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BLOW-UP OF SOLUTIONS OF SOME NONLINEAR HYPERBOLIC SYSTEMS

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ABSTRACT. We consider two hyperbolic systems: $u_{tt} = \Delta u + |v|^p$, $v_{tt} = \Delta v + |u|^q$ and $u_{tt} = \Delta u + |v_t|^p$, $v_{tt} = \Delta v + |u_t|^q$ in $\mathbf{R}^n \times (0, \infty)$ with $u(x,0) = f(x)$, $v(x,0) = h(x)$, $u_t(x,0) = g(x)$, $v_t(x,0) = k(x)$. We show that there exists a bound $B(n,p)$ such that if $1 < pq < B(n,p)$ all nontrivial solutions with compact support blow up in finite time.

1. Introduction. In this paper we study two systems of hyperbolic equations:

$$
\begin{aligned}
&u_{tt} = \Delta u + |v|^p, && v_{tt} = \Delta v + |u|^q, \\
&u(x,0) = f(x), && v(x,0) = h(x), \\
&u_t(x,0) = g(x), && v_t(x,0) = k(x), \\
&\qquad x \in \mathbf{R}^n, && t > 0,
\end{aligned}
\tag{1.1}
$$

and

$$
\begin{aligned}
&u_{tt} = \Delta u + |v_t|^p, && v_{tt} = \Delta v + |u_t|^q, \\
&u(x,0) = f(x), && v(x,0) = h(x), \\
&u_t(x,0) = g(x), && v_t(x,0) = k(x), \\
&\qquad x \in \mathbf{R}^n, && t > 0,
\end{aligned}
\tag{1.2}
$$

where $p, q \geq 1$ and $pq > 1$, and the initial values are compactly supported. Such systems are special cases of a significant class of quasilinear second order hyperbolic systems with application in physics and applied science, cf. [**5**].

Our main objective here is to establish blow-up theorems for systems (1.1) and (1.2). As an example of the type of results we wish to obtain,

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let us recall some known results for two related initial value problems of scalar equations.

$$
\begin{aligned}
u_{tt} &= \Delta u + |u|^p, \\
u(x,0) &= f(x), \qquad u_t(x,0) = g(x), \\
x &\in \mathbf{R}^n, \quad t > 0,
\end{aligned} \tag{1.3}
$$

and

$$
\begin{aligned}
u_{tt} &= \Delta u + |u_t|^p, \\
u(x,0) &= f(x), \qquad u_t(x,0) = g(x), \\
x &\in \mathbf{R}^n, \quad t > 0.
\end{aligned} \tag{1.4}
$$

Over the past few years, many authors have worked on (1.3), (1.4) and partially verified the following conjecture.

Conjecture W. *There exists a critical exponent $p_0(n)$ such that if $1 < p \leq p_0(n)$ every solution blows up in finite time, while there are nontrivial global small solutions if $p > p_0(n)$.*

Here $p_0(1) = +\infty$ for problem (1.3) with $n \geq 2$,

$$
p_0(n) = \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n-1)}, \tag{1.5}
$$

and for problem (1.4) with $n \geq 2$

$$
p_0(n) = \frac{n+1}{n-1}. \tag{1.6}
$$

The blow-up part is almost complete except the critical case for (1.3) with $n \geq 4$, see [**3, 7, 12, 14, 17**]. Of much greater difficulty is the global existence part. For problem (1.3) Glassey [**4**] and John [**6**] proved it for $n = 2$ and $n = 3$, respectively, Zhou [**16**] studied it for $n = 4$ in the Sobolev class, and Kubo [**8**] considered it with a radially symmetric restriction. For problem (1.4), Sideris [**13**] and Schaeffer [**11**] proved it for $n = 3$ and $n = 5$, respectively. Some other related results were obtained in [**9, 10**], and it is worth mentioning that Carpio [**1**] proved the existence of unbounded global solutions for the initial-boundary value problem:

$$\begin{gathered} u_{tt} = \Delta u + |u_t|^{p-1} u_t, \quad x \in \Omega, t > 0, \\ u = 0, \quad x \in \partial\Omega, t > 0, \\ u(x,0) = f(x), \qquad u_t(x,0) = g(x), \quad x \in \Omega, \end{gathered}$$

where Ω is a bounded domain and $p > 1$. For further details we refer the reader to the survey paper by Takamura [**15**] and the literature cited therein.

