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DUALITY OF THE BERGMAN SPACES ON SOME WEAKLY PSEUDOCONVEX DOMAINS

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1. Introduction and statement of results. Let D be a smoothly bounded pseudoconvex domain of finite type in $\mathbf{C}^2$, i.e., every one-dimensional complex submanifold of $\mathbf{C}^2$ which is tangent to the boundary bD has finite order of contact with bD. There are many equivalent formulations of finite type. For a more complete description the reader is referred to the survey article by D'Angelo [**6**] and the references therein.

For $1 \leq p < \infty$ and dV Lebesgue measure we consider the closed subspace $\mathcal{O}L^p(D)$ of $L^p(D, dV)$ consisting of holomorphic functions. These spaces are commonly referred to as the Bergman spaces. $\mathcal{O}L^p(D)$ is a Banach space with norm $\|f\|_p^p = \int_D |f(z)|^p \, dV(z)$. As usual, if X is a Banach space, we denote its dual by X^*. X^* is also a Banach space with norm $\|\cdot\|^*$. In this paper we prove the following

Theorem A. *For $1 < p < \infty$, the dual of $\mathcal{O}L^p(D)$ can be identified with $\mathcal{O}L^q(D)$ where $(1/p) + (1/q) = 1$. More precisely,*

(1) *If $g \in \mathcal{O}L^q(D)$, then g induces a bounded linear functional on $\mathcal{O}L^p(D)$ via the integral pairing*

$$\Phi(f) = \int_D f(z)\overline{g(z)} \, dV(z), \quad f \in \mathcal{O}L^p(D). \tag{1.1}$$

(2) *If $\Phi \in \mathcal{O}L^p(D)^*$, then there is a $g \in \mathcal{O}L^q(D)$ such that Φ is of the form* (1.1). *Moreover, the norms $\|g\|_p$ and $\|\Phi\|^*$ are equivalent.*

Fix a defining function $r(z)$ for D. We denote by ∇_r the complex normal derivative $\nabla_r = \sum_{j=1}^2 (\partial r/\partial \bar{z}_j)(\partial/\partial z_j)$. The Bloch space $\mathcal{B}(D)$ is the set of functions holomorphic on D satisfying $\|f\|_{\mathcal{B}} = \sup\{|r(z)\nabla_r f(z)| : z \in D\} < \infty$. We also prove the following

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Theorem B. *The dual of $\mathcal{O}L^1(D)$ can be identified with $\mathcal{B}(D)$ in the following way*:

(1) *If $b \in \mathcal{B}(D)$, then b induces a bounded linear functional on $\mathcal{O}L^1(D)$ via the integral pairing*

$$\Phi(f) = \int_D f(z)\overline{b(z)}\, dV(z), \quad f \in \mathcal{O}L^1(D) \cap \mathcal{O}L^2(D) \tag{1.2}$$

and $\mathcal{O}L^2(D)$ is dense in $\mathcal{O}L^1(D)$.

(2) *Every bounded linear functional $\Phi \in \mathcal{O}L^1(D)^*$ is induced by a function $b \in \mathcal{B}(D)$ via the integral pairing* (1.2). *Moreover, the norms $\|b\|_{\mathcal{B}}$ and $\|\Phi\|^*$ are equivalent.*

Theorem B is known to be true for other domains in $\mathbf{C}^n$. For strictly pseudoconvex domains in $\mathbf{C}^n$, Theorem B was proven by Ligocka [**10**], Coupet [**5**] and by Krantz and Ma [**9**]. For convex domains of finite type, Theorem B was proven by Krantz and Li [**8**]. The methods of [**9**] and [**8**] are different than those contained in [**10**], [**5**] or this paper. For bounded symmetric domains in $\mathbf{C}^n$, Zhu [**19**] has shown that the dual of $\mathcal{O}L^1(D)$ can be identified with a space of bounded Hankel operators and that this space is strictly larger than the space of Bloch functions except in the case where D is biholomorphically equivalent to the open unit ball. For proofs of Theorems A and B in the case of the open unit disk and references to results in higher dimensions, we refer the reader to [**20**].

