

**A COEFFICIENT PROBLEM
FOR UNIVALENT FUNCTIONS RELATED TO
TWO-POINT DISTORTION THEOREMS**

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ABSTRACT. We discuss a class of coefficient functionals over the set of normalized univalent functions on the unit disk. These functionals are related to symmetric, linearly invariant two-point distortion theorems for univalent functions due to Kim and Minda. Each of these theorems is necessary and sufficient for univalence. A special case is a distortion theorem of Blatter. Our approach is based on an application of Pontryagin's maximum principle to the Loewner differential equation. In the same fashion, two-point distortion theorems for bounded univalent functions are obtained. Related coefficient functionals are discussed, too.

1. Introduction. Let \mathcal{S} be the customary class of normalized univalent functions

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots$$

on the unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ into \mathbf{C} , and consider for a fixed number $p \in \mathbf{R}$ the functional $J_p : \mathcal{S} \rightarrow \mathbf{R}$ defined by

$$(1) \quad J_p(f) = J_p(a_2, a_3) := \operatorname{Re} \left(a_3 + \frac{p-3}{3} a_2^2 \right) + \frac{p+1}{3} |a_2|^2.$$

Every function $F \in \mathcal{S}$ maximizing J_p over \mathcal{S} is called an *extremal function* for J_p .

The coefficient functional J_p is related to the following one-parameter family of symmetric, linearly invariant two-point distortion theorems for (not necessarily normalized) univalent functions on the unit disk due to Blatter [1] and Kim and Minda [7].

Theorem 1.1. *Let $p > 0$ be a real number such that the Koebe function $K(z) := z/(1-z)^2 \in \mathcal{S}$ maximizes the functional (1) over the*

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class \mathcal{S} . If g is a (not necessarily normalized) univalent function on the unit disk \mathbf{D} , then

$$(2) \quad |g(a) - g(b)| \geq C(p, d_{\mathbf{D}}(a, b))[(1 - |a|^2)^p |g'(a)|^p + (1 - |b|^2)^p |g'(b)|^p]^{1/p}$$

for all $a, b \in \mathbf{D}$ where $d_{\mathbf{D}}$ denotes the hyperbolic metric in \mathbf{D} and

$$(3) \quad C(p, d) := \frac{1}{2} \frac{\sinh(2d)}{[2 \cosh(2pd)]^{1/p}}.$$

The inequality (2) is sharp for all $a, b \in \mathbf{D}$.

Condition (2) is necessary and sufficient for univalence. Whereas the fact that (2) is sufficient for univalence is almost immediate, the necessity of (2) for univalence is the hard part of the proof of Theorem 1.1.

Theorem 1.1 has its origin in a paper by Blatter [1] who showed that inequality (2) holds for all univalent functions g on \mathbf{D} in the case $p = 2$ by using the classical coefficient inequalities $|a_2| \leq 2$, $|a_3 - a_2^2| \leq 1$ and Loewner's [9] result $|a_3| \leq 3$ for functions in \mathcal{S} . Hence $p = 2$ is a possible choice in (2).

It is easy to see that if the Koebe function K maximizes the functional (1) over \mathcal{S} for one $p \geq 1$, then it also maximizes (1) over \mathcal{S} for all larger values of p . Kim and Minda [7] proved, using an inequality of Jenkins, that the Koebe function K maximizes the functional (1) in the class \mathcal{S} for $p = 3/2$, i.e., every $p \geq 3/2$ is a possible choice in (2). The limit case $p \rightarrow \infty$ constitutes an invariant form of the Koebe distortion theorem (cf. [7, p. 144]). On the other hand, Ruscheweyh has shown numerically, using the Schaeffer-Spencer formulas for the coefficient body

$$V_3 := \{(a_2, a_3) : f \in \mathcal{S}, f(z) = z + a_2 z^2 + a_3 z^3 + \dots\},$$

that for $p = 1$ the Koebe function is not an extremal function for the functional (1) in the class \mathcal{S} . Since the righthand side of (2) is a decreasing function of p for $p \geq 1$, Kim and Minda [7] posed the problem to find the smallest number $p > 1$ such that the Koebe function

maximizes the functional (1) in the class \mathcal{S} . We shall show that this optimal value is

$$p = p_0 := \frac{1}{2} \frac{2e^3 + 1}{e^3 - 1} \doteq 1.07859,$$

by establishing the following Theorem 1.2 which gives a complete picture of the functional (1) for all $p \in \mathbf{R}$.

Recently Jenkins [6] used the general coefficient theorem to show that (2) is true if and only if $p \geq 1$ whereas the method of Kim and Minda together with Theorem 1.2 only establishes inequality (2) if $p \geq p_0$.

Theorem 1.2. *Let $F \in \mathcal{S}$ maximize the functional (1) over \mathcal{S} for fixed $p \in \mathbf{R}$. Then $\mathbf{C} \setminus F(\mathbf{D})$ is an analytic Jordan arc extending to infinity.*

(a) *If $p \geq p_0$, then $F(z) = \pm K(\pm z)$.*

(b) *If $p \leq p_1 := 3/(4 \log 2) - 1/2 \doteq 0.58202$, then $F(z) = \pm iK(\mp iz)$.*

(c) *If $p_1 < p < p_0$, then F is not a rotation of the Koebe function. More precisely, there is a function $F_0 \in \mathcal{S}$ such that the complement of the image domain $\mathbf{C} \setminus F_0(\mathbf{D})$ is a curved analytic Jordan arc and F is one of the four functions $\pm F_0(\pm z)$, $\pm \overline{F_0(\pm \bar{z})}$.*

We study the problem of maximizing the coefficient functional (1) for fixed $p \in \mathbf{R}$ over \mathcal{S} by the use of Pontryagin's maximum principle applied to the Loewner differential equation in combination with some intricate but elementary calculations. Obviously, the problem is equivalent to maximizing the functional (1) over the coefficient body V_3 . However, although Schaeffer and Spencer [15] determined the boundary ∂V_3 quite explicitly, their formulas seem to be too complicated to be useful to tackle the problem in this way—the extremal functions for (1) lie on the ‘complicated’ part of ∂V_3 . On the other hand, certain analogies between Pontryagin's maximum principle and the Schaeffer-Spencer variational method (cf. [14]) have motivated our approach.

