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CHIUNGTZE C. TSEN (1898-1940) AND TSEN'S THEOREMS

SHISUN DING, MING-CHANG KANG AND ENG-TJIOE TAN

1. Introduction. Around the beginning of this century, the western education system was introduced into China to replace the traditional one. In 1917, Mingfu T. Hu (1891–1927) got his Ph.D. degree from Harvard University under the direction of W.A. Hurwitz [**33**]. He was the first person in China to obtain a Ph.D. degree in mathematics.[1] Subsequently, several persons who held Ph.D. degrees or were trained in some master programs in the United States, Europe or Japan returned to China and established university curriculums of modern mathematics. During 1928–1937 there arose a new generation of budding mathematicians with rather solid preparation as undergraduate students.[2] After graduation most of them went abroad for advanced studies. Among them, the most famous ones were Tsai-han Kiang (Zehan Jiang 1902–1994), Shiing-Shen Chern (born in 1911), Pao-lu Hsu (Baolu Xu, 1910–1970), etc. Although Loo-Keng Hua (Luogeng Hua, 1910–1985) and Wei-Liang Chow (1911–1995) were of the same age group, Hua was essentially self-taught and Chow had his undergraduate education outside of China, i.e., in the physics department of the University of Chicago. Chiungtze C. Tsen (1898–1940) belonged to this generation also.[3]

Somewhat older than other persons of this generation, Tsen got the bachelor degree two years before this period (1926). At the end of 1928 he went to Germany with a fellowship from the Chinese government. From the summer semester of 1929, he matriculated at Göttingen University. His mentor was Emmy Noether, one of the founders of modern algebra. The subject of his Ph.D. dissertation [**T2**] is central simple algebras. One of the main results of this dissertation is:

Theorem. *There is no nontrivial central division algebra over an algebraic function field in one variable over an algebraically closed field.*

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Except for the above theorem of Tsen, before the 1970's almost nothing was known about Tsen the person and his later research activities. Only in a passage in an article written in Chinese [**15**], did S.S. Chern mention Chiungtze C. Tsen, an old friend of his.

In his book on Emmy Noether in 1970, Dick wrote,

> Nothing is known so far about what became of the Chinese student, Chiungtze Tsen, whose dissertation was also inspired and tirelessly supported by E. Noether. However, he took his examinations after the emigration of his benefactress. In 1936 a paper by him "Zur Stufentheorie der quasi-algebraisch-Abgeschlossenheit kommutativer Körper", as reported by N. Jacobson in Theory of Rings (New York City, 1943) appeared in the Journal of the Chinese Mathematical Society (v. 1, pp. 81-92), from which one might conclude that Tsen may have returned to his country. [**20**, pp. 55–56]

About ten years later, Kimberling provided a partial answer to Dick's question [**40**, p. 41]. But the content of Tsen's third paper [**T3**] was not alluded to at all. It is in Lorenz's book on algebra [**49**] that results of this paper are presented in a complete form for the first time. He commented,

> It is remarkable that these results, being of such elementary nature, have no access to algebra literature. Moreover, the abovementioned publication of Tsen (i.e., the paper [**T3**]—authors' note) remains uncited mostly. ("Erstaunlicherweise haben diese Ergebnisse trotz ihrer elementaren Gestalt bisher so gut wie keinen Eingang in die algebraische Lehrbuchliteratur gefunden. Darüberhinaus ist auch sonst die erwähnte Publikation von Tsen weitgehnend unzitiert geblieben." [**49**, pp. 148–149])

Recently several articles about Tsen written in Chinese appeared [**47, 48, 76**]. There the content of [**T3**] was discussed; it was reported that the notion of C_i-fields introduced in Serge Lang's Ph.D. dissertation [**44**] had been defined in [**T3**]. Although such a statement is essentially correct, the notion of the level of a field ("Stufen") in [**T3**] is not exactly the same one as in [**44**]; on which, we shall elaborate in Section 5 of this paper.

Returning to Tsen's dissertation, the presentation of the proof of Tsen's theorem in most books [**69, 27, 71, 36, 64**] consists of two steps: Step 1. to show that a function field over an algebraically closed field is a C_1-field in Lang's sense; Step 2. to show that the Brauer group of a C_1-field is trivial. What a sneaky proof! In fact, the first reaction trying to prove a central simple algebra being trivial is either to establish an isomorphism of it onto the matrix algebra or to show that the associated 2-cocycle is cohomologously trivial, instead of the above "indirect" albeit clever argument. Indeed, the proof of Tsen's theorem in [**T1**] was different from the usual one and looked more natural, at least for psychological reasons. There were three proofs of Tsen's theorem in [**T2**]. The first one was Tsen's own proof and was the same as that presented in [**T1**], the second one was a simplified proof of Emmy Noether, the third one was a proof similar to that mentioned above which was due to Artin. (Step 1 with C_1-fields replaced by N_1-fields was still due to Tsen himself. For the definition of N_1-fields, see 5.2.)

In Chapter 3 of [**T2**], quite a lot of pages were devoted to studying the structure of central division algebras over $P(x)$, the rational function field in one variable over a real closed field P. A basic property observed by Tsen was reported in Deuring's book [**19**, p. 137]. Unfortunately, this property was misinterpreted in [**47, 48, 76**]. (See 4.6 of this paper.) Although Tsen's main results on $P(x)$ were superseded immediately by those of Witt [**82**], it seems worthwhile still to make known some of them.

In his short life, Tsen published only three research papers. The purpose of this paper is to examine results in these papers and to clarify some misunderstanding, besides an introduction of the life of Tsen. For convenience of the reader, we include a section to explain some terminology in central simple algebras and to describe briefly activities of this field in Germany before and when Tsen was in Göttingen. The life of Tsen is related in Section 2, based on information provided in [**47, 48, 76, 38**] and a page of Tsen's curriculum vitae [**Tb**]. It will be evident from the life and the career of Tsen how difficult it would be for a person in an under-developed country to become a mature mathematician and be recognized internationally.

2. The life of Tsen.

2.1. According to [**Tb**], Chiungtze C. Tsen was born on April

2, 1898, in Hsinkien (Xinjian), a city near Nanchang, the capital of the Province Chiangshi (Jiangxi), China.[4] The capital letter C in the middle of his full name refers to his first name, Chiung, which is the same character as the first one in the two characters of his nickname, Chiungtze. About his parents, there is inconsistent data. In [**Tb**], Tsen mentioned his father, Tshuuwun Tsen, being a merchant (Kaufmann). On the other hand, all available biographies written in China [**47, 48, 76, 38**] indicate that Tsen came from a financially poor fisherman family with two sons and several daughters. Being the eldest son of the family, he had been forced to take leave from school several times in order to work at a nearby mining factory or to help his father in fishing. This might explain his late entrance, at the age of 24, to the university. The following are chronological milestones of Tsen's education:

— 1905, entrance to elementary school.

— 1917, passed through the entrance examination to Nanchang First Normal College.

— 1922, Undergraduate study of mathematics at Wuchang Senior Normal College; graduation in May, 1926.

— 1926–1928, teaching career at a provincial normal college and two senior high-schools.

— Late 1928, stipend from Chiangshi Provincial government for study in Europe. Admission to Berlin University for language training.

2.2. From the summer semester of 1929, Tsen started his study of mathematics at Göttingen University. In [**Tb**] he mentioned as his academic teachers (Lehrers): Bernays, M. Born, Courant, Geiger, Herglotz, Hilbert, Landau, Lewy, Noether, R.W. Pohl, W. Weber and Weyl. However, besides Landau, the nearest and most influential person for Tsen's mathematical training is definitely Noether. He was one of the Noether boys (a description of Noether's followers used by Weyl in his memorial address at Bryn Mawr [**81**]) and the only Chinese among the many foreign visiting colleagues and students of Noether.[5] The presence of this Chinese Noether boy is well depicted in a group picture in [**40**], where Tsen was the only male person sitting in the presence of standing ladies.