This paper is organized as follows. In Section 2 we summarize some known information concerning domains of finite type in $\mathbf{C}^2$, prove some boundedness properties of the Bergman projection and prove Theorem A. In Section 3 we prove Theorem B as well as give a characterization of Bloch functions in terms of higher derivatives.

2. Domains of finite type and the Bergman projection.

We shall need the following information concerning the geometry of domains of finite type, due to Catlin [**3**]. Let z_0 be a point in bD. After renumbering, we may assume that $(\partial r/\partial z_2)(z) \neq 0$ for all z in a neighborhood U of z_0.

Theorem 1. *For each $z' = (z_1', z_2') \in U$ there exist numbers $d_k(z')$,*

$k = 0, 1, 2, \dots, m$, depending smoothly upon z' such that $d_0(z') \neq 0$ and such that in the new coordinates (ζ_1, ζ_2) defined by

$$(z_1, z_2) = \Phi'(\zeta) = (\Phi_1'(\zeta), \Phi_2'(\zeta))$$

where $\Phi_1'(\zeta) = z_1' + \zeta_1$ and $\Phi_2'(\zeta) = z_2' + d_0(z')\zeta_2 + \sum_{k=1}^m d_k(z')\zeta_1^k$, the function $\rho(\zeta) = r \circ \Phi'(\zeta)$ satisfies

$$\rho(\zeta) = r(z') + Re\,\zeta_2 + \sum_{\substack{j+k\leq m \\ j,k>0}} a_{jk}(z')\zeta_1^j\bar{\zeta}_1^k + O(|\zeta_1|^{m+1} + |\zeta_2||\zeta|). \tag{2.1}$$

Moreover, if $\Phi(\zeta) = (z_1' + \zeta_1, z_2' + e_0\zeta_2 + \sum_{k=1}^m e_k\zeta_1^k)$ has the property that $r \circ \Phi$ can be written expressed in the form (2.1) (*with possibly different numbers $a_{jk}(z')$*), *then $\Phi = \Phi'$.*

We now define, for $l = 2, 3, \dots, m$, functions $A_l(z')$ by $A_l(z') = \max\{|a_{jk}(z')| : j + k = l\}$. If the type of z_0 equals m, then it follows from results in [**1**] that $A_m(z_0) \neq 0$.

Since $\mathcal{O}L^2(D)$ is a closed subspace of the Hilbert space $L^2(D)$ there is an orthogonal projection $P : L^2(D) \to \mathcal{O}L^2(D)$. P is called the Bergman projection and is given by integration against the Bergman kernel $K_D(z, w)$, i.e.,

$$Pf(z) = \int_D f(w)K_D(z, w)\,dV(w), \quad f \in L^2(D).$$

Regularity of the Bergman projection is an important ingredient in the study of boundary behavior of biholomorphic maps. We recall that D satisfies Condition R if for every $\phi \in C^\infty(\overline{D})$ we also have $P\phi \in C^\infty(\overline{D})$. The fact that domains of finite type in $\mathbf{C}^2$ satisfy Condition R is a consequence of the subellipticity of the $\bar{\partial}$-Neumann problem in $\mathbf{C}^2$. It is a well-known result of Kerzman [**7**] that if D satisfies Condition R then for each fixed $z \in D$ one has $K_D(z, w) \in C^\infty(\overline{D})$. We will need the following theorem, due to McNeal [**11**], concerning the behavior of the Bergman kernel for domains of finite type in $\mathbf{C}^2$. Similar results have been obtained by Nagel, Rosay, Stein and Wainger [**14**]. In [**13**] McNeal and Stein extended these results to convex domains of finite type in $\mathbf{C}^n$. We use the notation $A \lesssim B$ if there is a constant c, independent of A and B, such that $A \leq cB$.