2. Pontryagin's maximum principle. The coefficient body V_3 can be described as the so-called reachable set of a control system arising from Loewner's differential equation [9]. In fact, if we consider

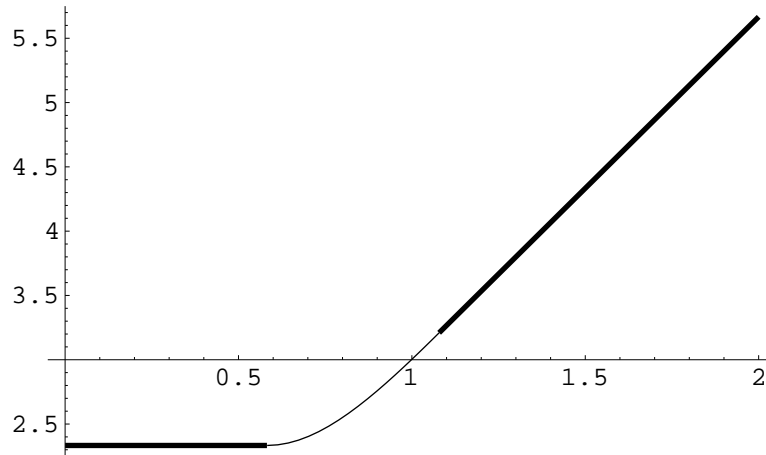


FIGURE 1. A plot of $\max_{(a_2, a_3) \in V_3} \operatorname{Re}(a_3 + ((p-3)/3)a_2^2) + ((p+1)/3)|a_2|^2$ as a function of p for $0 \leq p \leq 2$. The thick parts correspond to rotations of the Koebe functions.

for a fixed measurable function $u : [0, \infty) \rightarrow \mathbf{R}$ the initial value problem

$$(4) \quad \begin{aligned} \frac{d}{dt} a_2(t) &= -2e^{-t} e^{-iu(t)}, & a_2(0) &= 0, \\ \frac{d}{dt} a_3(t) &= -4e^{-t} e^{-iu(t)} a_2(t) \\ &\quad - 2e^{-2t} e^{-2iu(t)}, & a_3(0) &= 0, \end{aligned}$$

and denote the corresponding solution by $a_2(t, u(\cdot)), a_3(t, u(\cdot))$, then the so-called *entire reachable set*

$$\mathcal{R} := \{(a_2(\infty, u(\cdot)), a_3(\infty, u(\cdot))) : u : [0, \infty) \rightarrow \mathbf{R} \text{ measurable}\}$$

of (4) is dense in V_3 . See, for instance [3, Chapter 3].

Fix $p \in \mathbf{R}$ and let $F(z) = z + A_2 z^2 + A_3 z^3 + \dots \in \mathcal{S}$ be an extremal function for the functional J_p in \mathcal{S} , i.e.,

$$(5) \quad \max_{f \in \mathcal{S}} J_p(f) = J_p(F).$$

Then $\mathbf{C} \setminus F(\mathbf{D})$ consists of one or two piecewise analytic Jordan arcs (cf. [3, p. 304]) and it follows from Loewner's theory that a function

$u_0 : [0, \infty) \rightarrow \mathbf{R}$ exists with at most one point of discontinuity (which will be a discontinuity of the first kind) such that $A_2 = a_2(\infty, u_0(\cdot))$ and $A_3 = a_3(\infty, u_0(\cdot))$. In particular,

$$(6) \quad \max_{(a_2, a_3) \in \mathcal{R}} J_p(a_2, a_3) = J_p(A_2, A_3),$$

and the extremal problem (5) in the class \mathcal{S} is reduced to the ordinary optimal control problem (6). The standard approach of optimal control theory to deal with an extremal problem of type (6) is the Pontryagin maximum principle which we shall now describe briefly for functionals $J_p(a_2, a_3)$ on V_3 .

We denote by $\bar{\Psi}_2(t), \bar{\Psi}_3(t)$ the solution of the adjoint equation to (4) along $u_0(\cdot)$, that is,

$$(7) \quad \begin{aligned} \frac{d}{dt} \bar{\Psi}_2(t) &= 4e^{-t} e^{-iu_0(t)} \bar{\Psi}_3(t), & \bar{\Psi}_2(\infty) &= \frac{\partial J_p}{\partial a_2}(A_2, A_3), \\ \frac{d}{dt} \bar{\Psi}_3(t) &= 0, & \bar{\Psi}_3(\infty) &= \frac{\partial J_p}{\partial a_3}(A_2, A_3), \end{aligned}$$

where in our case

$$(8) \quad \begin{aligned} \frac{\partial J_p}{\partial a_2}(A_2, A_3) &= -\frac{3-p}{3} A_2 + \frac{p+1}{3} \bar{A}_2, \\ \frac{\partial J_p}{\partial a_3}(A_2, A_3) &= \frac{1}{2}. \end{aligned}$$

Then, we define the Hamiltonian function $H : [0, \infty] \times \mathbf{R} \rightarrow \mathbf{R}$ by

$$(9) \quad H(t, u) := \operatorname{Re} [-2e^{-t} e^{-iu} \bar{\Psi}_2(t) - 4e^{-t} e^{-iu} a_2(t, u_0(\cdot)) \bar{\Psi}_3(t) - 2e^{-2t} e^{-2iu} \bar{\Psi}_3(t)].$$

The initial value problems (4) and (7) imply that

$$(10) \quad 4a_2(t, u_0(\cdot)) \bar{\Psi}_3(t) + 2\bar{\Psi}_2(t) \equiv: A$$

is independent on t and therefore the Hamiltonian may be written as

$$(11) \quad \begin{aligned} H(t, u) &= H_A(t, u) \\ &:= -e^{-t} (\operatorname{Re} A \cos u + \operatorname{Im} A \sin u + 2e^{-t} \cos^2 u) + e^{-2t}. \end{aligned}$$

Inserting (8) into (10) for $t = \infty$ implies the important relation

$$(12) \quad A = \frac{2(2p+1)}{3} \operatorname{Re} A_2 - \frac{2}{3}i \operatorname{Im} A_2.$$

Now, Pontryagin's maximum principle (cf. [2, pp. 162–167]) reads as follows.

Theorem 2.1. *For fixed $p \in \mathbf{R}$, let $F(z) = z + A_2 z^2 + A_3 z^3 + \dots \in \mathcal{S}$ be an extremal function for the functional J_p , let $u_0 : [0, \infty) \rightarrow \mathbf{R}$ be a piecewise continuous function with at most one point of discontinuity which is of first kind such that $A_2 = a_2(\infty, u_0(\cdot))$ and $A_3 = a_3(\infty, u_0(\cdot))$, and let $A \in \mathbf{C}$ be given by (12). Then*

$$(13) \quad \max_{u \in \mathbf{R}} H_A(t, u) = H_A(t, u_0(t)) \quad \text{for all } t \in [0, \infty].$$

We shall use the necessary condition (13) of Theorem 2.1 to prove Theorem 1.2. It turns out that, for $A \in \mathbf{C} \setminus (\mathbf{R} \cup i\mathbf{R})$, condition (13) determines the function u_0 , i.e., $F \in \mathcal{S}$ uniquely. Therefore, it is convenient to introduce the following notion.

Definition 2.2. A function $f(z) = z + a_2 z^2 + a_3 z^3 + \dots \in \mathcal{S}$ is called *A-admissible* for some $A \in \mathbf{C}$ if a piecewise continuous function $u_0 : [0, \infty) \rightarrow \mathbf{R}$ exists with at most one point of discontinuity which is of first kind such that $a_2 = a_2(\infty, u_0(\cdot))$, $a_3 = a_3(\infty, u_0(\cdot))$ and which satisfies (13) for H_A given by (11).