Until recently, nothing was known about systems (1.1) and (1.2) from the point of view of critical exponents. In [**2**] we conducted a discussion for system (1.1) with $1 \le n \le 3$. Our argument strongly relies on the positivity of the fundamental solution of the wave equation and is quite different from all previous ones used in the verification of Conjecture W. We showed that there exists a bound $B(n,p)$, $\le \infty$, such that if $p, q > 0$ and $1 < pq < B(n,p)$, then all nontrivial solutions of (1.1) blow up in finite time. In this paper we shall extend similar results to (1.1) with $n \ge 4$ and to (1.2) with $n \ge 1$. In Section 2 we show the nonexistence of global solutions of (1.1), and we establish the blow-up result for (1.2) in Section 3.

2. Blow-up of solutions of system (1.1). In this section we establish the blow-up result for system (1.1) with $n \ge 4$. In such higher dimensional spaces, due to the fact that the Riemann function $R(t)$ for the wave equation is no longer a positive operator, the situation becomes very complicated. To overcome the difficulty, for problem (1.3) Sideris [**14**] averaged the Riemann function in time to yield a useful lower bound for time averages of the solution. In the sequel, we will first follow his idea to obtain a pair of second order ordinary differential inequalities. We then demonstrate the nonexistence of global solutions of these two inequalities via an integral equation argument. For definiteness, we may assume $p \le q$ throughout this section. And we note that (u,v) needs only to be a weak solution of (1.1) in the sense that on a time interval $0 \le t \le T < \infty$, $u(t) \in C([0,T]; L^q(\mathbf{R}^n)), v(t) \in C([0,T]; L^p(\mathbf{R}^n))$, and u, v satisfy the following integral equations

$$u(t) = u_0(t) + \int_0^t R(t-\tau) * |v(\tau)|^p \, d\tau,$$
$$v(t) = v_0(t) + \int_0^t R(t-\tau) * |u(\tau)|^q \, d\tau,$$

where u_0 and v_0 are solutions of $w_{tt} = \Delta w$ with the same initial data as u and v, respectively.

We begin by imposing several assumptions on the initial data.

(H1) (i) $f, g, h, k \in C_0^\infty(\mathbf{R}^n)$ with supp $\{f, g, h, k\} \subseteq \{|x| < d\}$;

(ii)

$$\int_{\mathbf{R}^n} |x|^{\rho-1} f(x) \, dx \geq 0,$$
$$\int_{\mathbf{R}^n} |x|^{\rho-1} h(x) \, dx > 0,$$
$$\int_{\mathbf{R}^n} |x|^{\rho} g(x) \, dx \geq 0,$$
$$\int_{\mathbf{R}^n} |x|^{\rho} k(x) \, dx > 0,$$

where $\rho = 0$ if n is odd and $\rho = 1/2$ if n is even.

As in [**14**], we also define that, for $t \geq d$,

$$\Phi(t) = \int_{t-d}^t (t-\tau)^m \int_{\mathbf{R}^n} u(x,\tau) \, dx \, d\tau, \tag{2.1a}$$

$$\Psi(t) = \int_{t-d}^t (t-\tau)^m \int_{\mathbf{R}^n} v(x,\tau) \, dx \, d\tau, \tag{2.1b}$$

where $m = (n-5)/2$ if n is odd and $m = (n-4)/2$ if n is even.

We then state the following two lemmas, whose proofs are identical to those of [**14**] and hence are omitted.

Lemma 2.1. *There exists a positive constant c_0 such that, for $t \geq d$,*

$$\Phi''(t) \geq c_0(t+d)^{-n(p-1)}|\Psi(t)|^p, \tag{2.2a}$$

$$\Psi''(t) \geq c_0(t+d)^{-n(q-1)}|\Phi(t)|^q. \tag{2.2b}$$

Lemma 2.2. *There exist a t_0 ($> d$) and a positive constant c_1 such that*

$$\Phi(t) \geq c_1(t+d)^{n+1-p(n-1)/2} \quad \text{for } t \geq t_0. \tag{2.3}$$

We now present the main result.