Theorem 2. *Let z_0 be a point of finite type m in the boundary of a smooth bounded pseudoconvex domain Ω in $\mathbf{C}^2$. For $z^1, z^2 \in \Omega$ near z^0, set $\zeta^i = \Phi(z^i)$, $\tilde{\Omega} = \Phi(\Omega)$ and $z' = \pi(z^1)$ where π is the projection onto $b\Omega$. Then there is a neighborhood U of z_0 such that for all two-indices α, β and $z^1, z^2 \in U$,*

$$|D^{\alpha}_{\zeta_1} D^{\beta}_{\zeta_2} K_{\tilde{\Omega}}(\zeta^1, \zeta^2)| \lesssim \sum_{l=2}^{m} \frac{A_l(z')^{C_1}}{[|r(\zeta^1)| + |r(\zeta^2)| + |\zeta^1_2 - \zeta^2_2| + \sum_{k=2}^{m} A_k(z')|\zeta^1_1 - \zeta^2_1|^k]^{C_2}}$$

where $C_1 = ((2 + \alpha_1 + \beta_1)/l)$, $C_2 = 2 + \alpha_2 + \beta_2 + ((2 + \alpha_1 + \beta_1)/l)$, (ζ_1, ζ_2) are the coordinates defined by Φ and r is a defining function for Ω.

We are now in a position to prove the following. Let ∇^j_r denote the derivative $\nabla_r \circ \nabla_r \circ \cdots \circ \nabla_r$ (j times).

Lemma 3. *Let $D \subset \mathbf{C}^2$ be a smoothly bounded pseudoconvex domain of finite type. Then*

(1) $\int_D |K_D(z,w)||r(w)|^{-\varepsilon}\, dV(w) \lesssim |r(z)|^{-\varepsilon}$, $0 < \varepsilon < 1$

(2) $\int_D |\nabla^j_r K_D(z,w)|\, dV(w) \lesssim |r(z)|^{-j}$, $j = 1, 2, \ldots$.

We will prove the estimate (2). The proof of (1) is similar. Since bD is compact, we may choose points $p_1, p_2, \ldots, p_N \in bD$ and neighborhoods U_k of p_k, $1 \le k \le N$, such that $bD \subset\subset \cup_{k=1}^{N} U_k$ and such that Theorems 1 and 2 hold on each U_k. It follows by standard arguments that we need only show that

$$\int_{D \cap U_k} |\nabla^j_r K_D(z,w)|\, dV(w) \lesssim |r(z)|^{-j}$$

for $z \in U_k$. For ease of notation, we now drop the subscript. To prove the estimate, assume that p is of type m, Φ_p is the biholomorphism from Theorem 1 and $\Omega = \Phi_p(D)$. If $\zeta^1 = \Phi_p(z)$ and $\zeta^2 = \Phi_p(w)$, then near $0 \in b\Omega$ the complex normal derivative is $(\partial/\partial\zeta_2)$. Since the Jacobian of Φ_p is bounded from above and below on U, the transformation formula for the Bergman kernel tells us we need only estimate

$$\int_{\Omega \cap W} \left| \frac{\partial^j}{(\partial \zeta^1_2)^j} K_{\Omega}(\zeta^1, \zeta^2) \right| dV(\zeta^2)$$

for $\zeta^1 \in W$, where $W = \Phi_p(U)$. By Theorem 2 we have

$$\left|\frac{\partial^j}{(\partial\zeta_2^1)^j} K_\Omega(\zeta^1,\zeta^2)\right| \lesssim \sum_{l=2}^m \frac{A_l(z')^{2/l}}{[|\rho(\zeta^1)|+|\rho(\zeta^2)|+|\zeta_2^1-\zeta_2^2|+\sum_{k=1}^m A_k(z^1)|\zeta_1^1-\zeta_1^2|^k]^{2+j+(2/l)}}.$$

We now introduce the following coordinate system from [**16**].