Therefore, Theorem 2.1 tells us that every extremal function $F(z) = z + A_2 z^2 + \dots \in \mathcal{S}$ for the functional J_p is *A-admissible* for A given by (12).

Definition 2.3. A function $f(z) = z + a_2 z^2 + \dots \in \mathcal{S}$ is called a *critical point* of the functional (1) if f is *A-admissible* for

$$(14) \quad A = \frac{2(2p+1)}{3} \operatorname{Re} a_2 - \frac{2}{3}i \operatorname{Im} a_2.$$

Obviously, this is a necessary condition for a function to be extremal for the functional J_p . Note that a function $f \in \mathcal{S}$ is a critical point if and only if $-f(-z)$, respectively $\overline{f(\bar{z})}$, is a critical point.

The strategy of the proof of Theorem 1.2 is as follows. We will show first (Lemmas 3.1 and 3.2) that an extremal function for J_p can be A -admissible only for $A \in \mathbf{C} \setminus (-4, 4)$ and that for $A \in i\mathbf{R}$ or $A \in \mathbf{R} \setminus (-4, 4)$ the only A -admissible functions are certain rotations of the Koebe function. If f is a critical point of the functional J_p which is A -admissible for the remaining case $A \in \mathbf{C} \setminus (\mathbf{R} \cup i\mathbf{R})$, then A lies on a curve contained in the annulus $1/3 < |A| < 4e^3/(e^3 - 1)$ and $p_1 < p < p_0$ (cf. Lemmas 3.4 and 3.5). Therefore, if $p \geq p_0$, then only a rotation of the Koebe function can be a critical point. From this it readily follows that the only extremal functions are $K(z)$ and $-K(-z)$. Similarly, if $p \leq p_1$, then the only extremal functions are $\pm iK(\mp iz)$. Finally, if $p \in (p_1, p_0)$, then no rotation of the Koebe function can be extremal. This will be shown in Lemma 3.6. In this case the maximal value of the functional J_p can be calculated numerically.

Remark 1. From the Schiffer variational theory for univalent functions we know that every extremal function $F \in \mathcal{S}$ for the functional J_p for fixed $p \in \mathbf{R}$ is a solution of the Schiffer differential equation

$$(15) \quad \left[\frac{zF'(z)}{F(z)} \right]^2 \frac{1 + Af(z)}{f(z)^2} = z^2 + Az + B_0 + \bar{A}z^{-1} + z^{-2},$$

where $2B_0 = J_p(F)$ and A is given by (12), cf. [3].

Remark 2. There is an intimate connection between Pontryagin's maximum principle (13) and the Schiffer differential equation (15), cf. [14]. A function $f \in \mathcal{S}$ is A -admissible if and only if it admits a piecewise analytic extension to $\bar{\mathbf{D}}$ such that

$$\left\{ \frac{zf'(z)}{f(z)} \right\}^2 \frac{1 + Af(z)}{f(z)^2}$$

is positive on $|z| = 1$, except for one or two points.

Remark 3. In terms of the Schiffer differential equation, the notion of A -admissibility was used by Pfluger in his study of the Fekete-Szegő

functional $a_3 - \lambda a_2^2$ for $\lambda \in \mathbf{C}$ in [11], [12], [13]. The critical points are exactly those functions which satisfy the extremality condition in Pfluger's terminology. Using an interesting global topological argument, Pfluger shows that, for $\lambda \neq 1$, there are exactly two functions satisfying the extremality condition, i.e., exactly two critical points. In the case of the functional (1) we shall show that this is only true for $p \notin (p_1, p_0)$. Consequently, the global reasoning of Pfluger cannot be adopted for our situation.

3. Proof of Theorem 1.2. According to Theorem 2.1 every extremal function for J_p is A -admissible. We will show first that we can restrict ourselves to the case $A \notin \mathbf{R} \cup i\mathbf{R}$ in the sequel. Otherwise, a rotation of the Koebe function will be extremal for (1).

Lemma 3.1. *Let $f \in \mathcal{S}$ be an A -admissible function.*

(a) *If $A \in i\mathbf{R} \setminus \{0\}$, then $f(z) = \pm iK(\mp iz)$.*

(b) *If $A \in \mathbf{R}$ and $|A| \geq 4$, then $f(z) = \pm K(\pm z)$.*

Proof. (a) If $\operatorname{Re} A = 0$, then the Hamiltonian function (11) becomes

$$\begin{aligned} H_A(t, u) &= -2e^{-t} \left(\frac{\operatorname{Im} A}{2} \sin u - e^{-t} \sin^2 u \right) - e^{-2t} \\ &= 2e^{-2t} \left(\sin u - \frac{1}{4} e^t \operatorname{Im} A \right)^2 - e^{-2t} - \frac{(\operatorname{Im} A)^2}{8}. \end{aligned}$$

Consider the case $\operatorname{Im} A > 0$. For fixed $t \geq 0$, the function $u \mapsto H_A(t, u)$ attains its maximum at $u = u_0(t)$ if and only if $\sin u_0(t) = -1$, i.e., $u_0(t) = -\pi/2$. Thus $f(z) = iK(-iz)$. If $\operatorname{Im} A < 0$ a similar argument shows that $f(z) = -iK(iz)$.

(b) If $A \in \mathbf{R}$, then the Hamiltonian function (11) takes the form

$$\begin{aligned} H_A(t, u) &= -e^{-t} (\operatorname{Re} A \cos u + 2e^{-t} \cos^2 u) + e^{-2t} \\ (16) \quad &= -2e^{-2t} \left(\cos u + \frac{\operatorname{Re} A}{4} e^t \right)^2 + e^{-2t} + \frac{(\operatorname{Re} A)^2}{8}. \end{aligned}$$

Therefore, if $A \geq 4$ or $A \leq -4$, (13) and (11) imply $u_0(t) = \pi$ or $u_0(t) = 0$, respectively, which leads to $f(z) = \pm K(\pm z)$. \square

Now we rule out the existence of an extremal function which is A -admissible for some $A \in (-4, 4)$.

Lemma 3.2. *Let $F(z) = z + A_2z^2 + A_3z^3 + \dots \in \mathcal{S}$ be an extremal function of (1), and let A be defined by (12). Then $A \notin (-4, 4)$.*

Proof. If $A \in (-4, 4)$, then the Hamiltonian function (11) has the form (16). We first consider the case $A = 0$. Then u_0 only takes the values $\pi/2$ and $-\pi/2$, i.e.,

$$F(z) = f_\mu(z) = \frac{z}{1 + 2\mu iz - z^2} = z - 2i\mu z^2 + \dots$$

for some $-1 \leq \mu \leq 1$. Now the relation $0 = \text{Im } A = -2\text{Im } A_2/3 = 4\mu/3$ leads to $\mu = 0$. This implies $F(z) = f_0(z) := z/(1 - z^2)$. However, f_0 is never an extremal function for the functional J_p because $J_p(f_0) = 1 < 7/3 = J_p(f_{\pm 1})$. Thus, $A = 0$ is not possible.

