ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 2, Spring 2001

GEODESIC LAMINATIONS ON COMPACT SURFACES AND HOMEOMORPHISMS OF THE CANTOR SET

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ABSTRACT. In this paper we investigate a connection between minimal geodesic laminations on compact hyperbolic surfaces and homeomorphisms of the Cantor set. Let M be a compact hyperbolic surface. To each minimal lamination $\mathcal{L} \subset M$ having no closed leaves, and to each compact curve C transverse to $\mathcal{L}$, we associate a group consisting of certain homeomorphisms on the intersection $C \cap \mathcal{L}$. This group is used to study various topological aspects of the lamination including orientability and existence of transverse measures.

0. Introduction. In [**8**] M. Urbański and I investigated a connection between circle maps, measured laminations on compact surfaces, and free actions of surface groups on **R**-trees. We showed that certain order preserving homeomorphisms f of the unit circle induce a measured lamination $(\mathcal{L}, \mu)$ on the torus T^2. The map f has no periodic points and no dense orbits; this is equivalent to saying that f is not topologically conjugate to a rotation. The resulting lamination $\mathcal{L}$ is minimal (each leaf is dense in $\mathcal{L}$), and each leaf and each complementary region of $\mathcal{L}$ is simply connected. Via results found in [**6**], the measured lamination $(\mathcal{L}, \mu)$ determines a free action (by isometries) of $\pi_1(T^2) = \mathbf{Z} \times \mathbf{Z}$ on an **R**-tree T. It is shown that T is isometric to **R** and that the ratio of the translation lengths of the standard generators (1,0) and (0,1) of $\mathbf{Z} \times \mathbf{Z}$ is equal to the rotation number of the homeomorphism f.

The basic idea in the construction of the lamination $\mathcal{L}$ on the torus is as follows. We begin with an essential simple closed curve C imbedded in T^2 together with an identification of C with the unit circle S^1. We then take an order preserving homeomorphism f of S^1 which is not topologically conjugate to a rotation and view it as a homeomorphism

1991 AMS *Mathematics Subject Classification.* 58F.
Research supported in part by NSF grant INT-9726708, and by a grant from the Texas Advanced Research Program.
Received by the editors on August 9, 1996, and in revised form on October 18, 1999.

of C. We let Δ denote the set of accumulation points of $\{f^n(x)\}_{n\in\mathbf{Z}}$ for some point $x \in C$. The set Δ is independent of the point $x \in C$ and is invariant under f. Furthermore, Δ is topologically a Cantor set, i.e., a nonempty perfect totally disconnected set. We then construct $\mathcal{L}$ by joining each point $p \in \Delta$ to the point $f(p)$ by a geodesic in T^2.

The lamination $\mathcal{L}$ is related to the homeomorphism f as follows. For each point $p \in \Delta$ the points p and $f(p)$ lie on the same leaf of $\mathcal{L}$ and the open segment $(p, f(p))$ contained in $\mathcal{L}$ is disjoint from C. It is shown in [**8**] that this sort of relationship between a lamination $\mathcal{L}$ on a compact surface M and an order preserving homeomorphism f of S^1 implies that the Euler characteristic of M is equal to zero. In this paper we propose to extend the above construction to more general hyperbolic surfaces.

Let M be a closed compact hyperbolic surface, and let $\mathcal{L} \subset M$ be a minimal geodesic lamination having no closed and no isolated leaves. Let $\mathcal{T}_{\mathcal{L}}$ be the set of all compact one-manifolds imbedded in M which meet the lamination $\mathcal{L}$ transversely and whose boundary, if nonempty, lies in the complement of $\mathcal{L}$. For each C in $\mathcal{T}_{\mathcal{L}}$, the intersection $\Delta = C \cap \mathcal{L}$ is topologically a Cantor set, i.e., a nonempty perfect, totally disconnected set. We associate to each $C \in \mathcal{T}_{\mathcal{L}}$ the group $\mathcal{G}_{\mathcal{L}}(C)$ consisting of all homeomorphisms f of $\Delta = C \cap \mathcal{L}$ satisfying the following two conditions:

(1) For each $x \in \Delta$, x and $f(x)$ belong to the same leaf of $\mathcal{L}$.