Lemma 4. *There is a neighborhood W of $0 \in b\Omega$ such that for $\zeta^1 \in W$ there are real coordinates $t_{\zeta^1}(\zeta^2) = (t_1(\zeta^2), t_2(\zeta^2), t_3(\zeta^2), t_4(\zeta^2))$ such that*

(i) $t_1(\zeta^2) = \rho(\zeta^2)$, $t_2(\zeta^2) = \mathrm{Im}\,(\zeta_2^1-\zeta_2^2)$ *and* $t_3(\zeta^2)+it_4(\zeta^2) = \zeta_1^1-\zeta_1^2$

(ii) $M^{-1} \le |J_{\mathbf{R}} t(\zeta^2)| \le M$ *for some positive constant M ($J_{\mathbf{R}}$ denotes the real Jacobian).*

(iii) $|t_j(\zeta^2)| < 1$ *for* $j = 1,2,3,4$.

Proof. The lemma will follow from the implicit function theorem once we prove that $d_{\zeta^2}\rho \wedge d_{\zeta^2}\mathrm{Im}\,(\zeta_2^1-\zeta_2^2) \ne 0$ at 0. From equation (2.1),

$$d_{\zeta^2}\rho(\zeta^2) = d_{\zeta^2}(\mathrm{Re}\,\zeta_2^2 + \sum a_{jk}(\zeta_1^2)^j(\bar\zeta_1^2)^k + O(|\zeta_1|^{m+1}+|\zeta_2||\zeta|).$$

Thus $d_{\zeta^2}\rho \wedge d_{\zeta^2}\mathrm{Im}\,(\zeta_2^1-\zeta_2^2) = -d_{\zeta^2}\mathrm{Re}\,\zeta_2^2 \wedge d_{\zeta^2}\mathrm{Im}\,\zeta_2^2 \ne 0$ at 0.

We introduce this special coordinate system (we may, at this point, have to shrink the original neighborhood U) and obtain

$$\int_{\Omega\cap W}\left|\frac{\partial^j}{(\partial\zeta_2^1)^j} K_\Omega(\zeta^1,\zeta^2)\right| dV(\zeta^2) \lesssim \sum_{l=2}^m \int_0^1\int_0^1\int_0^1\int_0^1 \frac{A_l(z')^{2/l}\,dt_1\,dt_2\,dt_3\,dt_4}{[|\rho(\zeta^1)|+t_1+t_2+A_l(z')|t_3+it_4|^l]^{-2+j+(2/l)}}.$$

Integrating in t_1 and then in t_2 yields

$$\int_{\Omega\cap W}\left|\frac{\partial^j}{(\partial\zeta_2^1)^j} K_\Omega(\zeta^1,\zeta^2)\right| dV(\zeta^2) \lesssim \sum_{l=2}^m \int_0^1\int_0^1 \frac{A_l(z')^{2/l}\,dt_3\,dt_4}{[|\rho(\zeta^1)|+A_l(z')|t_3+it_4|^l]^{j+2/l}}.$$

We now introduce polar coordinates $t_3 + it_4 = xe^{i\theta}$, and hence

$$\int_{\Omega\cap W}\left|\frac{\partial^j}{(\partial\zeta_2^1)^j}K_\Omega(\zeta^1,\zeta^2)\right|dV(\zeta^2) \lesssim \sum_{l=2}^{m}\int_0^{2\pi}\int_0^1 \frac{A_l(z')^{2/l}x\,dx\,d\theta}{[|\rho(\zeta^1)|+A_l(z')x^l]^{j+(2/l)}}.$$

Integrate in θ and make the change of variables $x = |\rho(\zeta^1)|^{1/l}y$ so that

$$\int_{\Omega\cap W}\left|\frac{\partial^j}{(\partial\zeta_2^1)^j}K_\Omega(\zeta^1,\zeta^2)\right|dV(\zeta^2) \lesssim \sum_{l=2}^{m}\frac{1}{|\rho(\zeta^1)|^j}\int_0^{\infty}\frac{A_l(z')^{2/l}y\,dy}{[1+A_l(z')y^l]^{j+(2/l)}}.$$

Finally, we make the change of variables $u = A_l(z')^{(1/l)}y$ and so

$$\int_{\Omega\cap W}\left|\frac{\partial^j}{(\partial\zeta_2^1)^j}K_\Omega(\zeta^1,\zeta^2)\right|dV(\zeta^2) \lesssim \frac{1}{|\rho(\zeta^1)|^j}\sum_{l=2}^{m}\int_0^{\infty}\frac{u\,du}{[1+u^l]^{j+(2/l)}}.$$

Each of the integrals in the sum is clearly convergent and so the lemma is proven.

