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A NOTE ON PI INCIDENCE ALGEBRAS

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ABSTRACT. The question when the incidence algebra of a locally finite partially ordered set over a commutative ring with identity is a polynomial identity algebra is discussed. When the incidence algebra is a PI algebra, all its minimal degree, linear, homogeneous identities are determined.

Let R be a commutative ring with identity and $R[X_1, X_2, \dots]$ the ring of polynomials in the noncommuting indeterminates $X_1, X_2, \dots$ over R. A polynomial of the form $rX_{i_1}^{e_1}X_{i_2}^{e_2}\cdots X_{i_n}^{e_n}$, with $r \in R$, i_j a positive integer and e_j a nonnegative integer for $j = 1, 2, \dots, n$, and $i_j \neq i_{j+1}$ for $j = 1, 2, \dots, n-1$, is called a monomial. It has coefficient r and degree $\sum_{j=1}^n e_j$. An element $f \in R[X_1, X_2, \dots]$ is called a polynomial. It is the sum of monomials and has degree, $\deg(f)$, the maximal degree of the monomials in f. If all monomials in f have the same degree, then f is a homogeneous polynomial. If none of the indeterminates X_i for $i > n$ appear in any monomial of f, we write $f = f(X_1, X_2, \dots, X_n)$. If A is an algebra over R, we write $p(r_1, r_2, \dots, r_n)$ for the element of R obtained when r_j is substituted for X_j, for $j = 1, 2, \dots, n$.

If A is an algebra over R, a nonzero polynomial $f = f(X_1, X_2, \dots, X_n) \in R[X_1, X_2, \dots]$ is an identity for A if for every $\alpha_1, \alpha_2, \dots, \alpha_n \in A$ we have $f(\alpha_1, \alpha_2, \dots, \alpha_n) = 0$. If, in addition, a monomial of f of highest degree has coefficient the identity of R, then f is a polynomial identity for A. The algebra A is a polynomial identity algebra, PI algebra, if A has a polynomial identity. If f is a polynomial identity for A, then from a result of Kaplansky [**4**], A satisfies a homogeneous polynomial identity of the form $X_1X_2\cdots X_n + p$, where p is the sum of monomials of the form $r_\sigma X_{\sigma(1)}X_{\sigma(2)}\cdots X_{\sigma(n)}$, with $r_\sigma \in R$, σ a nonidentity element of the symmetric group S_n, and $n \leq \deg(f)$. The polynomial $s(X_1, X_2, \dots, X_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) X_{\sigma(1)}X_{\sigma(2)}\cdots X_{\sigma(n)}$ is called the standard polynomial of degree n. Here $\operatorname{sgn}(\sigma)$ denotes the sign of σ. The famous Amitsur-Levitzki theorem [**1**] tells us that

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$M_n(R)$, the algebra of $n \times n$ matrices over R, is a PI algebra and satisfies the standard polynomial of degree $2n$.

In this note we are concerned with PI incidence algebras. Recall that a partially ordered set X is locally finite if $|\{y \in X | x \leq y \leq z\}| < \infty$ for each $x, z \in X$. If $I(X,R) = \{f : X \times X \to R | f(x,y) = 0 \text{ if } x \not\leq y\}$ with the operations

$$(f+g)(x,y) = f(x,y) + g(x,y)$$
$$fg(x,y) = \sum_{x \leq z \leq y} f(x,z)g(z,y)$$
$$(rf)(x,y) = rf(x,y)$$

for

$$f, g \in I(X,R), \quad r \in R, \quad x, y \in X,$$

then $I(X,R)$ is an R algebra, called the incidence algebra of X over R.

If $X = \{x_1, x_2, \dots, x_n\}$ is a set of cardinality n, then the map $\rho : I(X,R) \to M_n(R)$ given by $\rho(f)$ in the (i,j) position has value $f(x_i, x_j)$, is an injective homomorphism. We then see that, for $|X| = n$, $I(X,R)$ satisfies the standard polynomial of degree $2n$. If we had listed the elements of X so that when $x_j < x_k$, then $j < k$, then $\rho(f)$ is an upper triangular matrix for each $f \in I(X,R)$.

If X is a partially ordered set, call $x_1, x_2, \dots, x_n \in X$ a chain of length n if $x_1 < x_2 < \cdots < x_n$. The poset X is bounded if for some positive integer n it has a chain of length n and does not have any chain of length $n+1$. In that case we will say X has bound n. By making use of the Amitsur-Levitzki theorem, Feinberg [**3**], Leroux and Sarraillé [**5**] and Nacev [**6**] have each shown that the incidence algebra of a bounded locally finite poset of bound n satisfies the standard polynomial of degree $2n$ and does not satisfy any polynomial of lower degree. The standard polynomial of degree $2n$ is a homogeneous polynomial having $(2n)!$ terms. We show that this incidence algebra will satisfy a homogeneous polynomial, q, of degree $2n$ having only 2^n terms and that all minimal homogeneous polynomials satisfied by the incidence algebra are obtained from the polynomial q. Farkas [**2**] considers when the incidence algebra is algebraic and utilizes a polynomial allied to q. We define q in the following theorem.