Theorem 2.3. *Assume* (H1). *For $n \geq 4$ and $1 \leq p < p_0(n)$, $p_0(n)$ of* (1.5), *if $1 < pq < [(n+3)p+2]/[(n-1)p-2]$, every solution of* (1.1) *blows up in finite time.*

Proof. Assume to the contrary that (1.1) has a global solution (u, v). A combination of (2.2b) and (2.3) yields

$$\Psi''(t) \geq c_0 c_1^q (t+d)^{n+q-pq(n-1)/2} \quad \text{for } t \geq t_0. \tag{2.4}$$

Integrating (2.4) from t_0 to t, we have

$$\Psi'(t) - \Psi'(t_0) \geq c_0 c_1^q \int_{t_0}^{t} (\tau+d)^{n+q-pq(n-1)/2}\, d\tau. \tag{2.5}$$

Since $[(n-1)p-2]pq < (n+3)p+2 < 2(n+1)p$ for $p \geq 1 > 2/(n-1)$, $n+q-pq(n-1)/2 > -1$. Thus, from (2.5), it follows that $\Psi'(t)$ must be positive for $t \geq t_1 > t_0$, and we find

$$\begin{aligned} \Psi'(t) &\geq c_0 c_1^q \int_{t_1}^{t} (\tau+d)^{n+q-pq(n-1)/2}\, d\tau \\ &\geq c_2(t+d)^{n+1+q-pq(n-1)/2} \quad \text{for } t \geq t_2 \end{aligned} \tag{2.6}$$

for some $c_2 > 0$ and $t_2 > t_1$. Integration of (2.6) over (t_2, t) then leads to

$$\Psi(t) \geq c_3(t+d)^{n+2+q-pq(n-1)/2} \geq c_3(t+d)^{(n-1)/2} \quad \text{for } t \geq t_3 \tag{2.7}$$

for some $t_3 > t_2$, since $[(n-1)p-2]q < n+5$ for $p \geq 1$.

We now combine (2.2a) and (2.7) to obtain

$$\Phi''(t) \geq c_0 c_3^p (t+d)^{n-(n+1)p/2} \quad \text{for } t \geq t_3. \tag{2.8}$$

Since $n-(n+1)p/2 > -1$ for $p < p_0(n) \leq 2$, (2.8) implies that $\Phi'(t) > 0$ for $t \geq \hat{t} > t_3$. Hence, from (2.2), (2.3) and (2.7), we have that for $t \geq \hat{t}$,

$$\begin{aligned} \Phi(t) &\geq c_1 + c_0(t+d)^{-n(p-1)} \int_{\hat{t}}^{t} \int_{\hat{t}}^{\eta} \Psi^p(\tau)\, d\tau\, d\eta \\ &= c_1 + c_0(t+d)^{-n(p-1)} \int_{\hat{t}}^{t} (t-\tau)\Psi^p(\tau)\, d\tau, \end{aligned} \tag{2.9a}$$

$$\begin{aligned} \Psi(t) &\geq c_3 + c_0(t+d)^{-n(q-1)} \int_{\hat{t}}^{t} \int_{\hat{t}}^{\eta} \Phi^q(\tau)\, d\tau\, d\eta \\ &= c_3 + c_0(t+d)^{-n(q-1)} \int_{\hat{t}}^{t} (t-\tau)\Phi^q(\tau)\, d\tau. \end{aligned} \tag{2.9b}$$

Consider two cases.

Case 1. $p = 1$. Let $1/q < \theta < 1$. By (2.3) and (2.9), we find that, for $T \leq t \leq 2T$ with any positive number T $(\geq \hat{t})$

(2.10)

$$\begin{aligned} \Phi(t) &\geq c_1 + c_0^2 \int_T^t (\eta+d)^{-n(q-1)}(t-\eta) \int_T^\eta (\eta-\tau)\Phi^{\theta q}(\tau)\Phi^{(1-\theta)q}(\tau)\, d\tau \\ &\geq c_1 + c_0^2 c_1^{(1-\theta)q} 3^{-n(q-1)} T^{-\lambda} \int_T^t (t-\eta) \int_T^\eta (\eta-\tau)\Phi^{\theta q}(\tau)\, d\tau\, d\eta \\ &= c_1 + c_0^2 c_1^{(1-\theta)q} 3^{-n(q-1)} B(2,2) T^{-\lambda} \int_T^t (t-\tau)^3 \Phi^{\theta q}(\tau)\, d\tau, \end{aligned}$$

where $\lambda = n(q-1) - (n+3)(1-\theta)q/2$ and $B(2,2)$ is the Beta function. Thus, by comparison, $\Phi(t) \geq \varphi(t)$ on $[T, 2T]$, where