The estimate (1) was obtained, for strictly pseudoconvex domains in $\mathbf{C}^n$, by Phong and Stein [**15**]. In exactly the same way as in [**15**] we obtain

Corollary 5. *Let $D \subset \mathbf{C}^2$ be a smoothly bounded pseudoconvex domain of finite type. Then the Bergman projection is bounded on $L^p(D)$ for $1 < p < \infty$, i.e.,*

$$\|P\phi\|_p \le C_p\|\phi\|_p, \quad \phi \in L^p(D)\cap L^2(D).$$

For classes of domains of finite type in $\mathbf{C}^n$, Corollary 5 was proven by McNeal [**12**]. In Section 3 we will see that P is not bounded on $L^1(D)$ nor on $L^\infty(D)$. The estimate (1) also has another important application. If $(\cdot,\cdot)_D$ denotes the standard inner product on $L^2(D)$

then, since P is a bounded, self-adjoint operator, we have $(Pf, g)_D = (f, Pg)_D$ for all $f, g \in L^2(D)$. The proof that $\|Pf\|_p \leq C_p \|f\|_p$ and Hölder's inequality actually show

$$\int_D \left\{ \int_D |f(w)||K_D(z,w)|\, dV(w) \right\} |g(z)|\, dV(z) \leq C_p \|f\|_p \|g\|_q$$

for $f \in L^p(D) \cap L^2(D)$. The Fubini-Tonelli theorem allows us to interchange the order of integration and conclude that $(Pf, g)_D = (f, Pg)_D$ for all $f \in L^p(D) \cap L^2(D)$, $g \in L^q(D)$. Since $L^p(D) \cap L^2(D)$ is dense in $L^p(D)$, we conclude that $(Pf, g)_D = (f, Pg)_D$ for all $f \in L^p(D)$, $g \in L^q(D)$.

We may now proceed with the proof of Theorem A. It is clear, by using Hölder's inequality, that every $g \in \mathcal{O}L^q(D)$ induces a bounded linear functional on $\mathcal{O}L^p(D)$ via (1.1). To prove that every bounded linear functional is of this form, we consider first the case where $2 \leq p < \infty$. If $\Phi \in \mathcal{O}L^p(D)^*$, then by the Hahn-Banach theorem there is a $\hat{\Phi} \in L^p(D)^*$ with $\|\Phi\|^* = \|\hat{\Phi}\|^*$ and $\Phi = \hat{\Phi}$ on $\mathcal{O}L^p(D)$. By the Riesz representation theorem there is a $\phi \in L^q(D)$ with $\hat{\Phi}(f) = (f, \phi)_D$ and $\|\phi\|_q = \|\hat{\Phi}\|^*$. Since D is bounded, $L^p(D) \subset L^2(D)$ so that $Pf = f$ for $f \in \mathcal{O}L^p(D)$. Thus, we have

$$\Phi(f) = \Phi(Pf) = \hat{\Phi}(Pf) = (Pf, \phi)_D = (f, P\phi)_D.$$

By Corollary 5, $g = P\phi \in \mathcal{O}L^q(D)$ and so Φ is of the form (1.1). To prove the case where $1 < p < 2$ we use the fact that a subspace of a reflexive Banach space is reflexive, so that

$$\mathcal{O}L^p(D)^* = (\mathcal{O}L^q(D)^*)^* = \mathcal{O}L^q(D).$$

Corollary 6. *Let $D \subset \mathbf{C}^2$ be a smoothly bounded pseudoconvex domain of finite type. If $1 < p < 2$ and $p < r < q$, then $\mathcal{O}L^r(D)$ is dense in $\mathcal{O}L^p(D)$.*