Tsen got his Ph.D. degree on February 20, 1934. Attracted by Emil Artin, the next station of his career was Hamburg University. Having stayed there for one year only, Tsen returned to China in July, 1935.

2.3. When Tsen was an undergraduate student in Wuchang Senior Normal College (renamed as Wuhan University now), one of his teachers was Kien-Kwong Chen (Jiangong Chen, 1893–1971). With a Ph.D. degree from Tohoku Imperial University under Matsusaburô Fujiwara, Chen was a famous analyst and one of the pioneers of modern mathematics in China. From 1929, Chen taught at Zhejiang University. On returning to China, Tsen was invited by Chen to teach at the same university as an associate professor. Thus, a trio of mathematics at Zhejiang University took shape: Kien-Kwong Chen (analysis), Buchin Su (geometry) and Chiungtze Tsen (algebra) [**14**, p. 25]. He stayed there for two years. In the summer of 1937, Tsen accepted the offer of professorship from Beiyang Institute of Technology (formerly Beiyang University, and is renamed as Tienjin University now), then in Tienjin.[6] After the break-out of the Sino-Japan war in 1937, Tsen got married to Hesuei Qin who was a high-school teacher in chemistry.

The Sino-Japan war broke out near Beijing on July 7, 1937. Many universities in northern China were evacuated either southwards or westwards. Beiyang Institute of Technology was moved to Xian, an ancient capital of China. There in Xian, several evacuated universities, including Beiyang Institute of Technology, Beiping University (not to be confused with Peking University) and Beijing Normal University, were amalgamated into Xian Provisory University to which Tsen was affiliated. Because the Japanese military forces occupied nearby provinces around March 1938, this new university was moved to Hanzong and Chenggu, and was renamed as National Northwestern United University. However, this university split very soon; Beiyang Institute of Technology and the engineering colleges of Beiping University, Northeastern University and Jiaozhou Institute of Technology were combined to become Northwestern Institute of Technology and the campus was situated at Chenggu. Tsen became a professor of this new institute.

In 1939, a new institute, National Xikang Institute of Technology, was established in the suburb of Xichang, one of the major cities of the Province Xikang.[7] The president of this institute was a hydro-engineer, Xutien Li, who was a former president of Beiyang University. At the invitation of Li, Tsen and his wife endured an arduous journey to join this newly-founded institute. Miserably, Tsen's wife miscarried during travel due to road conditions. Life circumstances and the teaching environment in Xichang were very bad. Most of the campus offices were located in local temples. Tsen's health deteriorated with the hardship, especially the shortage of medicine. He died of stomach ulcer in November, 1940, in Xichang, at the age of 42. We find no account about the exact date of his death.

2.4. Tsen was survived by his wife; she died of larynx cancer around 1950. Since they remained childless throughout their marriage, a nephew of Tsen was adopted.

From S.S. Chern's recollections [**15**], Tsen was a cordial, sincere and open-minded person. According to Chern, "Tsen was well-liked by everybody."

Students of Tsen in Zhejiang University remembered that Tsen was a devoted teacher. He would arrive at the classroom before the class met and stopped lecturing until the class was over [**38**, p. 78]. He himself graded the homework of the students. Besides the ordinary classes, he offered several extra classes for students who were unable to catch up [**38**, p. 86].

3. Central simple algebras: an introduction.

3.1. Let K be a field. A finite-dimensional K-algebra A is, by definition, a finite-dimensional K-vector space A equipped with a ring structure so that $\alpha\cdot(ab) = (\alpha\cdot a)b = a(\alpha\cdot b)$ for any $\alpha \in K$, any $a, b \in A$. Denoting by 1 the multiplicative identity, then K is embedded in A by $\alpha \mapsto \alpha \cdot 1$ for any $\alpha \in K$. A central simple K-algebra A is a finite-dimensional K-algebra satisfying the properties: (i) the only two-sided ideals in A are $\{0\}$ and A, and (ii) the center of A is K, i.e., if $a \in A$ satisfies $ab = ba$ for any b, then $a \in K$. A central simple K-algebra A is called a central division K-algebra if A is a division ring when regarded as a ring.

Since the discovery of quaternions by Hamilton in 1843, there had been many activities to investigate the structure of finite-dimensional algebras over the real numbers or the complex numbers [**42, 59**].

We shall omit discussing the works of Cayley, B. Peirce, Frobenius, E. Cartan, Molien, Wedderburn, etc., and turn to the subject of searching for new division algebras.

3.2. In 1906 Dickson found a method to construct "new" central simple algebras [**21**] as follows:

Let K be any field, L a cyclic extension of K with $[L:K]=n$ and Galois group $\mathrm{Gal}\,(L/K)=\langle\sigma\rangle$. For any $a\in K\backslash\{0\}$, the cyclic algebra $(a,L/K,\sigma)$ is defined as

$$(a,L/K,\sigma):=\bigoplus_{i=0}^{n-1}Lx^i$$

where $x^n=a$ and $xu=\sigma(u)x$ for any $u\in L$.

The cyclic algebra $(a,L/K,\sigma)$ is a central simple K-algebra. It can be shown that $(a,L/K,\sigma)$ is isomorphic to $M_n(K)$, the matrix ring of degree n over K, if and only if $a=N_{L/K}(\alpha)$ for some $\alpha\in L$ where $N_{L/K}$ is the norm function from L into K. Moreover, $(a,L/K,\sigma)$ is a central division K-algebra if $n=\min\{i : 1\le i\le n)$ such that $a^i=N_{L/K}(\beta)$ for some $\beta\in L\}$.

A special case of cyclic algebras is the quaternion algebra (or the Hilbert symbol): Let K be any field with $\mathrm{char}\,K\neq 2$. For any $a,b\in K\backslash\{0\}$, the quaternion algebra $(a,b)_{2,K}$ is defined as

$$(a,b)_{2,K}:=K\cdot 1\oplus K\cdot u\oplus K\cdot v\oplus K\cdot uv$$

satisfying

$$u^2=a,\quad v^2=b,\quad vu=-uv.$$

It can be shown that $(a,b)_{2,K}$ is a central division K-algebra if and only if the homogeneous equation $aX^2+bY^2=Z^2$ has only the trivial solution $(0,0,0)$ over K.

3.3 Various properties of central simple algebras were studied in the 1920's. In particular, if A is a central simple K-algebra, then $\dim_K A=n^2$ for some positive integer n and $A\simeq M_r(D)$ for some integer r and some central division K-algebra D. The degree of A, denoted by $\deg(A)$ is defined to be $(\dim_K A)^{1/2}$. The index of a central division algebra D, denoted by $\operatorname{ind}(D)$, is simply $\deg(D)$.

Two central simple K-algebras A and B are called similar if $M_n(A) \simeq M_m(B)$ for some integers n and m, or equivalently, if $A \simeq M_r(D)$ and $B \simeq M_s(D)$ with the same central division K-algebra, D.

The class of all central simple K-algebras modulo the similarity relation forms the Brauer group of K, $Br(K)$, and the multiplication of the group $Br(K)$ is defined as

$$[A] \cdot [B] = [A \otimes_K B]$$

where $[A]$ denotes the similarity class determined by A [**10**]. The Brauer group is a torsion abelian group. The order of $[A]$ in $Br(K)$ is called the exponent of A, which will be denoted by $\exp(A)$.