(2) The map $\iota_f : \Delta \to \mathbf{N}$ defined by $\iota_f(x) = \mathrm{Card}\,([x, f(x)] \cap C) - 1$ is continuous, where $[x, f(x)]$ denotes the closed segment contained in the leaf of $\mathcal{L}$ joining x to $f(x)$.

The group structure is given by composition of mappings. We show that for each $C \in \mathcal{T}_{\mathcal{L}}$ the group $\mathcal{G}_{\mathcal{L}}(C)$ is nontrivial (cf. Theorem 1.13).

An element $f \in \mathcal{G}_{\mathcal{L}}(C)$ is called *irreducible* if a proper nonempty subset Δ' of Δ does not exist which is invariant under f and which is a finite union of closed intervals of Δ. We show that an element $f \in \mathcal{G}_{\mathcal{L}}(C)$ is irreducible if and only if it is minimal in the sense that the orbit of each point $x \in \Delta$ under the map f is dense in Δ (see Corollary 2.5). It will follow that if $\mathcal{L}$ is orientable then the first return map on Δ (with respect to the orientation on $\mathcal{L}$) defines an irreducible element of $\mathcal{G}_{\mathcal{L}}(C)$ (see Theorem 2.7). We establish a connection between irreducible elements of $\mathcal{G}_{\mathcal{L}}(C)$ and the existence of transverse measures on $\mathcal{L}$. Each irreducible element $f \in \mathcal{G}_{\mathcal{L}}(C)$ which carries discrete dynamics on $C \cap \mathcal{L}$ (in the sense of [**5**]) determines a transverse measure μ on $\mathcal{L}$.

1. Minimal laminations and homeomorphisms of the Cantor set. Let M be a closed compact hyperbolic surface (without boundary). We identify the universal cover $\tilde{M}$ of M with the Poincare disk $\mathbf{H}^2$. By a *geodesic* in M we mean the image in M under the covering map of a complete geodesic in $\mathbf{H}^2$. A geodesic in M is said to be *simple* if it has no transverse self intersections. A (*geodesic*) *lamination* in M is a nonempty closed subset $\mathcal{L}$ of M which is a disjoint union of simple geodesics. The geodesics contained in $\mathcal{L}$ are called the *leaves* of $\mathcal{L}$. A geodesic lamination $\mathcal{L}$ is said to be *minimal* if the closure of each leaf is all of $\mathcal{L}$. Clearly, such a lamination either consists of a single closed leaf or else contains no closed leaves (in which case $\mathcal{L}$ contains more than one leaf). A lamination is said to be *perfect* if it contains no isolated leaves. It is well known that for most surfaces such laminations exist, see Corollary 4.7.2 in [**1**]. In fact, there exist minimal perfect laminations $\mathcal{L}$ with the property that each leaf and each complementary region of $\mathcal{L}$ is simply connected. (See [**6**] and [**7**].) Such laminations fill up the surface in the following sense.

Proposition 1.1. *Let $\mathcal{L}$ be a geodesic lamination on M whose complementary regions are all simply connected. Then $\mathcal{L}$ is minimal.*

Proof. As each component of $M - \mathcal{L}$ is simple connected, it follows that $\mathcal{L}$ has no closed leaves. Thus, $\mathcal{L}$ is a disjoint union of closed isolated minimal sublaminations each of which consists of more than one leaf.

Let $\mathcal{L}_0$ be one such sublamination. We will show that $\mathcal{L} = \mathcal{L}_0$. Let N be a small ε-neighborhood of $\mathcal{L}_0$ disjoint from all other sublaminations and whose boundary is a union of simple closed curves. (See [**7**] for example.) By hypothesis each boundary component of N is trivial in the surface M. It follows that $\mathcal{L}$ contains no other sublaminations other than $\mathcal{L}_0$ for all other sublaminations would have to be contained in a union of disks, which is impossible. □

In what follows, $\mathcal{L}$ denotes a minimal geodesic perfect lamination in

M having no closed leaves.