Proof. By the Hahn-Banach theorem it suffices to show that if $\Phi \in \mathcal{O}L^p(D)^*$ and $\Phi = 0$ on $\mathcal{O}L^r(D)$ then $\Phi \equiv 0$. Suppose that such a Φ exists. Then there is a $g \in \mathcal{O}L^q(D)$ with $\Phi(f) = (f, g)_D$. Since $\mathcal{O}L^q(D) \subset \mathcal{O}L^r(D)$ we have $g \in \mathcal{O}L^r(D)$ so that

$$0 = \Phi(g) = (g, g)_D = \int_D |g(z)|^2\, dV(z).$$

Hence $g \equiv 0$ and so $\Phi \equiv 0$.

Corollary 7. *Let $D \subset \mathbf{C}^2$ be a smoothly bounded pseudoconvex domain of finite type. If $f \in \mathcal{O}L^p(D)$, $1 < p < \infty$, then*

$$f(z) = Pf(z) = \int_D f(w) K_D(z,w)\, dV(w).$$

Proof. Since D is bounded, $\mathcal{O}L^p(D) \subset \mathcal{O}L^2(D)$ for $2 \le p < \infty$ so the result is clearly true in this case. We consider the case $1 < p < 2$. If $f \in \mathcal{O}L^p(D)$ and $\varepsilon > 0$ there is an $h \in \mathcal{O}L^2(D)$ with $\|f - h\|_p < \varepsilon$. Since $Ph = h$, we have

$$\begin{aligned}\left| f(z) - \int_D f(w) K_D(z,w)\, dV(w) \right| \\ &\le |f(z) - h(z)| + \left| Ph(z) - \int_D f(w) K_D(z,w)\, dV(w) \right|.\end{aligned}$$

By the Cauchy estimates there is a constant C_z such that $|f(z) - h(z)| \le C_z \|f - h\|_p$. It is a well-known result of Kerzman [**7**] that for each $z \in D$ the function $k_z(w) := K_D(z,w)$ satisfies $k_z(w) = P\psi_z(w)$ where $\psi_z \in C_0^\infty(D)$. By Corollary 5, $k_z(w) \in L^q(D)$, in fact, since D satisfies Condition R, $k_z(w) \in C^\infty(\overline{D})$, so by Hölder's inequality $|Ph(z) - \int_D f(w) K_D(z,w)\, dV(w) \le \|h - f\|_p \|k_z\|_q$. Thus $|f(z) - \int_D f(w) K_D(z,w)\, dV(w)| \lesssim \varepsilon$ and, since ε is arbitrary, we are done.

An alternate way to prove density results is to use classical approximation techniques due to Kerzman and Lieb. In order to use these results we need L^p estimates for $\bar\partial$. Our purposes require the following theorem due to Bonami and Sibony [**2**].

Theorem 8. *Let $D \subset\subset \mathbf{C}^2$ be a smooth, bounded pseudoconvex domain of finite type. If f is a $\bar\partial$-closed $(0,1)$ form with coefficients in $L^1(D)$, then there is a $u \in L^1(D)$ that satisfies $\bar\partial u = f$.*

This result replaces an earlier (unpublished) result of Stein. Combining this result with the classical approximation techniques mentioned above leads to

Theorem 9. *Let $D \subset\subset \mathbf{C}^2$ be a smoothly bounded pseudoconvex domain of finite type. Then $A^\infty(D) = C^\infty(\overline{D}) \cap \mathcal{O}(D)$ is dense in $\mathcal{O}L^1(D)$.*

We conclude this section by observing that Corollary 7 tells us that integration against the Bergman kernel defines a bounded projection from $L^p(D)$ to $\mathcal{O}L^p(D)$ for $1 < p < \infty$. We recall that a closed subspace M of a Banach space X is complemented in X if there is a closed subspace N of X with $X = M + N$ and $M \cap N = \varnothing$. By taking $X = L^p(D)$, $M = \mathcal{O}L^p(D)$ and $N = \ker P$, we obtain

