_{G_1} + \sum_{j=s}^{r-1} (0, \beta_{2j+1})_{G_{2j+1}} + (\alpha_{2r+1}, \beta_{2r+1})_{G_{2r+1}} + \sum_{j=r+1}^{2r-s} (\alpha_{2j+1}, 0)_{G_{2j+1}} + (\alpha_{4r+1}, 0)_{G_{4r+1}} + \mathcal{O}(2N+3), \tag{4.37}$$

where $\alpha_{2r+2j+1} = 0$ or $\beta_{2s+2j+1} = 0$ for $j = 1, \ldots, r-s$ and $\alpha_{2r+1} = \pm 1$.

In particular, we consider the vector field

$$(0,1)_{G_1} + (0, \beta_{2s+1})_{G_{2s+1}} + \sum_{j=r}^{2r-s} (\alpha_{2j+1}, 0)_{G_{2j+1}} + (\alpha_{4r+1}, 0)_{G_{4r+1}} + \mathcal{O}(2N+3). \tag{4.38}$$

We take $n = \min\{j \in \mathbf{N} : 2(j+1)s \geq 2N+2\}$, and multiplying the vector field by

$$P(z, \bar{z}) = 1 - \beta_{2s+1}(z\bar{z})^s + \beta^2_{2s+1}(z\bar{z})^{2s} - \cdots + (-1)^n \beta^n_{2s+1}(z\bar{z})^{ns}. \tag{4.39}$$

So we obtain

$$(0,1)_{G_1} + \sum_{j=r}^{N} (\tilde{\alpha}_{2j+1}, 0)_{G_{2j+1}} + \mathcal{O}(2N+3). \tag{4.40}$$

This last system corresponds to item (i) of Theorem 4.3. Applying the reasoning carried out in case (a) before, we complete the proof. □

Remark 1. The result given in Takens [**16**] is obtained when we choose $\beta^*_{2r+1} = 0$.

Remark 2. In the case $r = +\infty$ that corresponds to a center at the origin for system (4.16), a hypernormal form under $\mathcal{C}^\infty$-equivalence is $f^* = (0,1)_{G_1} + \mathcal{O}(2N+3)$ for all $N \in \mathbf{N}$.

4.3. Some particular cases. In this subsection we present hypernormal forms up to order ∞ (using both $\mathcal{C}^\infty$-conjugation and $\mathcal{C}^\infty$-equivalence) for some particular cases, corresponding to lower degeneracies.

In these examples we show that our approach is useful not only to determine the structure of the hypernormal form, but also to compute its coefficients.

(a) Let us assume that the coefficients of the normal form (4.16) of the system (4.1) satisfy $\alpha_3 \neq 0$ (Proposition 2.2 of Chua and Kokubu [**7**] corresponds to this particular case). Then

(i) a hypernormal form up to order ∞ under $\mathcal{C}^\infty$-conjugacy is

$$\dot r = \alpha_3^* r^3 + \alpha_5^* r^5, \qquad \dot\theta = 1 + \beta_3^* r^2, \tag{4.41}$$

where

$$\alpha_3^* = \alpha_3, \qquad \alpha_5^* = \alpha_5, \qquad \beta_3^* = \beta_3.$$

(ii) Two different hypernormal forms up to order ∞ under $\mathcal{C}^\infty$-equivalence are

$$\dot r = \alpha_3^* r^3, \qquad \dot\theta = 1+\beta_3^* r^2 \quad \text{and} \quad \dot r = \alpha_3^* r^3 + \alpha_5^* r^5, \qquad \dot\theta = 1, \tag{4.42}$$

where

$$\alpha_3^* = \alpha_3, \qquad \alpha_5^* = \alpha_5 - \alpha_3\beta_3, \qquad \beta_3^* = \beta_3 - \frac{\alpha_5}{\alpha_3}.$$

In the following cases, there exist several possibilities, but for the sake of brevity we only present one of them.