If $y_1, y_2 \in X$ with $y_1 \leq y_2$, let $\delta_{y_1 y_2}$ denote the element of $I(X,R)$ given by $\delta_{y_1 y_2}(y_1, y_2) = 1$ and zero otherwise.

Theorem 1. *Let X be a locally finite partially ordered set and R a commutative ring with identity. Then $I(X,R)$ is a* PI *algebra if and only if X is bounded. If X has bound n, then*

$$q(X_1, X_2, \dots, X_n) = \prod_{i=1}^{n} (X_{2i-1}X_{2i} - X_{2i}X_{2i-1})$$

is a polynomial identity of degree $2n$ for $I(X,R)$, and no nonzero polynomial identity of lower degree is a polynomial identity for $I(X,R)$.

Proof. Suppose that $y_1 < y_2 < \cdots < y_m$ is a chain in X and there exists a polynomial identity for $I(X,R)$ of degree k, with $k < 2m$. By the theorem of Kaplansky, $I(X,R)$ then satisfies a homogeneous polynomial identity of degree at most k and hence a homogeneous polynomial identity of degree $2m-1$. If $p(X_1, X_2, \dots, X_{2m-1})$ denotes such an identity, we can assume that $p(X_1, X_2, \dots, X_{2m-1}) = X_1X_2 \cdots X_{2m-1} + \bar{p}(X_1, X_2, \dots, X_{2m-1})$ where $\bar{p}(X_1, X_2, \dots, X_{2m-1})$ is a homogeneous polynomial of degree $2m-1$ which does not contain a monomial of the form $rX_1X_2 \cdots X_{2m-1}$ for a nonzero $r \in R$. Then

$$\begin{aligned} p(\delta_{y_1y_1}, \delta_{y_1y_2}, \delta_{y_2y_2}, \dots, \delta_{y_{m-1}y_{m-1}}, \delta_{y_{m-1}y_m}, \delta_{y_my_m}) &\\ = \delta_{y_1y_1}\delta_{y_1y_2}\delta_{y_2y_2} \cdots \delta_{y_my_m} &\\ + \bar{p}(\delta_{y_1y_1}, \delta_{y_1y_2}, \delta_{y_2y_2}, \dots, \delta_{y_my_m}). \end{aligned}$$

Since the product of the delta functions is nonzero for the unique order immediately following the equal sign in the above equation, it follows that $\bar{p}(\delta_{y_1y_1}, \delta_{y_1y_2}, \dots, \delta_{y_my_m}) = 0$. This says that $p(\delta_{y_1y_1}, \delta_{y_1y_2}, \dots, \delta_{y_my_m})(y_1, y_m) = 1$, which contradicts the supposition of a polynomial identity of degree $2m-1$. We conclude that, when X has a chain of length m, any polynomial identity must be of degree at least $2m$. In particular, if X is unbounded, then $I(X,R)$ is not a PI algebra.

To complete the proof, it suffices to show that, when X is of bound n, then $q(X_1, X_2, \dots, X_{2n})$ is a polynomial identity for $I(X,R)$. Let $Z(I(X,R)) = \{f \in I(X,R) \mid f(x,x) = 0 \text{ for all } x \in X\}$. Then $Z(I(X,R))$ is an ideal of $I(X,R)$. Further, since X is of bound n, any product of the form $g_1g_2 \cdots g_n$, with $g_i \in Z(I(X,R))$ for $i = 1, 2, \dots, n$ is zero, and so $Z(I(X,R))$ is a nilpotent ideal with $(Z(I(X,R)))^n = 0$. If $f, g \in I(X,R)$, then $fg - gf \in Z(I(X,R))$ and so $q(f_1, f_2, \dots, f_{2n}) =$

$\prod_{i=1}^{n}(f_{2i-1}f_{2i} - f_{2i}f_{2i-1})$ is the product of n elements of $Z(I(X,R))$, and hence zero, for any $f_1, f_2, \dots, f_{2n} \in I(X,R)$. The result then follows. $\square$

We have seen that when X is finite of bound n, then both the standard polynomial $s(X_1, X_2, \dots, X_{2n})$ and $q(X_1, X_2, \dots, X_{2n})$ are each homogeneous, minimal degree polynomial identities for $I(X,R)$. The first of these polynomials is the sum of $(2n)!$ monomials while the second of these polynomials is the sum of only 2^n monomials. We now show that all homogeneous, minimal degree polynomial identities can be generated from $q(X_1, X_2, \dots, X_{2n})$. We will need to introduce some notation.