$$\varphi(t) = c_1 + c_4 T^{-\lambda} \int_T^t (t-\tau)^3 \varphi^{\theta q}(\tau)\, d\tau \quad \text{for } T \leq t \leq 2T \tag{2.11}$$

with $c_4 = c_0^2 c_1^{(1-\theta)q} 3^{-n(q-1)} B(2,2)$. Clearly, $\varphi(t)$ satisfies

$$\begin{aligned} \varphi^{(iv)}(t) &= 6c_4 T^{-\lambda}\varphi^{\theta q}(t), \quad T < t < 2T, \\ \varphi(T) &= c_1, \varphi'(T) = \varphi''(T) = \varphi'''(T) = 0. \end{aligned} \tag{2.12}$$

For $\lambda < 4$, if T is large enough, there exists a $\hat{T}$, $T < \hat{T} \leq 3T/2$, such that $\varphi(\hat{T}) = 2c_1$. Moreover,

$$\begin{aligned} \varphi''(t) &= 6c_4 T^{-\lambda}\int_T^t (t-\tau)\varphi^{\theta q}(\tau)\,d\tau \\ &\geq 6c_4 T^{-\lambda} t^{-1}\int_T^t (t-\tau)^2\varphi^{\theta q}(\tau)\,d\tau \\ &\geq T^{-1}\varphi'(t) \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} \varphi'''(t) &= 6c_4 T^{-\lambda}\int_T^t \varphi^{\theta q}(\tau)\,d\tau \\ &\geq 6c_4 T^{-\lambda} t^{-2}\int_T^t (t-\tau)^2\varphi^{\theta q}(\tau)\,d\tau \\ &\geq (T^{-2}/2)\varphi'(t). \end{aligned} \tag{2.14}$$

This equation in (2.12) together with (2.14) leads to

$$\varphi^{(iv)}(t)\varphi'''(t) \geq 3c_4 T^{-(\lambda+2)}\varphi^{\theta q}(t)\varphi'(t).$$

Integrating this relation from T to t then gives

$$\varphi'''(t) \geq cT^{-(\lambda+2)/2}(\varphi^{\theta q+1}(t) - c_1^{\theta q+1})^{1/2} \tag{2.15}$$

with $c = (6c_4/(\theta q+1))^{1/2}$. From now on, without causing any confusion, c may be used to denote various positive constants independent of T. By means of (2.13), one can see

$$\varphi'''(t)\varphi''(t) \geq cT^{-(\lambda+4)/2}(\varphi^{\theta q+1}(t) - c_1^{\theta q+1})^{1/2}\varphi'(t), \tag{2.16}$$

which, upon integration over (T,t) for $\hat{T} \leq t < 2T$, yields

$$\begin{aligned} (\varphi''(t))^2 &\geq cT^{-(\lambda+4)/2}\int_T^t (\varphi^{\theta q+1}(\tau) - c_1^{\theta q+1})^{1/2}\varphi'(\tau)\,d\tau \\ &\geq cT^{-(\lambda+4)/2}\bigg[(\theta q+1)^{1/2} c_1^{\theta q/2}\int_{c_1}^{2c_1}(\sigma - c_1)^{1/2}\,d\sigma \\ &\qquad + c_5\int_{2c_1}^{\varphi(t)}\sigma^{(\theta q+1)/2}\,d\sigma\bigg], \end{aligned}$$

where $c_5 = \min\{(\theta q+3)(\theta q+1)^{1/2}2^{-(\theta q+3)/2}/3, 2^{-(\theta q+1)/2}\}$. Hence, for $\hat{T} \le t < 2T$,

$$\varphi''(t) \ge cT^{-(\lambda+4)/4}\varphi^{(\theta q+3)/4}(t),$$

or

$$\varphi''(t)\varphi'(t) \ge cT^{-(\lambda+4)/4}\varphi^{(\theta q+3)/4}(t)\varphi'(t).$$

The above inequality implies that, for $3T/2 \le t < 2T$,

$$\begin{aligned}\varphi'(t) &\ge cT^{-(\lambda+4)/8}(\varphi^{(\theta q+7)/4}(t) - \varphi^{(\theta q+7)/4}(\hat{T}))^{1/2}\\ &\ge cT^{-(\lambda+4)/8}(\varphi^{(\theta q+7)/4}(t) - \varphi^{(\theta q+7)/4}(3T/2))^{1/2}.\end{aligned} \tag{2.17}$$