If $A \in (-4, 4) \setminus \{0\}$, then for fixed $t \geq 0$ the function $u \mapsto H_A(t, u)$ attains its maximum if and only if $u = u_0(t)$ where

$$(17) \quad \cos u_0(t) = \begin{cases} -(\text{Re } A/4)e^t & \text{for } 0 \leq t \leq \log |4/\text{Re } A|, \\ -\text{sign } \text{Re } A & \text{for } t \geq \log |4/\text{Re } A|. \end{cases}$$

Equation (4) for $u(t) = u_0(t)$ leads to

$$\text{Re } A_2 = -2 \int_0^\infty e^{-t} \cos u_0(t) dt = \frac{\text{Re } A}{2} \left(\log \left| \frac{4}{\text{Re } A} \right| + 1 \right).$$

Since $\text{Re } A \neq 0$, we obtain from (12)

$$(18) \quad |A| = 4 \exp \left(\frac{2(p-1)}{2p+1} \right),$$

i.e., in particular $-1/2 \leq p < 1$. Using equation (4) for $u(t) = u_0(t)$ and (12) we calculate

$$(19) \quad \begin{aligned} J_p(F) &= \text{Re}(A_3 - A_2^2) + \frac{2p+1}{3} A_2^2 \\ &= 1 - 4 \int_0^\infty e^{-2t} (\cos u_0(t))^2 dt + \frac{3}{4(2p+1)} A^2. \end{aligned}$$

Inserting (17) and (18), we find

$$(20) \quad J_p(F) = 1 + 2 \exp\left(\frac{4(p-1)}{2p+1}\right).$$

To finish the proof of Lemma 3.2, we therefore only have to present for each $p \in [-1/2, 1)$ a function $\hat{F} \in \mathcal{S}$ with $J_p(\hat{F}) > 1 + 2 \exp(4(p-1)/(2p+1))$. Evidently, $\hat{F}(z) = -iK(iz)$ is a suitable choice for $p < p_2 := (4 - \log(3/2))/(4 + 2 \log(3/2)) \doteq 0.74716$.

The case $p_2 \leq p < 1$ is more delicate. By composing a Pick function

$$P_m(z) := \frac{m(1-z)^2 + 2z - (1-z)\sqrt{m^2(1-z)^2 + 4mz}}{2z}, \quad m > 1,$$

with a Mobius transform

$$M_t(z) := e^{-it/2} \frac{z+1}{1-e^{-it}z}, \quad t \in \mathbf{R},$$

we construct a univalent mapping $f_{m,t,u} := F_m(M_t(e^{iu}z))$, $u \in \mathbf{R}$, from the unit disk onto the right half-plane minus a circular slit emerging from the origin tangentially to the positive real axis. Hence, $(f_{m,t,u})^2$ is a univalent slit-mapping. After renormalization we obtain a function $F_{m,t,u}(z) = z + a_2(m,t,u)z^2 + a_3(m,t,u)z^3 + \dots \in \mathcal{S}$ with coefficients

$$a_2(m,t,u) = \frac{e^{i(u-t)}}{2m} [3 + (4m-3)e^{it}],$$

$$a_3(m,t,u) = \frac{e^{2i(u-t)}}{m^2} [2 + (6m-5)e^{it} + (3m^2 - 6m + 3)e^{2it}].$$

For $p \in [p_2, 1)$, we choose the function $\hat{F} := F_{m_p, t_p, u_p}$ with

$$t_p := \frac{6(1-p)}{\pi}, \quad u_p := 3t_p/2,$$

$$m_p := \frac{3p+1 - 4 \cos t_p - 3(p+1) \cos 2t_p}{2 - 4p(\cos t_p + \cos 2t_p)}.$$

Notice that $m_p > 1$ for $p_2 < p < 1$. A straightforward calculation using the Taylor expansion shows

$$J_p(\hat{F}) = \frac{1}{6(-3p-1 + 4 \cos t_p + 3(p+1) \cos 2t_p)}$$

$$\times [4 - 8p + 11(p+1) \cos t_p + 4(10p+3) \cos 2t_p$$

$$+ 2(17p+3) \cos 3t_p + 4(4p-3) \cos 4t_p + 3(p-3) \cos 5t_p]$$

$$\geq \frac{8p+1}{3} + \frac{2(4p^2 - 11p + 7)}{3} t_p^2.$$

Finally,

$$\frac{8p+1}{3} + \frac{24(4p^2-11p+7)}{\pi^2}(1-p)^2 > 1 + 2 \exp\left(\frac{4(p-1)}{2p+1}\right)$$

if $p_2 < p < 1$ since the difference of these two functions is strictly monotonically decreasing and equality holds for $p = 1$. The assertion follows. \square

In the next lemma we shall deal only with A -admissible (not necessarily extremal) functions. We parametrize A in terms of $\varrho \in (0, 1]$ and $\varphi \in (-\pi, \pi]$ by

$$(21) \quad A = A(\varrho, \varphi) := \left(\varrho + \frac{1}{\varrho}\right)e^{i\varphi} + 2e^{-i\varphi}.$$

(This is the parametrization used by Schaeffer and Spencer [15, Chapter 13]). In view of Lemma 3.2, we only have to consider the case $A \in \mathbf{C} \setminus (-4, 4)$ for A -admissible functions.

Lemma 3.3. *If $A \in \mathbf{C} \setminus (-4, 4)$, then a uniquely determined A -admissible function $f_A(z) = z + a_2(A)z^2 + \dots \in \mathcal{S}$ exists. $\mathbf{C} \setminus f_A(\mathbf{D})$ is a single analytic arc extending to ∞ , and*

$$(22) \quad \begin{aligned} 2a_2(A) = & 4e^{-i\varphi} - A \log(1 + \varrho^2 + 2\varrho e^{-2i\varphi}) \\ & + A \log(1 - \varrho^2) + \bar{A} \log \frac{1 + \varrho}{1 - \varrho}. \end{aligned}$$

Equation (22) is exactly formula (13.5.8) in [15] for the part of the coefficient body V_3 which corresponds to one-slit mappings.

Proof. We show that, for $A \in \mathbf{C} \setminus (-4, 4)$, equation (13) determines the function u_0 and hence $a_2(A)$ uniquely. If $A \in i\mathbf{R}$ or $A \in \mathbf{R}$, $|A| \geq 4$, this and (22) have already been proved in Lemma 3.1. If $A = \pm 4$, then (22) has to be understood in the limit $\varrho \rightarrow 1$. Otherwise, we have to maximize the trigonometric polynomial (11) for fixed $A \notin \mathbf{R} \cup i\mathbf{R}$. This will be done by completing the square in (11) employing a clever

idea that was used by Tammi [16] and Haario [4] in their study of the functional $\operatorname{Re}(a_3 - a_2^2 + Ca_2)$, $C \in \mathbf{C}$.