A central simple K-algebra A is said to be trivial if A is isomorphic to the matrix ring. For any field extension L of K, $A \otimes_K L$ is a central simple L-algebra. Hence we have a group homomorphism from $Br(K)$ into $Br(L)$. L is called a splitting field of A if $A \otimes L$ is trivial. For any central division K-algebra D, there exists a separable splitting field L for D so that $[L : K] = \operatorname{ind}(D)$. If A has a cyclic splitting field L with $[L : K] = n$, then A is similar to a cyclic K-algebra of degree n.

The history of the theory of central simple algebras and its recent development are well documented in [**59, 42, 11, 5, 6, 65, 87**].

3.4. As pointed out by LaDuke [**42**, p. 153], it was Dickson who played a major role in determining the direction of most of the research in central simple algebras during the first three decades of this century. Dickson's book, *Algebras and their arithmetics* (published in 1923) summarized the main results of this area. This book was translated into German by A. Speiser in 1927 as [**22**]; this German translation contained much more new material and had great impact on the research of central simple algebras in Germany. However, it should be noticed that the focus of Dickson in central simple algebras is to find new division algebras or new types of central simple algebras [**37**, p. 21]. He was interested in results of proving that central simple algebras of degree 2 or 3 were cyclic algebras, while central simple algebras of degree 4 were crossed product but not necessarily cyclic algebras. On the other hand, Brauer, Noether and Hasse were interested in applications of central simple algebras in representation theory and class field theory [**13, 29**]. (Albert studies the structure of central simple algebras and also solved the problem of Riemann matrices [**1, 35**].)

3.5. The first paper of Noether on central simple algebras [**13**] was a joint work with Brauer. In this paper, they used the structure of central simple algebras to explain Schur's notion of indices of irreducible representations [**13, 43**]. Noether continued to establish a foundation of representation theory based on Wedderburn's theorem [**54, 37**].

Noether then started to investigate systematically the theory of central simple algebras. She lectured on this subject several times in Göttingen, 1924–1925, 1927–1928 [**42**, p. 156]. The paper [**56**] indicated some of the content of these courses, while the paper, "Nichtkommutative Algebren," reproduced in the Collected Works of Noether was her lecture notes in the summer semester of 1929 at Göttingen taken by Deuring [**37**, p. 20]. In these papers, all the key ideas in central simple algebras, e.g., crossed products, 2-cocycles, Skolem-Noether's theorem, splitting fields, etc., were presented.

For an appreciation of Noether's motivation to investigate central simple algebras, see her talks at the Bologna and Zürich International Congresses of Mathematicians [**53, 54**].

3.6. With the aid of central simple algebras, Hasse was able to give a conceptual definition of norm residue symbols [**29, 31**, pp. 274–275]. In [**28**] Hasse determined the structure of the Brauer group of a $\mathcal{P}$-adic field. In 1931, one of the most celebrated results in central simple algebras was obtained by Brauer, Hasse and Noether:

Theorem 3.7 [**12**]. *Every central simple algebra over an algebraic number field is a cyclic algebra.*

Remark. The above theorem was established independently by Albert and Hasse [**4**]. Although both [**12**] and [**4**] were published in 1932, according to the account of [**4**, p. 723], the theorem was proved in 1931. Several consequences of Theorem 3.8 can be easily obtained. For example,

Theorem 3.8. *Let A be a central simple algebra over an algebraic number field K. Then*

(i) $\exp(A) = ind(A)$ *and*

(ii) *A is trivial if and only if $A \otimes_K K_\nu$ is trivial for any place ν of K.*

3.9. Having obtained Theorem 3.7, with the aid of the structure of the Brauer groups of local fields, it is time to determine completely the structure of the Brauer group of an algebraic number field [**30**]. In the language of central simple algebras, Hilbert's product formula is equivalent to Hasse's sum theorem: for an algebraic number field K, an element in $\oplus_\nu Br(K_\nu)$ belongs to $Br(K)$ if and only if the sum of its Hasse invariants is zero.

4. The Ph.D. dissertation of Tsen.

4.1. In the summer semester of 1929, Chiungtze C. Tsen came to Göttingen University. Another Noether boy, E. Witt (1911–1991), came at the same time, who happened to have spent nine years of his childhood in China and would share a lot of mathematical interest together with Tsen in the future [**39**]. A favorite student of Noether, M. Deuring (1907–1984), just returned to Göttingen after studying with the Italian algebraic geometers F. Severi and F. Enriques for one year in Rome; Deuring would finish his Ph.D. degree the next year. Yet another person who came to Göttingen two years later was O. Teichmüller (1913–1943).

As to the faculty members of Göttingen in 1929, R. Courant was the director of the Mathematical Institute. E. Landau and G. Herglotz were full professors while H. Weyl would come to Göttingen the next year to succeed Hilbert's chair.

E. Noether (1882–1935) was then an associate professor (ausserordentlicher Professor); she remained so until her forced leave from Göttingen in 1933. However, this period was her prime time. She was one of the pioneers to revolutionize the foundation of algebra and was invited to give a talk on central simple algebra and representation theory in a special session at the Bologna International Congress in 1928. In 1932 she gave one of the 21 plenary talks at the Zürich International Congress; this time she talked on central simple algebras and algebraic number theory. The bible of abstract algebra, van der Waerden's *Moderne Algebra*, [**77**], was published in 1931, which was based on lectures of Noether and Artin. Another book bearing the stamp of Noether's mathematics, Deuring's *Algebren* [**19**], would appear in 1935. In 1932 she received with Artin the Alfred Ackermann-Teubner Memorial Prize (Alfred Ackermann-Teubner Gedächtnispreis) for advancement of mathematical science.

Despite various honors and recognitions, some people were still reserved about Noether's mathematical achievements. She was never elected to Göttinger Gesellschaft der Wissenschaften, perhaps due to prejudices about sex, race, political attitude, etc. Olga Taussky recalled that someone extended his distrust about Noether to Deuring, and a senior professor in Göttingen once talked very roughly to Noether [**72**, p. 84]. Although H. Weyl regarded Noether to be "his superior as a mathematician" and "the strongest center of mathematical activity" in Göttingen during 1930–1933 [**81**, p. 208], still he was skeptical about the abstract method advocated by Noether: "...the feeling is beginning to spread that the fertility of these abstracting methods is approaching exhaustion." [**81**, p. 214].

A. Weil made a colorful remark,

> Emmy Noether good-naturedly played the role of mother hen and guardian angel, constantly clucking away in the midst of a group from which van der Waerden and Grell stood out. Her courses would have been more useful had they been less chaotic, but nevertheless it was in this setting, and in conversations with her entourage, that I was initiated into what was beginning to be called "modern algebra" and, more specifically, into the theory of ideals in polynomial rings. [**80**, p. 51].

4.2. We now digress to the political events during Tsen's last years in Germany.

After winning the election of 1932 in Germany, Adolf Hitler, the leader of the Nazis, was made Chancellor of the Reich on January 30, 1933. On March 21, he announced the beginning of the Third Reich. Since then, the Nazi anti-Semitic policies were put into practice. Jews were systematically disqualified from taking part in German national and cultural life [**40**, p. 28].

According to a newspaper, the Göttinger Tageblatt of April 26, 1933, six professors of Göttingen University were placed on leave, and three of the names given were Born, Courant and Noether. In October, Noether boarded the "Bremen" leaving for the United States to become a visiting professor in Bryn Mawr. On November 2, a group of students in Göttingen boycotted the class of Landau; they claimed, "German students, at least the beginners, should not be taught by scientists of foreign races" [**66**, p. 5]. Among them was Teichmüller, a gifted student, who would produce significant works in quasi-conformal mappings and p-algebras [**73**] later. By Christmas of that year, Weyl also left Göttingen, accepting an offer from the Institute for Advanced Studies in Princeton. An account of Bieberbach's notorious lecture on mathematical styles, the so-called "Integrationstypus" ("I-type") and "Strahltypus" ("S-type"), was reported in a public press "Deutsche Zukunft" on April 8, 1934 [**67**]. Hasse was called to be the director of the Mathematical Institute of Göttingen in May. In the summer of 1934, Noether visited Göttingen from the United States to realize the political situation deteriorating immensely. The periodical "Deutsche Mathematik" was published in 1936 [**67**], where Teichmüller contributed quite a lot of articles. Finally, Artin left Germany in October, 1937.