Integrating (2.17) from $3T/2$ to $2T$ then gives

$$\begin{aligned}cT^{(4-\lambda)/8} &\le \int_{\varphi(3T/2)}^{\varphi(2T)} (\sigma^{(\theta q+7)/4} - \varphi^{(\theta q+7)/4}(3T/2))^{-1/2}\,d\sigma\\ &\le 2(\theta q+7)^{-1/2}\varphi^{-(\theta q+3)/8}(3T/2)\\ &\quad\cdot \int_{\varphi(3T/2)}^{2\varphi(3T/2)} (\sigma - \varphi(3T/2))^{-1/2}\,d\sigma\\ &\quad + 2^{(\theta q+7)/8}\int_{2\varphi(3T/2)}^{\infty} \sigma^{-(\theta q+7)/8}\,d\sigma\\ &= [4(\theta q+7)^{-1/2} + 16(\theta q-1)^{-1}]\varphi^{-(\theta q-1)/8}(3T/2)\\ &\le [4(\theta q+7)^{-1/2} + 16(\theta q-1)^{-1}]c_1^{-(\theta q-1)/8}.\end{aligned} \tag{2.18}$$

Since the constant c depends only on λ and θq, if T is sufficiently large, (2.18) yields a contradiction. This means that $\Phi(t)$, and hence the solution (u, v) of (1.1) cannot exist globally for $\lambda < 4$, which leads to the limitation $q < (n+5)/(n-3)$, because θq can be chosen arbitrarily close to one.

Case 2. $p > 1$. Let $1/pq < \theta < 1/q$. By (2.9a) and the inverse Hölder's inequality, we find that for $t \ge \hat{t}$,

$$\Phi^{\theta q}(t) \ge c_0^{\theta q}(t+d)^{-n(p-1)\theta q - 2(1-\theta q)} \int_{\hat{t}}^{t} (t-\tau)\Psi^{\theta pq}(\tau)\,d\tau, \tag{2.19}$$

which, combined with (2.3) and (2.9b) yields that, for $T \leq t \leq 2T$ with any positive number T $(\geq \hat{t})$

$$\begin{aligned}\Psi(t) &\geq c_3 + c_6 T^{-n(q-1)+(n+1-(n-1)p/2)(1-\theta)q} \\ &\qquad \cdot \int_T^t (t-\tau)\Phi^{\theta q}(\tau)\, d\tau \\ &\geq c_3 + c_7 T^{-\tilde{\lambda}} \int_T^t (t-\tau)^3 \Psi^{\theta p q}(\tau)\, d\tau,\end{aligned} \tag{2.20}$$

where $\tilde{\lambda} = n(q-1) - (n+1-(n-1)p/2)(1-\theta)q + n(p-1)\theta q + 2(1-\theta q)$. Thus, by comparison, $\Psi(t) \geq \psi(t)$ on $[T, 2T]$, where

$$\psi(t) = c_3 + c_7 T^{-\tilde{\lambda}} \int_T^t (t-\tau)^3 \psi^{\theta p q}(\tau)\, d\tau \quad \text{for } T \leq t \leq 2T.$$

As before, one can see that if T is large enough, $\psi(t)$ and hence $\Psi(t)$ cannot exist globally for $\tilde{\lambda} < 4$, which is equivalent to the following inequality

$$[(n-1)p-2]pq + [(n+1)p-2]\theta p q < 2(n+2)p. \tag{2.21}$$

Since $\theta p q$ can be chosen arbitrarily close to one, (2.21) leads to the bound

$$pq < \frac{(n+3)p+2}{(n-1)p-2}. \tag{2.22}$$

Remark. The hypothesis $p < p_0(n)$ is necessary because setting $q = p$ in (2.22) we obtain

$$(n-1)p^3 - 2p^2 - (n+3)p - 2 < 0,$$

that is,

$$(p+1)[(n-1)p^2 - (n+1)p - 2] < 0,$$

which is satisfied if $p < p_0(n)$.

3. Blow-up of solutions of system (1.2). In this section we establish the blow-up results for system (1.2) with $n \geq 1$. We will first follow Zhou's idea for problem (1.4) [**17**] to obtain two coupled integral inequalities and then show the nonexistence of global solutions of such inequalities. For that reason, we assume $u, v \in C^2(\mathbf{R}^n \times [0,T])$. And, for definiteness, we may again assume $p \le q$.