We fix $t \geq 0$, choose a parameter $q > 0$ (to be determined later) and use the identity $\cos^2 u = (q+1)\cos^2 u + q\sin^2 u - q$ to write the Hamiltonian H_A in the form

$$\begin{aligned} H_A(t, u) &= -e^{-t}(\operatorname{Re} A \cos u + \operatorname{Im} A \sin u + 2e^{-t}(q+1)\cos^2 u \\ &\quad + 2e^{-t}q\sin^2 u - 2e^{-t}q) + e^{-2t} \\ &= -2e^{-2t} \left(\cos u + \frac{\operatorname{Re} A}{4(q+1)} e^t \right)^2 (q+1) \\ &\quad - 2e^{-2t} \left(\sin u + \frac{\operatorname{Im} A}{4q} e^t \right)^2 q \\ &\quad + 2e^{-2t}q + e^{-2t} + \frac{(\operatorname{Re} A)^2}{8(q+1)} + \frac{(\operatorname{Im} A)^2}{8q} \\ &\leq 2e^{-2t}q + e^{-2t} + \frac{(\operatorname{Re} A)^2}{8(q+1)} + \frac{(\operatorname{Im} A)^2}{8q} \\ &=: f(t, q) \end{aligned}$$

with equality if and only if

$$(23) \quad \cos u = -\frac{\operatorname{Re} A}{4(q+1)} e^t \quad \text{and} \quad \sin u = -\frac{\operatorname{Im} A}{4q} e^t.$$

Now we choose $q = q_0(t)$ as follows. The function $q \mapsto f_q(t, q)$ is monotonically increasing on $(0, \infty)$ because $f_{qq}(t, q) > 0$. Moreover, $\lim_{q \rightarrow 0} f_q(t, q) = -\infty$ and $\lim_{q \rightarrow \infty} f_q(t, q) = 2e^{-2t}$, i.e., $q \mapsto f_q(t, q)$ has exactly one zero $q_0(t) > 0$ which is uniquely determined by A . The function $t \mapsto q_0(t)$ is continuously differentiable and strictly increasing because of

$$\frac{d}{dt} q_0(t) = \frac{4e^{-2t}}{f_{qq}(t, q_0(t))} > 0.$$

Now $f_q(t, q_0(t)) = 0$ implies

$$\left(\frac{\operatorname{Re} A}{4(q_0(t)+1)} e^t \right)^2 + \left(\frac{\operatorname{Im} A}{4q_0(t)} e^t \right)^2 = 1,$$

which shows that $u_0(t)$ is given by

$$\cos u_0(t) = -\frac{\operatorname{Re} A}{4(q_0(t)+1)} e^t \quad \text{and} \quad \sin u_0(t) = -\frac{\operatorname{Im} A}{4q_0(t)} e^t.$$

Thus u_0 is uniquely determined by A . In particular, $t \mapsto u_0(t)$ is a real analytic function and $\cos u_0(t) \neq 0$ and $\sin u_0(t) \neq 0$ for all $t \geq 0$. The maximal property of $u_0(t)$ implies $H_u(t, u_0(t)) = 0$ for all $t \geq 0$, i.e.,

$$(24) \quad \frac{\operatorname{Im} A}{\sin u_0(t)} - \frac{\operatorname{Re} A}{\cos u_0(t)} = 4e^{-t} > 0.$$

The properties of $q_0(t)$ show that $t \mapsto u_0(t)$ is a monotone function, and differentiation of (24) leads to the following equation for the inverse function $u \mapsto t(u)$

$$-4 \frac{dt}{du}(u) e^{-t(u)} = \operatorname{Im} A \frac{\cos u}{\sin^2 u} + \operatorname{Re} A \frac{\sin u}{\cos^2 u}.$$

This formula can be used to express

$$a_2(\infty, u_0(\cdot)) = -2 \int_0^\infty e^{-t} e^{-iu_0(t)} dt$$

in terms of $u_0(0)$, $u_0(\infty)$ and A only. Using (24) for $t = 0$ and $t = \infty$, one can express $u_0(0)$ and $u_0(\infty)$ as a function of A and this leads finally to (22).

To finish the proof of Lemma 3.3, we have to show that f_A is a one-slit mapping. This follows from a result of Kufarev [8] because $u'_0(t)$ is bounded on $[0, \infty)$. In fact, differentiation of (24) leads to

$$\begin{aligned} |u'_0(t)| &= \left| \frac{4e^{-t} \sin u_0(t) \cos u_0(t)}{(\operatorname{Re} A / \cos u_0(t)) + 4e^{-t} \cos^2 u_0(t)} \right| \\ &\leq \frac{4}{-4e^{-t} - (\operatorname{Re} A / \cos u_0(t))} \\ &= -4 \frac{\sin u_0(t)}{\operatorname{Im} A} \leq \frac{4}{|\operatorname{Im} A|}. \quad \square \end{aligned}$$

We now characterize the critical points of the functional J_p which are A -admissible for $A \notin (-4, 4)$.

Lemma 3.4. *If f is a critical point of the functional J_p which is A -admissible for $A = A(\varrho, \varphi) \in \mathbf{C} \setminus (-4, 4)$, then $f(\varrho, \varphi) = 0$ where*

$$(25) \quad \begin{aligned} f(\varrho, \varphi) &:= -4\varrho \sin \varphi - (1 + \varrho)^2 \cos \varphi \operatorname{Im} \log(1 + \varrho^2 + 2\varrho e^{-2i\varphi}) \\ &\quad - (1 - \varrho)^2 \sin \varphi \operatorname{Re} \log(1 + \varrho^2 + 2\varrho e^{-2i\varphi}) \\ &\quad + 2(1 - \varrho)^2 \sin \varphi \log(1 - \varrho) + 3(1 - \varrho)^2 \sin \varphi. \end{aligned}$$

Moreover, if $A \notin i\mathbf{R}$, then $p = p(\varrho, \varphi)$ where

$$(26) \quad p(\varrho, \varphi) = \frac{3}{4} \frac{\operatorname{Re} A(\varrho, \varphi)}{\operatorname{Re} a_2(A(\varrho, \varphi))} - \frac{1}{2}.$$

Proof. Inserting (22) into (14) we see that

$$-3 \operatorname{Im} A(\varrho, \varphi) = 2 \operatorname{Im} a_2(A(\varrho, \varphi)).$$

A straightforward calculation using (21) shows that this equation is equivalent to $f(\varrho, \varphi) = 0$. Inserting (22) into (14) and taking the real part, we obtain $3\operatorname{Re} A(\varrho, \varphi) = 2(2p + 1)\operatorname{Re} a_2(A(\varrho, \varphi))$. If $\operatorname{Re} A(\varrho, \varphi) \neq 0$, then $\operatorname{Re} a_2(A(\varrho, \varphi)) \neq 0$ and $p = p(\varrho, \varphi)$ follows. \square