4.3. Tsen published his first research paper [**T1**] just several months after Noether was dismissed. This paper is an excerpt of his dissertation [**T2**], the manuscript of which was finished on September 20, 1933. Although Noether was no more the official mentor of Tsen, it was quite possible that she had read this dissertation before she left Germany.

The title of Tsen's dissertation is *Algebren über Funktionenkörpern* (*Algebras over function fields*). One of the main results is

Theorem 4.4 [**T2**, Hauptsatz 2]. *Let Ω be an algebraically closed field, $\Omega(x)$ the rational function field in one variable over Ω, k a finite extension field of $\Omega(x)$. Then every central division k-algebra is trivial, i.e., $Br(k) = \{0\}$.*

4.5. The problem considered in the above theorem was suggested by Noether [**57**]. As an easy consequence of this theorem, it can be shown that, if P is a real closed field and k an algebraic function field in one variable over P, then every nontrivial central division k-algebra (if it exists at all) is of index 2, i.e., every central division k-algebra is a quaternion algebra $(a, b)_{2,k}$ for some $a, b \in k\backslash\{0\}$. In particular, this is the case when $k = P(x)$, the rational function field over P [**T2**, Satz 9]. Moreover, Tsen found that the similar statement as in Theorem 4.4 would be false if k is an algebraic function field in more than one variable over Ω; a simple example: the quaternion algebra $(x_1, x_2)_{2,k}$ is not trivial when $K := k(x_1, \dots, x_n)$ the rational field in n variables with $n \geq 2$ and k is any field with $\operatorname{char} k \neq 2$.

All the above together with Theorem 4.4 are the full content of Tsen's first paper [**T1**]. [**T1**] was submitted to Weyl and was presented on June 30, 1933, in Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen.

4.6. The result concerning $P(x)$ in the previous subsection, i.e., every nontrivial central division algebra over $P(x)$, where P is real closed, was of index two was included in Deuring's book [**19**, p. 137]. Unfortunately, this result is often misinterpreted as: there is precisely one nontrivial central division algebra over $P(x)$, which is of index two [**47, 48, 76**]. In fact, it is easy to see that, for any $a, b \in P$, the quaternion algebra $(x - a, -1)_{2,P(x)}$ is isomorphic to $(x - b, -1)_{2,P(x)}$ if and only if $a = b$. Thus there are infinitely many nontrivial division algebras over $P(x)$.

4.7. The oral examination required for the Ph.D. degree (in fact, Dr. rer. nat.) took place on December 6, 1933, with the examiners: F.K. Schmidt (Analysis), Franz Rellich (Geometry) and Georg Misch (Philosophy). The degree was conferred on February 20, 1934. In a letter of Noether [**57**], she evaluated this dissertation as "sehr gut" ("summa cum laude," very good). It was reported that the official reviewer of this dissertation was F.K. Schmidt [**48**], but we cannot find an official evidence to support this assertion, albeit the fact that in October 1933 Schmidt was invited to Göttingen in place of H. Weyl and stayed there for one year.[8]

Tsen dedicated this dissertation to his elder cousin Tsebu S. Lee (Zhibu Lei). Lei graduated from First Senior High School (in Tokyo) and Tokyo Imperial University. He was once the president of a provincial college of industry and technology in Chiangshi. Lei had been a financial benefactor to Tsen from Tsen's years as a college student.

4.8. The first chapter of [**T2**] is the preliminaries. The second chapter is devoted to proving Theorem 4.4. The strategy to prove it is rather natural, although somewhat devious. We shall summarize the main idea as follows.

Let the notations be the same as in Theorem 4.4. Suppose that A is a nontrivial central division k-algebra with $\operatorname{ind}(A) \geq 2$. Let K be a maximal separable subfield of A and L the minimal Galois extension of k containing K. Choose a prime number p dividing $\operatorname{ind}(A)$ and a p-Sylow subgroup H of the Galois group of L over k. Since we can find subfields $L_0, L_1, \dots, L_r$ such that

(i) $L := L_0 \supset L_1 \supset \cdots \supset L_r := L^H$, and

(ii) each L_{i-1} is a cyclic extension of L_i,

it follows that $A \otimes_k L$ is similar to a cyclic algebra over L_1. Note that each L_i is again an algebraic function field in one variable over Ω. If we can prove that any cyclic algebra over an algebraic function field in one variable over Ω is trivial, then $A \otimes_k L_1$ is trivial and $A \otimes_k L_2$ is similar to a cyclic algebra over L_2. Inductively, we can show that $A \otimes_k L_r$ is trivial. Thus L_r is a splitting field of A and $\operatorname{ind}(A) | [L_r : k]$. This leads to a contradiction.

Note that similar arguments as the above reduction process have been used in the proof of Brauer-Hasse-Noether's theorem [**10**, Reduktion 3, p. 401].

In conclusion, if we denote by k any algebraic function field in one variable over Ω, the proof of Theorem 4.4 is reduced to proving every cyclic k-algebra is trivial. The latter is equivalent to proving that $k = N_{L/k}(L)$ where L is any cyclic extension of k and $N_{L/k}$ is the norm function from L into k.

In fact, Tsen proved more than the above statement. He proved

Theorem 4.9 [**T2**, Satz 4]. *Let S be a finite-dimensional k-algebra. Assume that*

(i) *S is a finite division ring, or*

(ii) *k is an algebraic function field in one variable over an algebraically closed field, and S is either a field or a division ring.*

Then $k = N_{S/k}(S)$, where $N_{S/k}$ is the norm function of S into k when S is regarded as a finite-dimensional k-algebra.

Remark. Note that the above theorem implies not only Theorem 4.4, but also Wedderburn's theorem [**79**]: Every finite division ring is a field.[9] On the other hand, the proof of part (ii) of the above theorem may be reduced to the following

Theorem 4.10 [**T2**, Satz 5]. *Let Ω be an algebraically closed field, $f_1, \dots, f_r \in \Omega(x)[T_1, \dots, T_n]$ be homogeneous polynomials of the same degree d. If $n > rd$, then $f(T_1, \dots, T_n) = \cdots = f_r(T_1, \dots, T_n) = 0$ has a nontrivial solution over $\Omega(x)$.*

4.11. To prove Theorem 4.10, Tsen discovered an elementary but ingenious method which became one of the key ideas in [**T3, 44**], and is presented in some textbooks [**27**, (3.6) Theorem, pp. 22–23; **36**, Lemma 4, p. 652].[10] This concludes the proof of Theorem 4.4, which is the first proof presented in [**T2**].

The second proof of Theorem 4.4 is due to Noether by assuming that Part (ii) of Theorem 4.9 is valid when S is a central division k-algebra. Let $Nrd : S \to k$ be the reduced norm from S into k and $\operatorname{ind}(S) = n$. Then $N_{S/k}(\alpha) = \{Nrd(\alpha)\}^n$ for any $\alpha \in S$. Since $x \in \Omega(x) \subset k = N_{S/k}(S)$, it follows that $x^{1/n}, x^{1/n^2}, x^{1/n^3}, \dots$ are in k. Thus $[k : \Omega(x)]$ cannot be finite except when $n = 1$.