Theorem 10. *Let $D \subset \mathbf{C}^2$ be a smoothly bounded pseudoconvex domain of finite type. Then for $1 < p < \infty$, $\mathcal{O}L^p(D)$ is complemented in $L^p(D)$.*

3. The Bloch space. We denote by $\mathcal{B}(D)$ the space of all Bloch functions on D, those functions holomorphic on D that satisfy $\|f\|_{\mathcal{B}} = \sup\{|\nabla_r f(z) r(z)| : z \in D\} < \infty$ where $\nabla_r = \sum_{j=1}^{2}(\partial r/\partial \bar{z}_j)(\partial/\partial z_j)$ is the complex normal derivative. Bloch functions in several complex variables have been studied in [**17, 18, 9, 5, 19**]. In some of these papers Bloch functions have been defined in terms of the Bergman and Kobayashi metrics rather than the Euclidean distance (recall that for z close to the boundary of D, $|r(z)| \sim \operatorname{dist}(z, bD)$). In the case of smoothly bounded strictly pseudoconvex domains and domains of finite type in $\mathbf{C}^2$ these definitions are equivalent. The reader can easily check that the above definition agrees with the classical definition for Bloch functions on the open unit disk.

The main result of this section is Theorem B. In order to prove it we need only verify the two conditions of the following theorem from [**5**].

Theorem 11. *If $D \subset\subset \mathbf{C}^n$ has smooth boundary, then $\mathcal{O}L^1(D)^* = \mathcal{B}(D)$ if and only if the following two conditions are satisfied.*

(1) *$\mathcal{O}L^2(D)$ is dense in $\mathcal{O}L^1(D)$.*

(2) *$PL^\infty(D) = \{f = P\phi : \phi \in L^\infty(D)\} \subseteq \mathcal{B}(D)$.*

The proof of Theorem B follows directly from Theorem 11 by observing that $A^\infty(D) \subseteq \mathcal{O}L^2(D)$ so by Theorem 9 condition (1) is satisfied. Condition (2) is satisfied by Lemma 3 (with $j = 1$) since if $\phi \in L^\infty(D)$,

$$|\nabla_r P\phi(z)| \le \int_D |\phi(w)||\nabla_r K(z,w)|\, dV(w) \lesssim \|\phi\|_\infty |r(z)|^{-1}.$$

Thus $P\phi \in \mathcal{B}(D)$ and $\|P\phi\|_{\mathcal{B}} \lesssim \|\phi\|_\infty$.

We now give an equivalent formulation of the Bloch space. For $a \in D$ and $r > 0$ we let $B(a,r)$ denote the Euclidean ball centered at a with radius r and let $|B(a,r)|$ denote the volume of $B(a,r)$. For $B = B(a,r)$ we define f_B by $f_B = |B|^{-1}\int_B f(w)\, dV(w)$. We say that f is of bounded mean oscillation (BMO) if

$$\|f\|_{BMO} = \sup\left\{\frac{1}{|B(a,r)|}\int_{B(a,r)} |f - f_{B(a,r)}|\, dV : a \in D, B(a,r) \subset\subset D\right\} < \infty.$$

The space of holomorphic functions of bounded mean oscillation is denoted $BMOA$. In [**4**] it is shown that if $D \subset\subset \mathbf{C}^n$ has smooth boundary then $BMOA(D) = \mathcal{B}(D)$. Combining this result with Theorem B yields

Theorem 12. *Let $D \subset\subset \mathbf{C}^2$ be a smoothly bounded pseudoconvex domain of finite type. Then $\mathcal{O}L^1(D)^* = BMOA(D)$.*