We begin with two hypotheses on the initial values.

(H2) (i) $f, g, h, k \in C_0^\infty(\mathbf{R}^n)$ with supp $\{f, g, h, k\} \subseteq \{|x| < d\}$;

(ii)

$$\int_{\mathbf{R}^n} g(x)\,dx \ge 0, \qquad \int_{\mathbf{R}^n} k(x)\,dx > 0.$$

As in [**17**], when $n = 1$, we define

$$\Phi(y) = u(y, y+d), \qquad \Psi(y) = v(y, y+d) \tag{3.1a}$$

and, when $n \ge 2$, letting $x = (y, z) \in \mathbf{R} \times \mathbf{R}^{n-1}$, we define

$$\begin{aligned} \Phi(y) &= \int_{\mathbf{R}^{n-1}} u(y, z, y+d)\,dz, \\ \Psi(y) &= \int_{\mathbf{R}^{n-1}} v(y, z, y+d)\,dz. \end{aligned} \tag{3.1b}$$

We then state the following lemma, whose proof is similar to that of [**17**] and hence is omitted.

Lemma 3.1. *There exist positive constants c_0 and c_1 such that, for $t \ge d$,*

(3.2a)

$$\Phi(y) \ge c_0 \int_d^y (\xi + d)^{-(n-1)(p-1)/2} |\Psi(\xi)|^p \,d\xi,$$

(3.2b)

$$\Psi(y) \ge c_1 + c_0 \int_d^y (\xi + d)^{-(n-1)(q-1)/2} |\Phi(\xi)|^q \,d\xi.$$

We now present the main results.

Theorem 3.2. *Assume* (H2). *If* $1 < pq < \infty$, *for* $1 \le p < \infty$ *when* $n = 1$, *for* $1 \le p \le 2$ *when* $n = 2$, *and for* $p = 1$ *when* $n = 3$, *every solution of* (1.2) *blows up in finite time.*

Theorem 3.3. *Assume* (H2). *If* $1 < pq \le (n+1)p/[(n-1)p-2]$, *for* $2 < p \le 3$ *when* $n = 2$, *for* $1 < p \le 2$ *when* $n = 3$, *and for* $1 \le p \le (n+1)/(n-1)$ *when* $n \ge 4$, *every solution of* (1.2) *blows up in finite time.*

Proof of Theorem 3.2. Assume to the contrary that (1.2) has a global solution (u, v). Making use of (3.2) and applying Jensen's inequality, we find that, for $y \ge d$,

$$\begin{aligned}\Psi(y) &\ge c_1 + c_0(y+d)^{-(n+1)(q-1)/2}\left(\int_d^y \Phi(\xi)\,d\xi\right)^q \\ &\ge c_1 + c_0^{1+q}(y+d)^{-(n+1)(q-1)/2-(n-1)(p-1)q/2} \\ &\qquad \cdot \left(\int_d^y\int_d^\zeta \Psi^p(\xi)\,d\xi\,d\zeta\right)^q \\ &= c_1 + c_0^{1+q}(y+d)^{-(n+1)(q-1)/2-(n-1)(p-1)q/2} \\ &\qquad \cdot \left(\int_d^y (y-\xi)\Psi^p(\xi)\,d\xi\right)^q.\end{aligned} \tag{3.3}$$

Then, for any positive number Y $(\ge d)$, we have

$$\Psi^{1/q}(y) \ge c_2 + c_3 Y^{-\bar\lambda}\int_Y^y (y-\xi)(\Psi^{1/q}(\xi))^{pq}\,d\xi \quad \text{for } Y \le y \le 2Y, \tag{3.4}$$

with $\bar\lambda = (n+1)(q-1)/2q + (n-1)(p-1)/2$. Thus, by comparison, $\Psi^{1/q}(y) \ge \psi(y)$ on $[Y, 2Y]$, where

$$\psi(y) = c_2 + c_3 Y^{-\bar\lambda}\int_Y^y (y-\xi)\psi^{pq}(\xi)\,d\xi \quad \text{for } Y \le y \le 2Y. \tag{3.5}$$

Clearly, $\psi(y)$ satisfies

$$\begin{aligned}&\psi''(y) = c_3 Y^{-\bar\lambda}\psi^{pq}(y), \quad Y < y < 2Y, \\ &\psi(Y) = c_2, \psi'(Y) = 0.\end{aligned} \tag{3.6}$$