Lemma 1.2. *Let S^1_∞ denote the boundary of the Poincare disk $\mathbf{H}^2$. Then no point on S^1_∞ is an endpoint of infinitely many leaves of $\tilde{\mathcal{L}} \subset \mathbf{H}^2$.*

Proof. Actually the result of the lemma holds for arbitrary geodesic laminations $\mathcal{L} \subset M$ having no closed leaves (cf. [**1**, Lemma 4.4]). ◻

In fact, if a point $x \in S^1_\infty$ is an endpoint of a leaf $\tilde{\lambda}$ of $\tilde{\mathcal{L}}$, then it is either an endpoint of exactly one leaf or of exactly two leaves depending on whether λ is a regular leaf or a boundary leaf of $\mathcal{L}$.

Let $\mathcal{T}_{\mathcal{L}}$ be the set of all compact one-manifolds imbedded in M which meet $\mathcal{L}$ transversely and whose boundary if nonempty lies in the complement of $\mathcal{L}$. Let C be an element of $\mathcal{T}_{\mathcal{L}}$. Put $\Delta = C \cap \mathcal{L}$. The set Δ is topologically a Cantor set, i.e., a nonempty perfect, totally disconnected set. A point x in Δ is called a *boundary point* if it is isolated from one side. Otherwise, x is called a *regular point*. Both sets of points are dense in Δ. A subset J of Δ will be called a *closed interval* if it is the intersection of Δ with a closed interval I in C. The interior of J, denoted J°, is then $I^\circ \cap \Delta$.

Lemma 1.3. *Let I be a closed interval contained in C which meets the lamination $\mathcal{L}$ and whose endpoints lie in the complement of $\mathcal{L}$. Then there are only finitely many closed intervals $I' \subset C$ which are isotopic* (*relative to $\mathcal{T}_{\mathcal{L}}$*) *to the interval I.*

Proof. Let $\tilde{I}$ be a lift of I in $\mathbf{H}^2$. Since I meets the lamination $\mathcal{L}$, $\tilde{I}$ must meet infinitely many geodesics of $\tilde{\mathcal{L}}$ in $\mathbf{H}^2$. Let $\tilde{\gamma}$ denote one such geodesic, and let a and b denote the two endpoints of $\tilde{\gamma}$ on the circle at infinity. Now suppose to the contrary that there are infinitely many intervals I' isotopic to I. Then one can move $\tilde{I}$ indefinitely along $\tilde{\gamma}$ in at least one direction keeping the endpoints at all times in the complement of $\tilde{L}$. Since the Euclidean metric of $\tilde{I}$ must tend to zero as we move $\tilde{I}$ towards the circle at infinity, it follows that either a or b must be an endpoint of infinitely many leaves of $\tilde{\mathcal{L}}$. This contradicts the result of Lemma 1.2. ◻

In what follows, let $\Delta' \subset \Delta$ be a finite union of closed sub-intervals of Δ.

Definition 1.4. Let $f : \Delta' \to \Delta$ be a continuous function. We say that f is weakly supported by $\mathcal{L}$ if for each point $x \in \Delta'$, the points x and $f(x)$ lie on the same leaf of $\mathcal{L}$.

For example, given a transverse orientation on the curve C, the first return map to C is weakly supported by $\mathcal{L}$. More generally, in the same way any continuous vector field parallel to $\mathcal{L}$ on Δ determines a map f which is weakly supported by $\mathcal{L}$. In order to obtain the converse, we need to impose an additional condition on the map f.

Let $f : \Delta' \to \Delta$ be weakly supported by $\mathcal{L}$. Associated to f is a map $\iota_f : \Delta' \to \mathbf{N}$ defined as follows: for $x \in \Delta'$,

$$\iota_f(x) = \mathrm{Card}\,([x, f(x)] \cap C) - 1$$

where $[x, f(x)]$ denotes the closed segment contained in the leaf of $\mathcal{L}$ joining x to $f(x)$. Thus, $\iota_f(x) = 0$ if and only if x is a fixed point of f.

Definition 1.5. A map $f : \Delta' \to \Delta$ is said to be supported by $\mathcal{L}$ if it is weakly supported by $\mathcal{L}$ and if the associated map $\iota_f : \Delta' \to \mathbf{N}$ is also continuous.