In the same way that we obtained Corollary 7 we obtain

Corollary 13. *Let $D \subset\subset \mathbf{C}^2$ be a smoothly bounded pseudoconvex domain of finite type. If $f \in \mathcal{O}L^1(D)$, then*

$$f(z) = \int_D f(w) K(z,w)\, dV(w).$$

It is important to observe that even though the Bergman kernel reproduces holomorphic L^1 functions, integration against the kernel does not yield a bounded projection from $L^1(D)$ to $\mathcal{O}L^1(D)$. To see this, assume, in order to reach a contradiction, that there is a constant $C > 0$ such that $\|Pf\|_1 \le C\|f\|_1$ for all $f \in L^2(D)$, which is dense in $L^1(D)$. If $h \in L^\infty(D)$, then the linear functional defined on $L^1(D)$ by $\Phi_h(f) = \int_D Pf\bar{h}\,dV$ satisfies $|\Phi_h(f)| \le \|h\|_\infty \|Pf\|_1 \le C\|h\|_\infty \|f\|_1$. Thus Φ_h defines a bounded linear functional on $L^1(D)$. But we also have $\Phi_h(f) = \int_D f\overline{Ph}\,dV$. By the Riesz representation theorem we must have $Ph \in L^\infty(D)$. We know that $PL^\infty(D) = \mathcal{B}(D)$ so in order to reach a contradiction we need only construct an unbounded Bloch function. Note that this will also show that P isn't bounded on $L^\infty(D)$.

To construct this function, choose a point $p \in bD$ such that $|p| = \sup\{|z| : z \in D\}$. Then there is an **R**-linear function $l(z) = \sum_{j=1}^n \alpha_j z_j + \sum_{j=1}^n \beta_j \bar{z}_j$ such that $l(p) = 0$ and $l(z) < 0$ for $z \in D$. Since l is real-valued, we must have $\beta_j = \bar{\alpha}_j$ for $1 \le j \le n$. Thus $l(z) = \operatorname{Re} h(z)$ where $h(z) = 2\sum_{j=1}^n \alpha_j z_j$ is **C**-linear. By multiplying h by a suitably large constant we may assume that $|h(z)| \ge |r(z)|$, and a **C**-linear change of coordinates puts $h(z)$ in the form $h(z) = \alpha z_n$. We define $f \in \mathcal{O}(D)$ by $f(z) = \log h(z)$. Then f is unbounded and $|\partial f/\partial z_n| \lesssim |r(z)|^{-1}$, i.e., $f \in \mathcal{B}(D)$.

We conclude with another formulation of the Bloch space, corresponding to the characterization given in [**21**] for bounded symmetric domains. For $j = 1, 2, 3, \ldots$, let

$$\|f\|_{\mathcal{B},j} = \sup\{|\nabla_r^j f(z)||r(z)|^j : z \in D\}.$$

We define the spaces $\mathcal{B}_j(D)$ by $\mathcal{B}_j(D) = \{f \in \mathcal{O}(D) : \|f\|_{\mathcal{B},j} < \infty\}$. Clearly $\mathcal{B}_1(D) = \mathcal{B}(D)$.

Theorem 14. *Let $D \subset\subset \mathbf{C}^2$ be a smoothly bounded pseudoconvex domain of finite type. Then, for $j = 2, 3, \ldots$, the norms $\|\cdot\|_{\mathcal{B},j}$ and $\|\cdot\|_{\mathcal{B}}$ are equivalent and hence $\mathcal{B}_j(D) = \mathcal{B}(D)$.*

Proof. It is easy to see that $\|f\|_{\mathcal{B},j} \lesssim \|f\|_{\mathcal{B},j+1}$ for $j = 1, 2, 3, \ldots$. Thus $\mathcal{B}_j(D) \subseteq \mathcal{B}(D)$. To prove the reverse inclusion, let $f \in \mathcal{B}$, then $f = P\phi$ for some $\phi \in L^\infty(D)$. BY Lemma 3,

$$|\nabla_r^j f(z)| \le \int_D |\phi(w)||\nabla_r^j K(z,w)|\,dV(w) \lesssim \|\phi\|_\infty |r(z)|^{-j}.$$

Thus, $\|f\|_{\mathcal{B},j} \lesssim \|\phi\|_\infty \lesssim \|f\|_{\mathcal{B}}$.

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