Obviously, $f(\varrho, \pm\pi/2) = 0$ if and only if $\varrho = 1/3$ and $\varphi = \mp\pi/2$ corresponds to $\mp iK(\pm iz)$. Moreover, $f(1, \varphi) = 0$ if and only if $\varphi = 0$ or $\varphi = \pi$ which corresponds to $K(z)$ or $-K(-z)$ and $p(1, 0) = p(1, \pi) = 1$. By symmetry it is therefore sufficient to study the equation $f(\varrho, \varphi) = 0$ in detail for $0 < \varrho < 1$ and $0 \leq \varphi < \pi/2$ and to calculate $p(\varrho, \varphi)$ for $f(\varrho, \varphi) = 0$. Obviously, $f(\varrho, 0) = 0$ for all $0 < \varrho < 1$ and $\varphi = 0$ corresponds to $a_2(A) = 2$, i.e., to the Koebe function $K(z)$. However, for certain values of ϱ another critical point may occur.

Lemma 3.5. *If $f(\varrho, \varphi) = 0$ with $0 < \varrho < 1$ and $0 \leq \varphi < \pi/2$, then either*

(a) $\varphi = 0$ and $p(\varrho, \varphi) = (3 + 2\varrho + 3\varrho^2)/(8\varrho) \in (1, \infty)$ or

(b) $\varphi > 0$. In that case $1/3 < \varrho < (1 - e^{-3/2})/(1 + e^{-3/2}) \doteq 0.63515$ and $\varphi = \varphi(\varrho)$ is a differentiable and strictly decreasing mapping onto $(0, \pi/2)$. The function $p = p(\varrho, \varphi(\varrho))$ is differentiable and strictly increasing and takes its values in (p_1, p_0) .

Proof. We only have to prove (b). To make computations easier we adopt the following transformations. We introduce functions

$$g(v, x) := \frac{1}{v} - 4 - \frac{1}{v} \sqrt{\frac{1+x}{1-x}} T + \frac{1}{2} L,$$

$$q(v, x) := 1 - v - v \sqrt{\frac{1-x}{1+x}} T - \frac{1}{2} L - \log v,$$

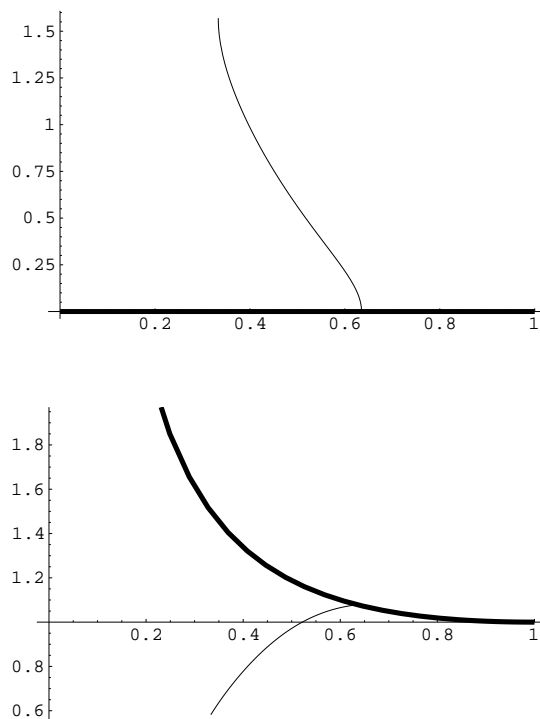


FIGURE 2. The locus of the zeros of $f(\varrho, \varphi)$ consisting of two curves in the ϱ - φ -plane (on the top) and the values of p as a function of ϱ along these curves (on the bottom). The thick parts correspond to the Koebe function.

defined on $(v, x) \in (0, 1) \times (-1, 1)$ where we used the shorthand notations

$$T := \arctan \frac{\sqrt{1-x^2}}{(1+v)/(1-v)+x}, \quad L := \log \frac{(1+v^2)+(1-v^2)x}{2}.$$

The following estimate on T will be useful later

$$(27) \quad \frac{(1-v)\sqrt{1-x^2}}{2} < T < \frac{\sqrt{1-x^2}}{(1+v)/(1-v)+x}.$$

The first inequality in (27) may be obtained by comparing the partial derivatives with respect to v for fixed x , the second one readily follows from $\arctan y < y$ for $y > 0$.

For fixed x we compute

$$(28) \quad \lim_{x \rightarrow 1^-} g(v, x) = -3 - \log v, \quad \lim_{x \rightarrow 1^-} q(v, x) = 1 - v.$$

Hence, both functions g and q may be continuously extended to the rectangle $X := (0, 1) \times (-1, 1]$.

By the transformation

$$(29) \quad v = v(\varrho) := \left(\frac{1 - \varrho}{1 + \varrho} \right)^2, \quad x = x(\varphi) := \cos 2\varphi,$$

we define a bijection $(\varrho, \varphi) \mapsto (v(\varrho), x(\varphi))$ of $(0, 1) \times [0, \pi/2)$ onto X . A straightforward calculation leads to the relations

$$(30) \quad \begin{aligned} \sin \varphi g(v(\varrho), x(\varphi)) &= \frac{-1}{(1 - \varrho)^2} f(\varrho, \varphi), \\ q(v(\varrho), x(\varphi)) &= \frac{3}{2p(\varrho, \varphi) + 1} \end{aligned}$$

between f and p and the new functions g and q . Therefore, we ought to minimize $q(v, x)$ for $(v, x) \in X$ with $g(v, x) = 0$.

We claim that the locus of the zeros of $g(v, x)$ is a curve $\gamma : t \mapsto (t, x(t))$, $t \in (e^{-3}, 1/4]$, with

$$\lim_{t \rightarrow 1/4^+} \gamma(t) = (1/4, -1), \quad \lim_{t \rightarrow e^{-3}^-} \gamma(t) = (e^{-3}, 1),$$

where $x'(t) < 0$ is continuous. The existence of such a curve γ is guaranteed by the implicit function theorem since the partial derivatives

$$(31) \quad \begin{aligned} g_v(v, x) &= \frac{1}{v^2} \left(-1 + \sqrt{\frac{1+x}{1-x}} T \right), \\ g_x(v, x) &= \frac{(1+x)(1-v) - 2\sqrt{(1+x)/(1-x)} T}{2v(1-x^2)}, \end{aligned}$$

of g are negative on $(0, 1) \times (-1, 1)$ by (27) and

$$\lim_{v \rightarrow 0^+} g(v, x) = +\infty, \quad \lim_{v \rightarrow 1^-} g(v, x) = -3,$$

for fixed x . A computation of the limit $\lim_{x \rightarrow -1^+} g(v, x) = 1/v - 4$ for fixed v together with (28) proves the statement about the endpoints of γ .