The third proof is due to Artin. He analyzed Tsen's proof and found

Theorem 4.12 (Artin [**T2**, Satz 7]). *If K is an N_1-field, so is any finite extension field of K.*

Remark. We shall define the notion of N_1-fields in 5.2. By Lang's theorem, see remark (iii) of Theorem 5.4, the notion of C_1-fields and that of N_1-fields are equivalent. Assuming Theorem 4.10 and

Theorem 4.12, Artin then showed that $Br(K) = \{0\}$ for any N_1-field K, and therefore Theorem 4.4 was proved. This is almost the same proof given in most textbooks [**69, 27, 71, 36, 64**] except that N_1-fields are replaced by C_1-fields. Chevalley described briefly the argument using C_1-fields, which was identical with the proof we found in most textbooks nowadays [**16**]. (Theorem 4.12 was called Artin's transition theorem by Tsen.)

4.13. Assuming $\operatorname{char} k = 0$, [**85**, Satz 4] provides a proof of Part (ii) of Theorem 4.9 when S is a field. So, by the arguments at the end of 4.8, this furnishes another proof of Tsen's theorem when $\operatorname{char} k = 0$. It is remarkable that Witt's proof is analytic in nature; he used only Abel's theorem and Jacobi inversion theorem in the theory of Riemann surfaces. It seems that there is some flaw in this proof because the division by n in a Jacobian variety is unique only up to an n-division point. Thus the assertion in [**82**, p. 9, line 22] looks dubious.

4.14. We now turn to the third chapter of [**T2**]. In this chapter, P is denoted as a real closed field and Tsen tried to understand central simple algebras over $P(x)$, the rational function field in one variable over P.

Since every central division algebra over $P(x)$ is of index two, the main task is equivalent to the calculation of Hilbert symbols. Indeed, after lengthy calculations [**T2**, pp. 12–15], Tsen was able to prove the following

Theorem 4.15 [**T2**, Satz 13]. *For any $\alpha, \beta \in P(x)\backslash\{0\}$, the Hilbert symbol $(\alpha, \beta)_{2,P(x)}$ is nontrivial if and only if there exists some $c \in P$ so that both α, β are well-defined at c and $\alpha(c) < 0$, $\beta(c) < 0$.*

Remark. In [**T2**, p. 16], Tsen included another proof of the above theorem, which was due to Witt. With the help of this theorem, the local-global principle is established. Namely,

Theorem 4.16 [**T2**, Satz 16]. *Let A be a central simple algebra over $P(x)$. Then A is trivial if and only if $A \otimes_{P(x)} P(x)_\nu$ is trivial where*

ν is any P-place of $P(x)$, i.e., ν is a valuation on $P(x)$ such that ν is trivial on P.

Remark. It is not difficult to show that Theorem 4.15 and Theorem 4.16 are equivalent.

4.17. In [**T2**, Satz 17], Tsen proposed to study the structure of universal splitting fields. A field extension K containing k is called a universal splitting field for $Br(k)$ if the group homomorphism $Br(k) \to Br(K)$ is the zero map. For example, $P(\sqrt{-1})(x)$ is a universal splitting field of $Br(P(x))$. For any $f(x) \in P(x)\backslash\{0\}$, Tsen showed that $P(x, \sqrt{-f(x)})$ is a universal splitting field if and only if $f(x)$ is positive-definite, i.e., there is no $c \in P$ such that $f(x)$ is defined at c and $f(c) < 0$. [**T2**, Satz 18] says that if K is a Galois extension of $P(x)$ of even degree and has no real point, i.e., the residue field of any P-place of K is isomorphic to $P(\sqrt{-1})$, then K is a universal splitting field. Note that Witt again provided another proof of this result in a more general situation. (See the paragraph after the proof of Satz 4 in [**82**].)

4.18. It seems that the results in the third chapter of [**T2**] are rather incomplete, because all of them were superseded immediately by [**82**]. (Note that the manuscript of [**82**] was finished on New Year's Eve of 1934, just about three months after Tsen finished his manuscript.) Thus we would like to remind the reader of relevant results of Witt and the subsequent developments. Theorem 4.16 was generalized to the case of real function fields as follows:

Theorem 4.19 [**82**, pp. 4–5]. *Let k be an algebraic function field in one variable over exact constant field $\mathbf{R}$, the field of real numbers.*

(i) *If $\alpha \in k$, then α is positive-definite if and only if $\alpha = \beta^2 + \gamma^2$ for some $\beta, \gamma \in k$.*

(ii) *If A is any central simple k-algebra, then A is trivial if and only if $A \otimes_k k_\nu$ is trivial where ν is any $\mathbf{R}$-place of k.*

Remarks. (1) In Theorem 4.19, the constant field $\mathbf{R}$ can be replaced by any real closed field. In fact, the generalized version for (i) is established in [**41**, (4.1) Theorem]. Now we shall prove the generalized version of (ii). By Theorem 4.4, every central simple k-algebra is split by $k(\sqrt{-1})$, and thus is similar to $(\alpha,-1)_{2,k}$ for some $\alpha \in k\backslash\{0\}$. Using the generalized version of (i), we find that $(\alpha,-1)_{2,k}$ is trivial, $\Leftrightarrow \alpha = \beta^2+\gamma^2$ for some $\beta,\gamma \in k$, $\Leftrightarrow \alpha$ is positive-definite, $\Leftrightarrow \alpha X^2 - Y^2 = Z^2$ has a nontrivial solution over k_ν for any P-place ν on k.

(2) For a modern treatment of results in [**82**], see [**41**] and [**17**].

4.20. There is yet another generalization of Theorem 4.16. Thanks to Witt's algebraic theory of quadratic forms in [**84**], the triviality of the Hilbert symbol $(\alpha,\beta)_{2,P(x)}$ is equivalent to the hyperbolicity of the quadratic form $X^2-\alpha Y^2-\beta Z^2+\alpha\beta U^2$ over $P(x)$. Now the local-global principle for quadratic forms, in particular that for the forms $X^2-\alpha Y^2-\beta Z^2+\alpha\beta U^2$, over $K(x)$ where K is any field with $\operatorname{char} K \neq 2$ follows from the following theorem of Milnor on the Witt rings of quadratic forms.

Theorem 4.21 [**50**, Theorem 5.3]. *Let K be any field with $\operatorname{char} K \neq 2$, $K(x)$ the rational function field over K. Denote by $W(K)$, $W(K(x))$ the Witt rings of quadratic forms over K and $K(x)$, respectively. Then the following is a short exact sequence*

$$0 \longrightarrow W(K) \longrightarrow W(K(x)) \longrightarrow \oplus W(\overline{K(x)}_\pi) \longrightarrow 0$$

where the summation extends over all monic irreducible polynomials π in $K[x]$.

5. The level of a field.

5.1. When Tsen took the oral examination and prepared his Ph.D. dissertation in 1933, Noether had been dismissed from Göttingen University. Witt got the Ph.D. degree in June of this year. From 1933 until 1938, Witt conducted a seminar in Göttingen [**39**]. After Hasse came to Göttingen at the end of June of 1934, Witt became his assistant. Participants of Witt's seminar included Hasse, H.L. Schmid and Teichmüller. We don't know whether Tsen ever attended this seminar.

After getting the Ph.D. degree in 1934, Tsen moved to Hamburg University for postdoctoral study. He got a one-year financial support as a research fellow from the China Foundation for the Promotion of Education and Culture.[11] Emil Artin was there. Obviously, Artin's attraction was the reason for Tsen's move to Hamburg. There he met S.S. Chern who was still a graduate student of W. Blashke then and was to become one of the outstanding mathematicians of this century. Two years later Chern went to Paris to work with Elie Cartan for the postdoctoral study, again sponsored by the same Foundation.