Lemma 1.6. *Let $f : \Delta' \to \Delta$ be supported by $\mathcal{L}$. For each point $x \in \Delta'$, let $\nu(x)$ be the unit tangent vector to $\mathcal{L}$ at x in the direction from x to $f(x)$. Then ν defines a continuous vector field on Δ'. Conversely, let ω be a continuous vector field on Δ' and let $\iota : \Delta' \to \mathbf{N}$ be any continuous function. Then the pair (ω, ι) determines a map f on Δ' which is supported by $\mathcal{L}$ with $\iota_f = \iota$.*

Proof. The first assertion follows immediately from the continuity of the map f together with the fact that the lamination $\mathcal{L}$ has no closed leaves. As for the second assertion, we define $f : \Delta' \to \Delta$ as follows: for $x \in \Delta'$, if x lies on some leaf $\lambda \subset \mathcal{L}$, we move along λ in the direction $\omega(x)$ until we cross the curve C $\iota(x)$ times and take $f(x)$ to be the stopping point. The continuity of ω and ι on Δ' together ensure the continuity of f. □

Remark 1.7. The continuity of the map ι_f does not in general follow from the continuity of f.

Proof. To see this we construct an example of a map f on Δ which is weakly supported by $\mathcal{L}$, and whose associate map ι_f is discontinuous. Let x_0 be a regular point of Δ, and let γ_0 be the leaf of $\mathcal{L}$ through the point x_0. Choose a transverse orientation ν of C and let y_0 denote the point of Δ obtained by starting at x_0 and moving along γ_0 in the direction of $\nu(x_0)$ until we meet C again for the first time. Now let $\{I_k\}_{k\geq 0}$ be a sequence of nonempty disjoint closed intervals contained in Δ converging to x_0 whose endpoints consist of boundary points of Δ. In addition, if $I_k = [a_k, b_k]$, we require that the point a_k be isolated from the left while the point b_k is isolated from the right. Similarly choose a sequence of nonempty disjoint closed intervals $\{J_k\}_{k\geq 0}$ contained in Δ converging to y_0 whose endpoints consist of boundary points of Δ.

For each $k = 0, 1, 2, \ldots,$ there exists a continuous function f_k on I_k having the following properties:

(1) $f_k : I_k \to J_k$,

(2) for each $x \in I_k$, we have that x and $f_k(x)$ lie on the same leaf of $\mathcal{L}$,

(3) the map $\iota_{f_k} : I_k \to \mathbf{N}$ is continuous,

(4) for each point $z \in I_k$, we have $\iota_{f_k}(z) > \text{maximum}\,\{\iota_{f_{k-1}}(x) \mid x \in I_{k-1}\}$.

In fact, for each $k \geq 0$, and for each point $x \in I_k$, the point x lies on some leaf γ of $\mathcal{L}$ which meets the interior of J_k at infinitely many points. Let $y \in \gamma \cap J_k^\circ$, and set $n = \text{Card}\,([x, y] \cap C) - 1$. For each $x' \in I_k$ sufficiently close to x, if x' is contained in a leaf $\gamma' \subset \mathcal{L}$, then traveling along γ' parallel to $[x, y]$, the nth crossing with C will occur in J_k°.

We next define f on Δ as follows: if $x \in I_k$ for some k, we set $f(x)$ equal to $f_k(x)$, while if $x \in \Delta - \cup_{k\geq 0} I_k$, then $f(x)$ is the first return map to C along the leaf γ through x in the direction $\nu(x)$. Then f is weakly supported by $\mathcal{L}$, in fact, since the intervals J_k converge to $y_0 = f(x_0)$, it follows that f is continuous at x_0. On the other hand, ι_f is unbounded in any neighborhood of the point x_0. □

Lemma 1.8. *Let $f : \Delta' \to \Delta$ be a map supported by $\mathcal{L}$. There exists finite decomposition $\Delta' = J_1 \cup \cdots \cup J_n$ into pairwise disjoint closed intervals such that, for each $1 \leq i \leq n$, ι_f is constant on J_i, the restriction of f to J_i is a homeomorphism of J_i onto $f(J_i)$ and $f(J_i)$ is a closed interval isotopic to J_i. Moreover, if f has no fixed points, then the intervals J_i can be chosen so that $f(J_i) \cap J_i = \varnothing$.*