Now we shall prove that $q(v, x)$ is decreasing on γ . To do so, we consider

$$(32) \quad \frac{dq}{dt}(t, x(t)) = q_v(t, x(t)) + q_x(t, x(t))x'(t).$$

Differentiation of the identity $g(t, x(t)) = 0$ leads to

$$x'(t) = -\frac{2(1-x^2)}{v} \frac{-1 + \sqrt{(1+x)/(1-x)}T}{(1-v)(1+x) - 2\sqrt{(1+x)/(1-x)}T} \Big|_{x=x(t)}^{v=t}$$

and allows us to replace $x'(t)$ on the righthand side of (32). With (31) and the formulas

$$q_v(v, x) = -1 - \sqrt{\frac{1-x}{1+x}}T,$$

$$q_x(v, x) = -\frac{(1-x)(1-v) - 2v\sqrt{(1-x)/(1+x)}T}{2(1-x^2)}$$

for the partial derivatives of q , we obtain

$$(33) \quad \frac{dq}{dt}(t, x(t)) = \frac{-1+x-v-xv}{v} \times \frac{v-1+v\sqrt{(1-x)/(1+x)}T + \sqrt{(1+x)/(1-x)}T}{(v-1)(1+x) + 2\sqrt{(1+x)/(1-x)}T} \Big|_{x=x(t)}^{v=t}.$$

Another application of (27) shows $(dq/dt)(t, x(t)) < 0$. Hence,

$$1 - e^{-3} = q(e^{-3}, -1) \leq q(v, x) < q(1/4, 1) = \log 4$$

on γ , since q is continuous on X .

Translating our result via (29) and (30) to the functions $f(\varrho, \varphi)$ and $p(\varrho, \varphi)$ we obtain the assertion. \square

In view of Lemmas 3.2, 3.4 and 3.5 (b), any function other than a rotation of the Koebe function might be extremal for (1) only if

$p_1 < p < p_0$. This function, which we denote by F_0 , is $A(\varrho, \varphi(\varrho))$ -admissible for $\varrho \in (1/3, (1-e^{-3/2})/(1+e^{-3/2}))$ such that $p = p(\varrho, \varphi(\varrho))$, i.e., uniquely determined up to symmetry, and maps the unit disk onto the complement of a single analytic Jordan arc extending to ∞ . To finish the proof of Theorem 1.2, we show

Lemma 3.6. *If $p_1 < p < p_0$, then $J_p(F_0) > J_p(\pm K(\pm z)) = (8p+1)/3$ and $J_p(F_0) > J_p(\pm iK(\mp iz)) = 7/3$.*

Proof. Fix $p \in (p_1, p_0)$ and, by Lemma 3.5 (b), $\varrho \in (1/3, (1-e^{-3/2})/(1+e^{-3/2}))$ such that $p = p(\varrho, \varphi(\varrho))$. The value of the functional J_p for the $A(\varrho, \varphi)$ -admissible function F_0 can be expressed in terms of $A(\varrho, \varphi)$ only. In fact, the reasoning in the proof of Lemma 3.3 to show that the second coefficient of F_0 can be expressed as a function of $A(\varrho, \varphi)$ applied to (19) for $F = F_0$ leads to the remarkably simple formula

$$J_p(F_0) = \frac{1}{\varrho} + \varrho + \cos 2\varphi.$$

We adopt the notations in the proof of Lemma 3.5. Then, using the transformation (29), $J_p(F_0) > (8p+1)/3$ is equivalent to

$$(34) \quad j(v, x) := \frac{4-4v}{3+v+x-xv} < q(v, x)$$

where $v = v(\varrho)$, $x = x(\varphi(\varrho))$. We shall prove (34) for any $(v, x) \in X$. For $v \in (0, 1)$ fixed

$$\lim_{x \rightarrow 1^-} j(v, x) - q(v, x) = 0,$$

using (28). Now by the aid of (27) we estimate the difference of the partial derivatives

$$\begin{aligned} j_x(v, x) - q_x(v, x) &= -\frac{4(1-v)^2}{(3+v+x-vx)^2} \\ &\quad + \frac{(1-x)(1-v) - 2v\sqrt{(1-x)/(1+x)}T}{2(1-x^2)} \\ &= \frac{(1-v)^2}{2} \left(\frac{1}{1+x} - \frac{8}{(3+v+x-vx)^2} \right) \geq 0. \end{aligned}$$

This proves (34).

Similarly, the inequality $J_p(F_0) > 7/3$ is equivalent to $j(t, x(t)) < 6/5$, where $t = v(\varrho) \in (e^{-3}, 1/4)$. To prove this we show that the lefthand side is an increasing function in t :

(35)

$$\begin{aligned} \frac{dj}{dt}(t, x(t)) &= j_v(t, x(t)) + j_x(t, x(t))x'(t) \\ &= \frac{8(1 + v - x + vx)}{v(3 + v + x - xv)^2} \\ &\quad \times \frac{(1 + v + x - vx)\sqrt{(1+x)/(1-x)}T - (1 - v + x - vx)}{1 - v + x - xv - 2\sqrt{(1+x)/(1-x)}T}. \end{aligned}$$

The numerator and denominator of the second fraction are positive by the estimate (27) on T . This, together with $j(1/4, v(1/4)) = j(1/4, -1) = 6/5$ gives the assertion. \square

4. Remarks.

1. For $p > 0$ we denote the maximum of the functional J_p in (1) by M_p . An examination of the proof of Theorem 1.1 in [7] shows that (2) remains valid for any univalent function f on \mathbf{D} if $C(p, d)$ is replaced by

$$\begin{aligned} \tilde{C}(p, d) &:= \frac{1}{2P(p)} \frac{\sinh(2P(p)d)}{[2 \cosh(2pP(p)d)]^{1/p}}, \\ P(p) &:= \sqrt{\frac{6M_p - 2}{16p}}. \end{aligned} \tag{36}$$

For $p \geq p_0$ we have $P(p) = 1$ by Theorem 1.2 and therefore $\tilde{C}(p, d) = C(p, d)$.

For smaller values of p the new distortion theorem will not be sharp. Nevertheless, we now obtain two-point distortion theorems of a similar fashion as (2) also for $0 < p < 1$. A straightforward calculation shows $\tilde{C}(p, d) < C(p, d)$ for all $d > 0$ and $p \geq 1/3$ as soon as $M_p > (8p + 1)/3$. We omit the details.

2. Kim and Minda [7, Theorem 1], studied the functional

$$K_p(f) = K_p(a_2, a_3) := \left| a_3 + \frac{p-3}{3}a_2^2 \right| + \frac{p}{3}|a_2|^2, \quad p \in \mathbf{R}, \quad f \in \mathcal{S},$$

and computed its maximum value. Regardless of its similarity to J_p , this functional is easier to handle since (12) now has to be replaced by $A = 4p/3\operatorname{Re} A_2$. For large values of $|p|$ a rotation of the Koebe function is extremal, K if $p \geq 3/2$, respectively $-K(-z)$ if $p \leq 0$, whereas in the intermediate case a two-slit mapping K_0 is extremal, which doesn't belong to the 'complicated' part of ∂V_3 . This allows us to calculate $K_p(K_0) = 1 + 2 \exp((2p-3)/p)$ as has been done by Kim and Minda using Jenkin's description of the part of ∂V_3 which belongs to two-slit mappings or similarly as the reasoning which leads to (20).