In 1935 the Chinese Mathematical Society was established. The next year the first volume of "Journal of Chinese Mathematical Society" was published. In this volume appeared Tsen's third paper [**T3**]. Results in this paper were obtained when he was in Hamburg (1934–1935); and the manuscript was finished on May 15, 1936. He dedicated this paper in memory of Noether who died on April 14, 1935.

It seems that just a handful of people have ever read this paper. So far as we know, the only research papers or textbooks which list [**T3**] as a reference are [**2, 34, 20, 40, 49, 63**] and papers by Pfister, say [**60, 61, 62**]. In fact, both Albert and Teichmüller wrote reviews of this paper [**3, 74**]; moreover, Nakayama mentioned this paper in his review of Serge Lang's dissertation [**52**]. We believe that around the 1970's this paper was discovered and studied by several German mathematicians, including Albrecht Pfister and Falko Lorenz.[12] Pfister remarked,

> Apparently his paper has been forgotten during the war, so that his results had to be rediscovered by Lang before they reached the mathematical community. [**61**, p. 485].

5.2. The influence of Artin on this paper is clear. After Artin analyzed Tsen's proof of Theorem 4.4, he proposed to call a field K quasi algebraically-closed if for any homogeneous polynomial $f(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$ with $n > \deg f$, f has a nontrivial zero over K. The idea of quasi algebraic-closedness plays a crucial role both in Tsen's paper [**T3**] and Lang's dissertation [**44**].

We should note a difference between Tsen's formulation and Lang's formulation, however. Lang considered both the case of only one equation (homogeneous or just a polynomial without constant term) and the case of a system of equations, while Tsen insisted to work with a system of equations.

To be precise, for any real number α, a field K is called a T_α-field, respectively an ST_α-field, if for any homogeneous polynomials, respectively polynomials without constant terms, $f_1, \dots, f_r \in K[x_1, \dots, x_n]$ such that $n > \sum_{i=1}^r (\deg f_i)^\alpha$, $f_1(x_1, \dots, x_n) = f_2(x_1, \dots, x_n) = \cdots = f_r(x_1, \dots, x_n) = 0$ has a nontrivial solution over K.

On the other hand, for any nonnegative integer i, a field K is called a C_i-field, respectively an SC_i-field, if for any homogeneous polynomial, respectively a polynomial without constant term, $f \in K[x_1, \dots, x_n]$ with $n > (\deg f)^i$, $f(x_1 \cdots x_n) = 0$ has a nontrivial solution over K.

Similarly, for any nonnegative integer i, a field K is called an N_i-field, respectively an SN_i-field, if for any homogeneous polynomials, respectively polynomials without constant terms, $f_1, \dots, f_r \in K[x_1, \dots, x_n]$ such that $n > rd^i$ where d is the common degree of $f_1, \dots, f_r$, respectively d is the maximal degree of $f_1, \dots, f_r$, $f_1(x_1, \dots, x_n) = f_2(x_1, \dots, x_n) = \cdots = f_r(x_1, \dots, x_n) = 0$ has a nontrivial solution over K.

Except for C_i-fields, the names of T_α-field, ST_α-fields, SC_i-fields, N_i-fields and SN_i-fields are introduced by us, following [**49**]. (An ST_i-field is called a T_i-field in [**49**].) An ST_α-field is called a field of level α by Tsen [**T3**], and the investigation of the level of a field is the theme of this paper. An SC_i-field is called a strongly C_i-field by Lang [**44**]. N_i-fields and SN_i-fields are named after Nagata [**51**]. By Nagata's theorem, a field is a C_i-field, respectively an SC_i-field, if and only if it is an N_i-field, respectively an SN_i-field. See Remark (iii) of Theorem 5.5. Note that in current mathematical literature, the notion of the level of a field K is completely different from Tsen's level: A positive integer s is called the level of a field K if s is the minimal integer n such that -1 can be written as a sum of n squares in K; if it is impossible to find such an integer, the level of K is defined to be ∞. This notion is due to H. Kneser [**63**, Chapter 3].

For a nonnegative integer i, it is clear that $T_i \Rightarrow N_i \Rightarrow C_i$ and $ST_i \Rightarrow SN_i \Rightarrow SC_i$. However, it is not straightforward at all whether $C_i \Rightarrow T_i$ and $SC_i \Rightarrow ST_i$. In his review of Lang's paper, Nakayama didn't compare the definitions and the corresponding results of ST_i-fields and C_i-fields carefully and therefore made some confusing remarks [**52**].

Note that in Tsen's definition, a field can be an ST_α-field where α

is any real number, actually a nonnegative real number, not just a nonnegative integer. However, Tsen remarked at the end of his paper that he could not find an example of a field K which was an ST_α-field, but not an $ST_{[\alpha]}$-field, $[\alpha]$: the integer part of α. Note that Nagata also conceived the possibility of C_α-fields when α is just a real number, and proposed a somewhat similar problem [**51**, Problem 5].

To allow the parameter α in an ST_α-field being any real number will cause extra difficulties sometimes. Suppose that a field K is an ST_α-field for some real number α and define β to be the greatest lower bound of the following set

$$\{\alpha \in \mathbf{R} : K \text{ is an } ST_\alpha\text{-field}\}.$$

We may thus call this field K a genuine ST_β-field temporarily. Caution: There is no guarantee that a genuine ST_β-field should be an ST_β-field. Unfortunately, in his paper, Tsen called an ST_α-field a field of level α in the definition and in some situations, while he meant a genuine ST_α-field for a field of level α on other occasions [**T3**]. Even Albert and Teichmüller were confused. They decided to call the level of a field α, if it was a genuine ST_α-field [**3, 74**]. Thus we shall abandon the terminology of a genuine ST_α-field in the sequel.

5.3. Before discussing possible relations among C_i-fields, T_i-fields, etc., the notion of a normic form for a field K was introduced both in [**T3**] and [**44**]. For a field K and a positive integer m, a normic form of order m over K is a homogeneous polynomial $f \in K[x_1, \dots, x_n]$ with $\deg f \geq 2$ and $n = (\deg f)^m$ such that $f(x_1, \dots, x_n) = 0$ has only the trivial solution over K.

Theorem 5.4. (i) [**T3**, Satz 7]. *An SC_1-field K satisfies the following property*:

For any polynomials $f_1, \dots, f_r \in K[x_1, \dots, x_n]$ with the same degree d and without constant terms, if $n > rd$, then $f_1(x_1, \dots, x_n) = \cdots = f_r(x_1, \dots, x_n) = 0$ has a nontrivial solution over K.

(ii) [**T3**, Satz 8]. *If K is an SC_1-field and has a field extension of degree p for any given prime number p, then K is an ST_1-field.*

Remarks. (i) In [**T3**, Satz 7], Tsen actually proved that an SC_1-field is an SN_1-field. The assumption that the field K has a field extension of any given prime degree in [**T3**, Satz 8] was used by Tsen to guarantee that the field K had a normic form of any given degree ≥ 2.

(ii) In [**T3**, Satz 8], Tsen mistakenly wrote a C_1-field for an SC_1-field. However, from his proof it is easy to see that he meant an SC_1-field.

(iii) Serge Lang proved that a C_1-field is an N_1-field, and an SC_1-field is an SN_1-field without assuming the existence of normic forms [**44**, Corollary, p. 376].

Theorem 5.5. *Let i be a positive integer.*

(i) [**T3**, Satz 9]. *If a field K is an SC_i-field and has a normic form of order i, then K satisfies the following properties*:

For any polynomials $f_1, f_2, \ldots, f_r \in K[x_1, \ldots, x_n]$ with the same degree d and without constant terms, if $n > rd^i$, then $f_1(x_1, \ldots, x_n) = f_2(x_1, \ldots, x_n) = \cdots = f_r(x_1, \ldots, x_n) = 0$ has a nontrivial solution over K.