Multiplying the equation in (3.6) by $\psi'(y)$ and integrating from Y to y, we obtain

$$\psi'(y) = c_4 Y^{-\bar{\lambda}/2}(\psi^{pq+1}(y) - \psi^{pq+1}(Y))^{1/2}. \tag{3.7}$$

Integration of this relation over $(Y, 2Y)$ then leads to

$$\begin{aligned} c_4 Y^{(2-\bar{\lambda})/2} &= \int_{\psi(Y)}^{\psi(2Y)} (\sigma^{pq+1} - \psi^{pq+1}(Y))^{-1/2}\, d\sigma \\ &\le (pq+1)^{-1/2}\psi^{-pq/2}(Y) \int_{\psi(Y)}^{2\psi(Y)} (\sigma - \psi(Y))^{-1/2}\, d\sigma \\ &\quad + 2^{(pq+1)/2} \int_{2\psi(Y)}^{\infty} \sigma^{-(pq+1)/2}\, d\sigma \\ &= 2[(pq+1)^{-1/2} + 2(pq-1)^{-1}]c_2^{(1-pq)/2}. \end{aligned} \tag{3.8}$$

For $\bar{\lambda} < 2$, if Y is sufficiently large, (3.8) yields a contradiction, which means that $\Psi(y)$, and hence the solution (u, v) of (1.2), cannot exist globally. The restriction $\bar{\lambda} < 2$ is equivalent to the following inequality

$$[(n-1)p - 2]q < n + 1. \tag{3.9}$$

In view of the conditions on p for $1 \le n \le 3$, (3.9) is valid, and hence the proof is completed.

Proof of Theorem 3.3. We again assume to the contrary that (1.2) has a global solution (u, v). Using $\Psi(y) \ge c_1 > 0$, from (3.2b), (3.2a) yields

$$\Phi(y) \ge c_5 (y+d)^{1-(n-1)(p-1)/2} \quad \text{for } y \ge 2d.$$

Combining this with (3.2b), it follows that

$$\Psi(y) \ge c_0 c_5^q \int_{2d}^{y} (\xi + d)^{-(n-1)(q-1)/2 + q(1-(n-1)(p-1)/2)}\, d\xi. \tag{3.10}$$

Then $q \le (n+1)/[(n-1)p - 2]$ implies that $-(n-1)(q-1)/2 + q[1 - (n-1)(p-1)/2] \ge -1$, and we have

$$\Psi(y) \ge c_0 c_5^q \int_{2d}^{y} (\xi + d)^{-1}\, d\xi = c_0 c_5^q \log\left(\frac{y+d}{3d}\right) \quad \text{for } y \ge 2d. \tag{3.11}$$

Hence, we find that, for $Y \leq y \leq 2Y$ with any positive number Y $(\geq 9d^2)$

(3.12)
$$\Psi^{1/q}(y) \geq 2^{-(1/q)-1} c_0^{1/q} c_5 (\log Y)^{1/q} + 2^{(1/q)-1} 3^{-\bar{\lambda}} c_0^{(1/q)+1} Y^{-\bar{\lambda}} \cdot \int_Y^y (y-\xi)(\Psi^{1/q}(\xi))^{pq}\, d\xi,$$

and, by comparison, $\Psi^{1/q}(y) \geq \psi(y)$ on $[Y, 2Y]$, where

$$\psi(y) = c_6 (\log Y)^{1/q} + c_7 Y^{-\bar{\lambda}} \int_Y^y (y-\xi)\psi^{pq}(\xi)\, d\xi \quad \text{for } Y \leq y \leq 2Y. \tag{3.13}$$

Proceeding essentially the same as in the proof of Theorem 3.2, we finally obtain

(3.14)
$$c_8 Y^{(2-\bar{\lambda})/2} \leq 2[(pq+1)^{-1/2} + 2(pq-1)^{-1}] c_6^{(1-pq)/2} (\log Y)^{-(pq-1)/2q}.$$

For $\bar{\lambda} \leq 2$ which is equivalent to $pq \leq (n+1)p/[(n-1)p-2]$, if Y is sufficiently large, (3.14) yields a contradiction. This indicates that $\Psi(y)$ and hence the solution (u, v) of (1.2) cannot exist globally.

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