Proof. For each regular point $x \in \Delta'$, there exists a neighborhood $U \subset \Delta$ of x on which ι_f is constant, and on which the map f is determined by a transverse orientation of U together with the natural number $\iota_f(x)$ (see Lemma 1.6). By choosing U sufficiently small, we can ensure that $f|_U$ maps U homeomorphically onto $f(U)$ and that, for each closed interval $I \subset U$, I is isotopic to $f(I)$. Moreover, if f is fixed point free, U can be chosen so that U and $f(U)$ are disjoint. The result of the lemma now follows from the compactness of Δ'. ◻

Lemma 1.9. *Let $f : \Delta' \to \Delta$ be a map supported by $\mathcal{L}$. Let $X = f(\Delta') \cap (\Delta - \Delta')$. Then there exists a map $f' : \Delta' \cup X \to \Delta$ with the following properties*:

(1) *f' is supported by $\mathcal{L}$,*

(2) *f' is an extension of f,*

(3) *$f'(x) \subset \Delta' - (f(\Delta') \cap \Delta')$.*

Thus, if X contains a nondegenerate closed interval, then so does $\Delta' - (f(\Delta') \cap \Delta')$.

Proof. In case the set X is empty there is nothing to show. Otherwise, if X is nonempty, it follows from Lemma 1.8 that X contains a nondegenerate closed interval. Let $\Delta' = J_1' \cup \cdots \cup J_k'$ be a partition of Δ' given by Lemma 1.8. Let

$$P = \cup\{P(e) \mid e \text{ is an endpoint of some } J_i'\}$$

where

$$P(e) = \{f^l(e) \mid l \in \mathbf{Z}^+\} \cap X.$$

The cardinality of $P(e)$ is at most one for each endpoint e. The finite set P determines a partition $X = X_1 \cup \cdots \cup X_r$ into disjoint closed intervals. By construction, the interior of each interval X_j is disjoint from the images under f of all the endpoints of the intervals J'_i. Thus the preimage $f^{-1}(X_j)$ is a closed interval contained in one of the intervals J'_i. A similar argument shows that the interval $f^{-1}(X_j)$ is either contained in $f(\Delta')$ or disjoint from it. In case it is contained in $f(\Delta')$, then by a similar argument we obtain that the interval $f^{-2}(X_j)$ is contained in one of the intervals J'_i and is again either contained in $f(\Delta')$ or disjoint from it.

Thus, for each $1 \leq j \leq r$, there exists a smallest positive integer t_j such that $f^{-t_j}(X_j) \subset \Delta' - (f(\Delta') \cap \Delta')$, for otherwise we would obtain an infinite sequence of disjoint closed intervals

$$X_j, f^{-1}(X_j), f^{-2}(X_j), \ldots$$

each of which is isotopic to the interval X_j contrary to the result of Lemma 1.3. We define the map $f' : \Delta' \cup X \to \Delta$ by $f'|_{\Delta'} = f$ and $f'|_{X_j} = f^{-t_j}(X_j) \subset \Delta' - (f(\Delta') \cap \Delta')$ for each $1 \leq j \leq r$. □

The following two corollaries are immediate consequences of Lemma 1.9.

Corollary 1.10. *Let $f : \Delta' \to \Delta$ be supported by $\mathcal{L}$. Then Δ' cannot be a proper subset of $f(\Delta')$.*

Proof. Again, by Lemma 1.8, if $f(\Delta') - \Delta'$ were nonempty, it would contain a nondegenerate closed interval. The assertion now follows from the last statement of Lemma 1.9. □

Corollary 1.11. *Let $f : \Delta' \to \Delta$ be a map supported by $\mathcal{L}$. If f is one-to-one, then $f(\Delta')$ cannot be a proper subset of Δ'. In particular, if $\Delta' = \Delta$, then f is a homeomorphism of Δ onto itself.*

Proof. Let $\Delta_0 = f(\Delta')$. Suppose to the contrary that Δ_0 is a proper subset of Δ'. Define $g : \Delta_0 \to \Delta$ by $g(x) = f^{-1}(x)$ for all $x \in \Delta_0$. Then g is a map supported by $\mathcal{L}$ and $g(\Delta_0) = \Delta'$ properly contains Δ_0 contradicting Corollary 1.10. □

Definition 1.12. Let $\mathcal{G}_{\mathcal{L}}(C)$ denote the collection of all homeomorphisms $f : \Delta \to \Delta$ which are supported by $\mathcal{L}$. Then $\mathcal{G}_{\mathcal{L}}(C)$ is a group under composition of maps.