(b) Let us assume that the coefficients of the normal form (4.16) of system (4.1) satisfy $\alpha_3 = 0$, $\alpha_5 \neq 0$. Then

(i) A hypernormal form up to order ∞ under $\mathcal{C}^\infty$-conjugacy is

$$\dot{r} = \alpha_5^* r^5 + \alpha_9^* r^9, \qquad \dot{\theta} = 1 + \beta_3^* r^2 + \beta_5^* r^4, \tag{4.43}$$

where

$$\alpha_5^* = \alpha_5, \qquad \alpha_9^* = \alpha_9 - \frac{\alpha_7^2}{\alpha_5}, \qquad \beta_3^* = \beta_3, \qquad \beta_5^* = \beta_5 - \frac{\beta_3 \alpha_7}{\alpha_5}.$$

(ii) A hypernormal form up to order ∞ under $\mathcal{C}^\infty$-equivalence is

$$\dot{r} = \alpha_5^* r^5, \qquad \dot{\theta} = 1 + \beta_5^* r^4, \tag{4.44}$$

where

$$\alpha_5^* = \alpha_5, \qquad \beta_5^* = \beta_5 + \frac{\alpha_7^2 - \beta_3 \alpha_5 \alpha_7 - \alpha_5 \alpha_9}{\alpha_5^2}.$$

(c) Let us assume that the coefficients of the normal form (4.16) of system (4.1) satisfy $\alpha_3 = \alpha_5 = 0$, $\alpha_7 \neq 0$. Then

(i) A hypernormal form up to order ∞ under $\mathcal{C}^\infty$-conjugacy is

$$\dot{r} = \alpha_7^* r^7 + \alpha_{13}^* r^{13}, \qquad \dot{\theta} = 1 + \beta_3^* r^2 + \beta_5^* r^4 + \beta_7^* r^6, \tag{4.45}$$

where

$$\alpha_7^* = \alpha_7, \qquad \alpha_{13}^* = \alpha_{13} + \frac{\alpha_9^3 - 2\alpha_7 \alpha_9 \alpha_{11}}{\alpha_7^2},$$
$$\beta_3^* = \beta_3, \qquad \beta_5^* = \beta_5 - \frac{\beta_3 \alpha_9}{2\alpha_7},$$
$$\beta_7^* = \beta_7 - \frac{\beta_3 \alpha_7 \alpha_{11} - \beta_3 \alpha_9^2 + \beta_5 \alpha_7 \alpha_9}{\alpha_7^2}.$$

(ii) A hypernormal form up to order ∞ under $\mathcal{C}^\infty$-equivalence is

$$\dot{r} = \alpha_7^* r^7, \qquad \dot{\theta} = 1 + \beta_7^* r^6, \tag{4.46}$$

where $\alpha_7^* = \alpha_7$,

$$\beta_7^* = \beta_7 - \frac{\beta_5 \alpha_7^2 \alpha_9 + \alpha_7^2 \alpha_{13} - 2\alpha_7 \alpha_9 \alpha_{11} - \beta_3 \alpha_7 \alpha_9^2 + \beta_3 \alpha_7^2 \alpha_{11} + \alpha_9^3}{\alpha_7^3}.$$

(d) Let us assume that the coefficients of the normal form (4.16) of system (4.1) satisfy $\alpha_3 = \alpha_5 = \alpha_7 = 0$, $\alpha_9 \neq 0$. Then

(i) A hypernormal form up to order ∞ under $\mathcal{C}^\infty$-conjugacy is

$$\dot{r} = \alpha_9^* r^9 + \alpha_{17}^* r^{17}, \qquad \dot{\theta} = 1 + \beta_3^* r^2 + \beta_5^* r^4 + \beta_7^* r^6 + \beta_9^* r^8, \tag{4.47}$$

where $\alpha_9^* = \alpha_9$,

$$\alpha_{17}^* = \alpha_{17} + \frac{-2\alpha_9^2\alpha_{11}\alpha_{15} - \alpha_{11}^4 + 3\alpha_9\alpha_{11}^2\alpha_{13} - \alpha_9^2\alpha_{13}^2}{\alpha_9^3},$$

$$\beta_3^* = \beta_3, \qquad \beta_5^* = \beta_5 - \frac{\beta_3\alpha_{11}}{3\alpha_9},$$
$$\beta_7^* = \beta_7 - \frac{12\beta_5\alpha_9\alpha_{11} + 9\beta_3\alpha_9\alpha_{13} - 8\beta_3\alpha_{11}^2}{18\alpha_9^2},$$

$$\begin{aligned}\beta_9^* = \beta_9 &\\ &- \frac{\beta_5\alpha_9(\alpha_9\alpha_{13} - \alpha_{11}^2) + \beta_7\alpha_9^2\alpha_{11} + \beta_3(\alpha_9^2\alpha_{15} - 2\alpha_9\alpha_{11}\alpha_{13} + \alpha_{11}^3)}{\alpha_9^3}.\end{aligned}$$