3. In [10] Ma and Minda proved the following analogue of Theorem 1.1 for nonnormalized *bounded* univalent functions.

Theorem 4.1. *If g is univalent on \mathbf{D} and $g(\mathbf{D}) \subseteq \mathbf{D}$, then*

$$d_{\mathbf{D}}(g(a), g(b)) \geq \frac{1}{4} \log \left(\frac{[1 + e^{-4pd_{\mathbf{D}}(a,b)}]^{1/p} + \Delta_p(g, a, b)}{[1 + e^{-4pd_{\mathbf{D}}(a,b)}]^{1/p} + e^{-4d_{\mathbf{D}}(a,b)} \Delta_p(g, a, b)} \right)$$

for all $a, b \in \mathbf{D}$ and all $p \geq 3/2$, where

$$\begin{aligned} \Delta_p(g, a, b) := & \left[\left(\frac{[(1-|a|^2)|g'(a)|/(1-|g(a)|^2)]}{1 - [(1-|a|^2)|g'(a)|/(1-|g(a)|^2)]} \right)^p \right. \\ & \left. + \left(\frac{[(1-|b|^2)|g'(b)|/(1-|g(b)|^2)]}{1 - [(1-|b|^2)|g'(b)|/(1-|g(b)|^2)]} \right)^p \right]^{1/p}. \end{aligned}$$

The proof relies on the following coefficient inequality for bounded univalent functions established by Ma and Minda [10].

Theorem 4.2. *If $f(z) = a_1z + a_2z^2 + a_3z^3 + \dots$ is univalent on \mathbf{D} and $f(\mathbf{D}) \subseteq \mathbf{D}$, then for $p \geq 3/2$,*

$$(37) \quad \left| 3 \left[\frac{a_3}{a_1} - \left(\frac{a_2}{a_1} \right)^2 \right] + \frac{p+|a_1|}{1-|a_1|} \left(\frac{a_2}{a_1} \right)^2 \right| + \frac{1+p}{1-|a_1|} \left| \frac{a_2}{a_1} \right|^2 \leq (8p+1+|a_1|)(1-|a_1|).$$

Equality holds if and only if f is a rotation of the Pick function.

Using the procedure in Section 3, we can show that the constant $3/2$ in Theorem 4.2 can be replaced by

$$(38) \quad p_b(|a_1|) := \frac{1}{2} \cdot \frac{2e^3 + 1}{e^3 - 1} - \frac{1}{2} \cdot \frac{e^3 + 2}{e^3 - 1} |a_1|,$$

but by no smaller number. Thus Theorem 4.1 remains valid for all univalent functions $g : \mathbf{D} \rightarrow \mathbf{D}$ and all $p \geq p_b(g)$ where

$$p_b(g) := \frac{1}{2} \cdot \frac{2e^3 + 1}{e^3 - 1} - \frac{1}{2} \cdot \frac{e^3 + 2}{e^3 - 1} \cdot \max_{z \in \mathbf{D}} \frac{|g'(z)|(1 - |z|^2)}{1 - |g(z)|^2}.$$

4. We close this paper by briefly considering the extremal problem

$$(39) \quad \max_{(a_2, a_3) \in V_3} I_c(a_2, a_3)$$

for the functional

$$(40) \quad I_c(a_2, a_3) = \operatorname{Re}(a_3 - ca_2^2) + c|a_2|^2, \quad c \in \mathbf{R} \text{ fixed,}$$

which looks very similar to the functional (1).

The extremal problem (39) has been studied by Jakubowski and Zyskowska [5]. They obtained the sharp upper bound for $c \notin [1/2, 1]$ and a nonsharp upper bound for $c \in [1/2, 1]$ by an application of the Landau-Valiron lemma. Using the same method as for the functional (1), we can complete their results.

Theorem 4.3. *Let $F \in \mathcal{S}$ be an extremal function for I_c for fixed $c \in \mathbf{R}$. Then $\mathbf{C} \setminus F(\mathbf{D})$ is an analytic Jordan arc extending to infinity.*

(a) *If $c \geq c_0 := e/(2e - 2) \doteq 0.790988$, then $F(z) = \pm iK(\mp iz)$.*

(b) *If $c \leq 1/2$, then $F(z) = \pm K(\pm z)$.*

(c) *If $1/2 < c < c_0$ then F is not a rotation of the Koebe function. More precisely, there is a function $F_0 \in \mathcal{S}$ such that $\mathbf{C} \setminus F_0(\mathbf{D})$ is a curved analytic Jordan arc and there are exactly four extremal functions, namely $\pm F_0(\pm z)$ and $\pm \overline{F_0(\pm \bar{z})}$.*

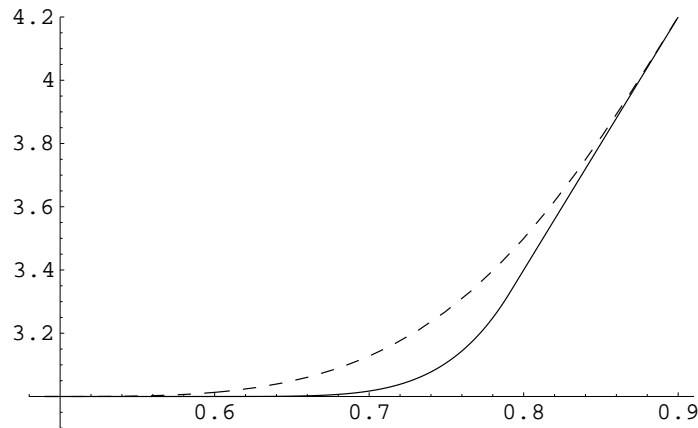


FIGURE 3. A plot of $\max_{(a_2, a_3) \in V_3} \operatorname{Re}(a_3 - ca_2^2) + c|a_2|^2$ as a function of c for $0.5 \leq c \leq 0.9$. The Jakubowski-Zyskowska estimate is shown dashed.

5. We are not able to give an explicit formula for $M_p = \max_{f \in \mathcal{S}} J_p(f)$ if $p_1 < p < p_0$. However, the differential equations (33) and (35) for $q(t)$ and $j(t)$, $t \in (e^{-3}, 1/4]$ can be solved numerically and lead, after the substitution $p(t) := 3/(2q(t)) - 1/2$, $M_p(t) := 4/j(t) - 1$, to a parametrization of the curve $(p, M_p(t))$, $p_1 < p < p_0$, in terms of t which has been used to produce Figure 1. A similar approach has been used for Figure 3.

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