Let n be a positive integer and $A = a_1 a_2 \cdots a_{2n}$, $B = b_1 b_2 \cdots b_{2n}$ permutations of $\{1, 2, \dots, 2n\}$. Here $\{a_1, a_2, \dots, a_{2n}\} = \{b_1, b_2, \dots, b_{2n}\} = \{1, 2, \dots, 2n\}$. We will call the permutation A neighbor similar to the permutation B if $\{a_{2i-1}, a_{2i}\} = \{b_{2i-1}, b_i\}$ for $i = 1, 2, \dots, n$. Neighbor similarity is an equivalence relation, and the equivalence class of A contains a unique permutation $\overline{A} = \bar{a}_1 \bar{a}_2 \cdots \bar{a}_{2n}$ such that $\bar{a}_{2i-1} < \bar{a}_{2i}$ for $i = 1, 2, \dots, n$. We refer to $\overline{A}$ as the canonical representative of the equivalence class of A. Each equivalence class has 2^n elements and there are $(2n)!/2^n$ classes. The permutation A can be associated with the element $\sigma_A \in S_{2n}$ given by $\sigma_A(i) = a_i$ for $i = 1, 2, \dots, 2n$. Distinct permutations are associated with distinct elements of S_{2n}. Let $C = \{\sigma_{\overline{A}} \mid A \text{ is a permutation of } \{1, 2, \dots, 2n\}\}$. Further, for $\sigma \in C$, we let q^σ denote the polynomial $q(X_{\sigma(1)}, X_{\sigma(2)}, \dots, X_{\sigma(2n)})$. Certainly, as $q(X_1, X_2, \dots, X_{2n})$ is a polynomial identity for $I(X,R)$ so is q^σ. Since $s(X_1, X_2, \dots, X_{2n})$, the standard polynomial in $2n$ variables, equals $\sum_{\sigma \in C} \operatorname{sgn}(\sigma) q^\sigma$, which is a linear combination of q polynomials, we have retrieved the result of Feinberg, Leroux and Sarraillé, and Nacev. In this direction we consider which polynomials are linear combinations of q polynomials. For the given incidence algebra $I(X,R)$, let $LHP(X_1, X_2, \dots, X_{2n})$ denote the set of all linear homogeneous polynomials in $X_1, X_2, \dots, X_{2n}$, which are identities for $I(X,R)$.

Theorem 2. *Let X be a locally finite, bounded, partially ordered set with bound n, and R a commutative ring with identity. Then $LHP(X_1, X_2, \dots, X_{2n})$ is a free R module with basis $Q = \{q^\sigma \mid \sigma \in C\}$.*

Proof. Since the monomials in distinct elements of Q are linearly independent in $R[X_1, X_2, \ldots, X_{2n}]$, we have that the elements of Q are linearly independent. We must check that they span $LHP(X_1, X_2, \ldots, X_{2n})$. Let $\phi(X_1, X_2, \ldots, X_{2n}) \in LHP(X_1, X_2, \ldots, X_{2n})$. Then ϕ is a linear homogeneous polynomial of degree $2n$, and we can write

$$\phi(X_1, X_2, \ldots, X_{2n}) = \sum_{\tau \in S_{2n}} r_\tau X_{\tau(1)} X_{\tau(2)} \cdots X_{\tau(2n)},$$

where $r_\tau \in R$. Suppose that $\tau_1, \tau_2 \in S_{2n}$ are associated with neighbor similar permutations A_1 and A_2 of $\{X_1, X_2, \ldots, X_{2n}\}$. Here $A_1 = X_{\tau_1(1)} X_{\tau_1(2)} \cdots X_{\tau_1(2n)}$, $A_2 = X_{\tau_2(1)} X_{\tau_2(2)} \cdots X_{\tau_2(2n)}$, with $\{\tau_1(2i-1), \tau_1(2i)\} = \{\tau_2(2i-1), \tau_2(2i)\}$, for $i = 1, 2, \ldots, n$. Let $N(\tau_1, \tau_2)$ equal the number of integers i with $1 \leq i \leq n$ such that $\tau_1(2i-1) \neq \tau_2(2i-1)$. If $q^\sigma \in Q$ is the unique polynomial having a monomial of the form $cX_{\tau_1(1)} X_{\tau_1(2)} \cdots X_{\tau_1(2n)}$ with $c \neq 0 \in R$, then $(-1)^{N(\tau_1, \tau_2)} c X_{\tau_2(1)} X_{\tau_2(2)} \cdots X_{\tau_2(2n)}$ is another monomial in q^σ. Since c is a unit in R, in particular 1 or -1, in order to verify the theorem it is sufficient to check that $r_{\tau_1} = (-1)^{N(\tau_1, \tau_2)} r_{\tau_2}$. We check this by induction on $N(\tau_1, \tau_2)$.