(ii) A hypernormal form up to order ∞ under $\mathcal{C}^\infty$-equivalence is

$$\dot{r} = \alpha_9^* r^9, \qquad \dot{\theta} = 1 + \beta_9^* r^8, \tag{4.48}$$

where $\alpha_9^* = \alpha_9$,

$$\begin{aligned}\beta_9^* = \beta_9 - (&\alpha_9^3\alpha_{17} - 2\beta_3\alpha_9^2\alpha_{11}\alpha_{13} - \alpha_{11}^4 + \beta_3\alpha_9^3\alpha_{15} - \alpha_9^2\alpha_{13}^2 + \beta_3\alpha_9\alpha_{11}^3\\ &- \beta_5\alpha_9^2\alpha_{11}^2 + 3\alpha_9\alpha_{11}^2\alpha_{13} + \beta_5\alpha_9^3\alpha_{13} + \beta_7\alpha_9^3\alpha_{11} - 2\alpha_9^3\alpha_{11}\alpha_{15})/\alpha_9^4.\end{aligned}$$

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REFERENCES

1. A. Algaba, E. Freire and E. Gamero, *On the hypernormal form for the Takens-Bogdanov singularity*, preprint, 1998.

2. A. Algaba, E. Freire and E. Gamero, *Hypernormal form for the Hopf-zero bifurcation*, accepted for publication in Internat. J. Bifur. Chaos, 1997.

3. M. Ashkenazi and Shui-Nee Chow, *Normal forms near critical points for differential equations and maps*, IEEE Trans. Circ. Syst. **35** (1988), 850–862.

4. A. Baider and J.A. Sanders, *Further reduction of the Takens-Bogdanov normal form*, J. Differential Equations **99** (1992), 205–244.

5. S.N. Chow and J.K. Hale, *Methods of bifurcation theory*, Springer-Verlag, New York, 1982.

6. L.O. Chua and H. Kokubu, *Normal forms of nonlinear vector fields*, Part I: *Theory and algorithm*, IEEE Trans. Circ. Syst. **35** (1988), 863–880.

7. ———, *Normal forms of nonlinear vector fields*, Part II: *Applications*, IEEE Trans. Circ. Syst. **36** (1989), 51–70.

8. A. Deprit, *Canonical transformations depending on a small parameter*, Celest. Mechan. **1** (1969), 12–324.

9. F. Dumortier, R. Roussarie and J. Sotomayor, *Generic three-parameter families of planar vector fields, unfolding of saddle, focus and elliptic singularities with nilpotent linear parts*, Lecture Notes in Math. **1480** (1991), 1–64.

10. C. Elphick, E. Tirapegui, M.E. Brachet, P. Coullet and G. Iooss, *A simple global characterization for normal forms of singular vector fields*, Physica D **29** (1987), 95–127.

11. E. Gamero, E. Freire and E. Ponce, *On the normal forms for planar systems with nilpotent linear part*, in *Bifurcation and chaos: Analysis, algorithms, applications*, Internat. Ser. Numer. Math. **97**, 1991, 123–127.

12. E. Gamero, E. Freire and A.J. Rodríguez-Luis, *Hopf-zero bifurcation: Normal form calculation and application to an electronic oscillator*, Proceedings of Equadiff '91, 1993, 517–524.

13. E. Gamero, E. Freire, A.J. Rodríguez-Luis, E. Ponce and A. Algaba, *Hypernormal form calculation for triple-zero degeneracy*, accepted for publication in Bulletin of the Belgian Mathematical Society–Simon Stevin, 1993.

14. J. Guckenheimer and P.J. Holmes, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, Springer, Berlin, 1983.

15. A.J. Lichtenberg and M.A. Liebermann, *Regular and chaotic dynamics*, Second edition, Springer, New York, 1992.

16. F. Takens, *Normal forms for certain singularities of vector fields*, Ann. Inst. Fourier **23** (1973), 163–195.

17. S. Ushiki, *Normal forms for singularities of vector fields*, Japan J. Appl. Math. **1** (1984), 1–37.

18. A. Vanderbauwhede, *Centre manifolds, normal forms and elementary bifurcations*, Dynamics Reported **2** (1989), 89–169.

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