(ii) [**T3**, Satz 10]. *If an SC_i-field has a normic form of order i with any given degree ≥ 2, then it is an ST_i-field.*

Remarks. (i) Again, what Tsen actually proves in [**T3**, Satz 9] is that an SC_i-field with a normic form of order i is an SN_i-field.

(ii) Lang proved that if K is a C_i-field, respectively an SC_i-field, and has a normic form of order i, then it is an N_i-field, respectively an SN_i-field [**44**, Theorem 3]. Also Lang proved that if K is a C_i-field, respectively an SC_i-field, and has a normic form of order i with any given degree ≥ 2, then K is a T_i-field, respectively an ST_i-field [**44**, Theorem 4].

(iii) Nagata showed that a C_i-field, respectively an SC_i-field, is an N_i-field, respectively an SN_i-field, without any assumption on the existence of normic forms [**44**, Theorem 1a and Theorem 1b].

(iv) [**T3**, Satz 7], i.e., Part (i) of Theorem 5.4 was called "Artin's equivalence theorem" by Tsen, while in [**44**, Theorem 3] it was called "Artin's criterion" by Lang.

5.6. Although it is still unknown whether a T_i-field may impose stronger conditions than a C_i-field, the "transition properties" of both are quite similar.

First of all, Tsen showed that if a field was an ST_α-field for some real number α, then $\alpha \geq 1$ or $\alpha = 0$. Clearly, a field is an ST_0-field, or a C_0-field, if and only if it is algebraically closed. Then the nontrivial results:

Theorem 5.7. (i) [**T3**, Satz 1]. *Let α be a positive real number. If a field is an ST_α-field, so is any finite extension of it.*

(ii) [**44**, Theorem 5]. *Let i be a positive integer. If a field is a C_i-field, respectively an SC_i-field, and has a normic form of order i, respectively a normic form of order i for any given degree ≥ 2, so is any finite extension of it.*

(iii) [**44**, Theorem 2a and Theorem 2b]. *Let i be any positive integer. If a field is a C_i-field, respectively an SC_i-field, so is any finite extension of it.*

Theorem 5.8. (i) [**T3**, Satz 5]. *Let α be a nonnegative real number. If a field K is an ST_α-field, then $K(x)$ is an $ST_{\alpha+1}$-field.*

(ii) [**44**, Theorem 6]. *Let i be a nonnegative integer. If a field K is a C_i-field, respectively an SC_i-field, and has a normic form of order i, respectively a normic form of order i with any given degree ≥ 2, then $K(x)$ is a C_{i+1}-field, respectively an SC_{i+1}-field. Similar results generalize to T_i-fields and ST_i-fields.*

(iii) [**44**, Theorem 2a and Theorem 2b]. *Let i be a nonnegative integer. If a field K is a C_i-field, respectively an SC_i-field, then $K(x)$ is a C_{i+1}-field, respectively an SC_{i+1}-field.*

5.9. As a corollary of the above theorem, Tsen deduced that, for any positive integer i, the rational function field $K(x_1, \dots, x_{i-1})$ is an ST_i-field and not an ST_α-field for any $\alpha < i$, provided that K is an ST_1-field and is not algebraically closed [**T3**, Satz 6]. The key point to proving this result is the property: If i is any positive integer, if K has a normic form of order i, then $K(x)$ has a normic form of order $i+1$ [**T3**, Satz 4; **44**, p. 377].

As indicated in Theorem 5.5, SC_i-fields had already appeared in

Satz 9 and Satz 10 in [**T3**], although Tsen didn't develop the formal properties of these fields systematically. We don't know whether he had ever tried to work on some conjectures of Artin, e.g., is a local field necessarily a C_2-field?

In Lang's paper, in order to prove the transition theorem, i.e., Theorem 5.8. (ii), it is necessary to have the equivalence theorem first [**44**, Theorem 3]. It is not the same situation in the case of ST_α-fields. Tsen first studied various properties of ST_α-fields and then proceeded to the equivalence theorem.

Another remark about Tsen's third paper. We should like to point out that there are many misprints in this paper, although it is not difficult for an expert to detect them. Even the definitions of levels [**T3**, p. 82, line 11] and normic forms [**T3**, p. 86, line 10] were misprinted. Perhaps one may have a sympathetic understanding towards these irritating misprints, if he takes into account the situation in China then: there were not enough experienced printers for scientific papers (in western languages!), China had engaged in wars with Japan now and then since 1931, the over-all war between China and Japan broke out one year after this paper was published, the Journal of Chinese Mathematical Society published only two volumes (1936 and 1940) and ceased to publish afterwards.

5.10. Due to the similarity of definitions and results of Tsen's ST_α-fields and Lang's C_i-fields, one may wonder whether Artin and/or Lang knew Tsen's paper [**T3**]. We thus sent a preliminary version of this article to Serge Lang and asked his comments. Prof. Lang replied immediately on April 8, 1996. He said, "Obviously I did not know of his paper in Ch. Math. Soc. 1936. I'll take it into account from now on."

6. Epilogue.

6.1. Theorem 4.4 is often quoted as Tsen's theorem. It is the foundation to study $Br(\mathbf{Q}(x))$ or $Br(K(x))$ where $\mathbf{Q}$ is the field of rational numbers and K is any field. It is a pity that Tsen didn't pursue along this line, perhaps due to his untimely death and also due to the fact that the necessary machinery, e.g., homological algebra, corestriction algebra, was not available at that time. This problem awaited the next generation, D.K. Fadeev (1951), M. Auslander and Brumer (1968), to attack. See [**24**, p. 51] for the solution and relevant results.

During the late 1950's, the theory of Galois cohomology and étale cohomology was built up by Lang, Serre, Tate, Grothendieck and his school, etc. [**68, 18, 25**]. It is in this context that Tsen's theorem becomes even more important. Tsen's theorem provides a vanishing theorem for the second étale cohomology of an algebraic curve over an algebraically closed field with the multiplicative group as its coefficient group. In case of a torsion coefficient, one gets information with the aid of the Kummer sequence. Without Tsen's theorem, the Kummer theory would not give very much. In the higher dimensional case, some useful information can be extracted by the method of fibering by curves. In short, Tsen's theorem and Hilbert Satz 90 form two basic vanishing theorems for étale cohomology. Combined with various devissage reduction steps, they give many fundamental results, e.g., the proper base change theorem [**18; 25**, p. 3].

For other applications of Tsen's theorem see, for example, [**70**, p. 24].

6.2. An application of ST_i-fields, or C_i-fields, is in quadratic form. Pfister found the theory of multiplicative forms, or Pfister forms, in 1964. With the aid of Theorem 5.8, he showed in 1967 that if $K := R(x_1, \dots , x_n)$ where R is a real closed field, then every element in K which is a sum of squares can be written as a sum of 2^n squares [**61**].

6.3. As indicated before, when Artin analyzed the first proof of Tsen's theorem [**T1**; 4.8–4.10 of this paper], he found that something more could be said. He formulated the notion of quasi algebraically-closed fields (= C_1-fields or N_1-fields) and showed that the Brauer group of such a field was trivial. This provided another method to prove $Br(K) = 0$ for some field K. In view of Wedderburn's theorem that the Brauer group of any finite field was trivial [**79**], it led Artin to conjecture that any finite field is a C_1-field. This was confirmed immediately by Chevalley that a finite field was not only a C_1-field, but also an ST_1-field [**16**]. Warning then showed that the number of solutions was $\equiv 0 \pmod p$ where p is the characteristic of this finite field [**78**]. Ax found a sharper estimation than Warning's [**7**].