Suppose that $N(\tau_1, \tau_2) = 1$ with τ_1 and τ_2 associated with the neighbor similar permutations A_1 and A_2 of $\{X_1, X_2, \ldots, X_{2n}\}$. Then there is a unique integer i, with $1 \leq i \leq n$, such that $X_{\tau_1(2i-1)} \neq X_{\tau_2(2i-1)}$. It then follows that

$$\begin{aligned} &r_{\tau_1} X_{\tau_1(1)} X_{\tau_1(2)} \cdots X_{\tau_1(2n)} + r_{\tau_2} X_{\tau_2(1)} X_{\tau_2(2)} \cdots X_{\tau_2(2n)} \\ &\quad = X_{\tau_1(1)} X_{\tau_1(2)} \cdots X_{\tau_1(2i-2)} (r_{\tau_1} X_{\tau_1(2i-1)} X_{\tau_1(2i)} \\ &\qquad + r_{\tau_2} X_{\tau_1(2i)} X_{\tau_1(2i-1)}) X_{\tau_1(2i+1)} X_{\tau_1(2i+1)} X_{\tau_1(2i+2)} \cdots X_{\tau_1(2n)}. \end{aligned}$$

Let $y_1 < y_2 < \cdots < y_n$ be a chain in X. We will evaluate $\phi(X_1, X_2, \ldots, X_{2n})$ when $X_{\tau_1(1)} = \delta_{y_1 y_1}$, $X_{\tau_1(2)} = \delta_{y_1 y_2}$, ..., $X_{\tau_1(2i-3)} = \delta_{y_{i-1} y_{i-1}}$, $X_{\tau_1(2i-2)} = \delta_{y_{i-1} y_i}$, $X_{\tau_1(2i-1)} = \delta_{y_i y_i}$, $X_{\tau_1(2i)} = \delta_{y_i, y_i}$, $X_{\tau_1(2i+1)} = \delta_{y_i, y_{i+1}}$, $X_{\tau_1(2i+2)} = \delta_{y_{i+1}, y_{i+1}}$, ..., $X_{\tau_1(2n-1)} = \delta_{y_{n-1} y_n}$, $X_{\tau_1(2n)} = \delta_{y_n y_n}$. Because of the choice of the variables, if $\tau \in S_{2n}$, with $\tau \neq \tau_1$ and $\tau \neq \tau_2$, then the value of the monomial $r_\tau X_{\tau(1)} X_{\tau(2)} \cdots X_{\tau(2n)}$ is zero at this specification of the variables. It then follows that the value of ϕ, at this choice of variables, is $r_{\tau_1} + r_{\tau_2}$.

Since ϕ is an identity for $I(X,R)$, we have that $r_{\tau_1} = -r_{\tau_2}$. This establishes the result if $N(\tau_1,\tau_2)=1$.

Suppose now that τ_1 and τ_2 are associated with neighbor similar permutations A_1 and A_2 of $\{X_1,X_2,\dots,X_{2n}\}$ and that $N(\tau_1,\tau_2)\geq 2$. Proceeding by induction, we assume we know the result for smaller values of $N(\lambda_1,\lambda_2)$, where λ_1 and λ_2 are the elements of S_{2n} associated to neighbor similar permutations of $\{X_1,\dots,X_{2n}\}$. Let i be the smallest positive integer such that $\tau_1(2i-1)\neq\tau_2(2i-1)$, and let A_3 be the permutation of $\{X_1,X_2,\dots,X_{2n}\}$ given by

$$X_{\tau_1(1)}X_{\tau_1(2)}\cdots X_{\tau_1(2i-2)}X_{\tau_2(2i-1)}X_{\tau_2(2i)}X_{\tau_1(2i+1)}X_{\tau_1(2i+2)} \cdots X_{\tau_1(2n-1)}X_{\tau_1(2n)}.$$

Further, suppose τ_3 is the element of S_{2n} associated with A_3. Then A_1, A_2 and A_3 are all equivalent permutations and $N(\tau_1,\tau_3)=1$, $N(\tau_3,\tau_2)=N(\tau_1,\tau_2)-1$. By induction we have that $r_{\tau_1}=(-1)^{N(\tau_1,\tau_3)}r_{\tau_3}$ and $r_{\tau_3}=(-1)^{N(\tau_3,\tau_2)}r_{\tau_2}$, and thus $r_{\tau_1}=(-1)^{N(\tau_1,\tau_2)}r_{\tau_2}$. This establishes the result. □

Corollary. *In $LHP(X_1,X_2,\dots,X_{2n})$, $q(X_1,X_2,\dots,X_{2n})$ is the unique polynomial containing the monomial $X_1X_2\cdots X_{2n}$ and having a minimal number of terms.*

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