Artin also formulated other conjectures [**46**, pp. 245–250]. For example, if K is a field which is complete under a discrete valuation with perfect residue class field, Artin conjectured that (i) the maximal unramified extension of K was a C_1-field and (ii) K is a C_2-field provided its residue class field is a finite field. The first conjecture was proved by Lang [**44**]. A counterexample to the second conjecture was given by Terjanian [**75**]. For various results of Terjanian, Schanuel, Ax and Kochen, see [**71**, Chapter 4].

6.4. Along the rationale of Artin's conjecture on finite fields, Tsen went further. He asked the following question:

Question [**T3**, p. 86, lines 2–3]. Let K be a field with $Br(K) = 0$. Is K an ST_α-field for some $\alpha < 2$?

In [**51**, Problem 6], Nagata showed that, if K is a C_1-field, then $N_{L/K}(L) = K$ for any finite extension L of K. He then asked whether the converse was true, i.e., $N_{L/K}(L) = K$ for any finite extension L would imply that K is a C_1-field. (Note that the assumption $N_{L/K}(L) = K$ for any L will imply $Br(K) = 0$ by the argument in 4.8.)

In 1962 Serre found that if K was a C_1-field and $\operatorname{char} K = p > 0$, then $[K : K^p] \leq p$. Thus, let K be the separable closure of $k(x, y)$ where k is any field with $\operatorname{char} k = p > 0$ and $k(x, y)$ is the rational function field in two variables over k. Then every algebraic extension of K has trivial Brauer group, while K is not a C_1-field [**68**]. For a counterexample in characteristic zero to Nagata's question, the example Y in Ax's construction [**8**, pp. 1215–1216] will work.

It turns out that there is no easy way to characterize a C_1-field or an ST_α-field for some $\alpha < 2$. Thanks to an anonymous referee of this paper who pointed out the negative answer to Tsen's question. By an example due to M. Auslander [**69**, Exercise 1 of 3.1, p. 89], we can find a field K with $\operatorname{char} K = 0$ such that $Br(K) = 0$ and the dimension of K is greater than 1. In particular, there is a finite extension field K' of K such that $Br(K') \neq 0$. Suppose that K is an ST_α-field for some $\alpha < 2$. Then K' is also an ST_α-field with the same α by Theorem 5.7

(i). Now choose a central division K'-algebra of degree d with $d > 1$. The reduced norm of it is a homogeneous polynomial of degree d in d^2 variables over K'. Since this polynomial has no nontrivial zero, K' is not an ST_β-field for any $\beta < 2$. We get a contradiction.

6.5. There are other generalizations of the notions of C_i-fields. For example, Lang defines a field K to be an oddly C_i-field if for every homogeneous polynomial $f \in K[x_1, \dots, x_n]$ with odd degree so that $n > \deg f$, $f(x_1, \dots, x_n) = 0$ has a nontrivial solution over K [**45**]. M. Amer defines a field K to be a C_i^q-field if for every quadratic form $f_1, \dots, f_r \in K[x_1, \dots, x_n]$ with $n > r \cdot 2^i$, $f_1(x_1, \dots, x_n) = \dots = f_r(x_1, \dots, x_n) = 0$ has a nontrivial solution over K [**62**].

6.6. Tsen returned to China in July, 1935. Although he had good training in abstract algebra, a fashionable subject at that time, it seemed that he didn't have a profound impact on the development of modern mathematics in China. From the article of Chuan-Chih Hsiung, a differential geometer of Lehigh University, we know that Tsen had offered a course of modern algebra based on van der Waerden's book, and a course of group theory based on A. Speiser's book [**32**]. It seems that nobody in China then pursued study of Tsen's research subject, the theory of central simple algebras, because of his teaching.[13]

In the meantime, Witt and Teichmüller continued to work on central simple algebras, besides other research directions [**83, 85, 86, 73**]. There is no knowing whether Tsen had access to their on-going progress. But the sudden death ended everything.

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ENDNOTES

1. The first Chinese to get a Ph.D. degree in physics is Fuji Li (1907, Bonn University).

2. The period 1928–1937 is a "golden decade" in modern China for the first 50 years of this century. During these ten years China was reunited and the political situation became relatively more stable. For an appreciation of the rapid development of education in China then, Peking University in particular, see [**23**], for example.

3. Although Chiungtze Tsen was almost of the same age as Kien-Kwong Chen (Jiangong Chen, 1893–1971) and Buchin Su (Buqing Su, born in 1902), we will count neither Chen nor Su as members of this generation because Chen graduated from Hangzhou Senior Normal College in 1913 and Su studied in Tokyo Senior Industrial School until 1920.

4. Some references reported that Tsen was born in 1897. See [**38**, p. 108] in particular.

5. The only Noether boys from Asia were: Tsen from China, Kenjirô Shoda (1902–1977) and Shinzirô Mori (1893–1979) from Japan [**40**, p. 42; **26**, pp. 141–142]. Although Zyoiti Suetuna (1898–1970) frequented Noether's circle [**20**, p. 65], his research interest was not the same as that of Noether.

6. We don't know the reason why Tsen accepted the offer of Beiyang Institute of Technology. A conjecture which tried to explain it was reported in [**14**, p. 25; **47**, p. 63].

7. Another source reported the year as 1938 [**47**, p. 63].

8. Pages 46–47 of [**38**] is a Chinese translation of [**Tb**]. On [**38**, p. 47, lines 2–4] it contains the following passage, which was obviously translated from German, "I should like to thank my thesis advisor Prof. Dr. F.K. Schmidt." However, we cannot find similar passages in [**Tb**].

9. Wedderburn's theorem on finite division algebras was regarded as a nontrivial result then. Besides this proof by Tsen, there are several other proofs due to Wedderburn, Artin, Witt, Chevalley, etc. [**58**].

10. Prof. S.S. Chern provided an anecdote about Theorem 4.4. Tsen used to tell his friends that Theorem 4.4 was related to an exercise in Bôcher's book [**9**]. Perhaps Tsen meant that he got the inspiration of the proof of Theorem 4.10 from some exercise of [**9**]. We guess that the exercise in Tsen's mind is [**9**, Exercise 3, pp. 238–239].

11. The China Foundation for the Promotion of Education and Culture was established in 1925 by the American government owing to a remission of the Boxer indemnity.

12. We should like to thank Prof. A. Pfister who informed us of the story of how he found Tsen's level:

> In December 1967 Witt invited me to give a colloquium talk in Hamburg. As usual I attributed the results on C_i-fields for $i > 1$ to Serge Lang, but Witt immediately replied, "No, these results are due to my "friend" Tsen. You can find them in the boxes of separata in the library in Göttingen." Thus I got to know these wonderful papers.

Prof. I. Kersten also told us that Witt always talked about Tsen's level and Tsen's theorem in his algebra lectures.

13. Among Tsen's students in Zhejiang University are: Zhengguo Bai (differential geometry, later at Hangzhou University), Chuan-Chih Hsiung, Sucheng Zhang (algebraic topology, later at Academia Sinica) and Fuzu Zhu (arithmetic theory of quadratic forms, later at East China Normal University). Tsen taught the course "higher mathematics" in Xikang Institute of Technology, which was supposed to be a service course.

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Department of Mathematics, Peking University, Beijing, China
E-mail address: `dingss@sun.ihep.ac.cn`

Department of Mathematics, National Taiwan University, Taipei, Taiwan, China
E-mail address: `kang@math.ntu.edu.tw`

Department of Mathematical Sciences, Chengchi University, Taipei, Taiwan, China
E-mail address: `engt@cc.nccu.edu.tw`