Theorem 1.13. *Let M be a closed compact hyperbolic surface without boundary, and let $\mathcal{L} \subset M$ be a minimal lamination having no closed leaves. Let C be a compact one-manifold in M which meets the lamination $\mathcal{L}$ transversely and whose boundary if nonempty lies in the complement of $\mathcal{L}$. Then the group $\mathcal{G}_{\mathcal{L}}(C)$ defined above is nontrivial. In fact, there exists an element $F \in \mathcal{G}_{\mathcal{L}}(C)$ which is fixed point free, i.e., $F(x) \neq x$ for each $x \in C \cap \mathcal{L}$.*

Proof. Let $\Delta = C \cap \mathcal{L}$. Let ν be any transverse orientation of C. Then the first return map determines a map $f : \Delta \to \Delta$ which is supported by $\mathcal{L}$ (cf. Lemma 1.6). Let $\Delta = J_1 \cup J_2 \cup \cdots \cup J_n$ be the decomposition given by Lemma 1.8. Since $\mathcal{L}$ has no closed leaves, it follows that f is fixed point free, and therefore each J_i can be taken so that $f(J_i) \cap J_i = \varnothing$ (cf. Lemma 1.8). For each $1 \leq m \leq n$, set

$$\Delta_m = J_1 \cup \cdots \cup J_m.$$

We show by induction that for each m there exists a fixed point free, one-to-one map

$$F_m : \Delta_m \to \Delta$$

which is supported by $\mathcal{L}$. However, F_{m+1} will not necessarily be an extension of F_m. It follows then by Corollary 1.11 that the map $F = F_n : \Delta = \Delta_n \to \Delta$ is a nontrivial homeomorphism of Δ onto itself.

For $m = 1$ we take $F_1 = f|_{J_1}$. Next suppose that F_m is defined on Δ_m having the above properties. We show how to define F_{m+1} on $\Delta_{m+1} = \Delta_m \cup J_{m+1}$ having the required properties. Let $X = F_m(\Delta_m) \cap J_{m+1}$ and $Y = J_{m+1} - X$. By Lemma 1.10, there exists a map F'_{m+1} on $\Delta_m \cup X$ with the following properties:

(1) F'_{m+1} is supported by $\mathcal{L}$

(2) F'_{m+1} is an extension of F_m

(3) $F'_{m+1}(X) \subset \Delta_m - (F_m(\Delta_m) \cap \Delta_m)$.

We are now ready to define the map F_{m+1} on all of Δ_{m+1}. First, set

$$Z = F'^{-1}_{m+1}(f(Y)) \cap F'_{m+1}(\Delta_m \cup X),$$

and

$$W = \Delta_m \cup X - Z.$$

Set

$$F_{m+1}|_W = F'_{m+1}|_W, \tag{1.1}$$

$$F_{m+1}|_Z = f^{-1} \circ F'_{m+1}|_Z, \tag{1.2}$$

and

$$F_{m+1}|_Y = f|_Y. \tag{1.3}$$

Note that F_{m+1} is not necessarily an extension of F_m; in fact, if for some $x \in \Delta_m$, $F_m(x)$ should equal to $f(y)$ for some point $y \in Y$, then $F_{m+1}(x) = y \neq F_m(x)$. We further note that $f(Y) \cap Y = \varnothing$ since $Y \subset J_{m+1}$.

It follows from (1.1), (1.2) and (1.3) that $F_{m+1} : \Delta_m \to \Delta$ defines a one-to-one map which is supported by $\mathcal{L}$. Moreover, F_{m+1} is fixed point free; in fact, if $x \in \Delta_m$, then F_{m+1} maps x either to $F_m(x)$ or to a point in Y. On the other hand, points in J_{m+1} are mapped either to Δ_m or to $F(Y)$, both of which are disjoint from J_{m+1}.

Let $F = F_n : \Delta = \Delta_n \to \Delta$. By Corollary 1.11, it follows that F is the desired nontrivial element of $\mathcal{G}_{\mathcal{L}}(C)$. □

We note that in the proof of Theorem 1.13, for each $1 \leq m \leq n$, the map F_m is "locally" an iterate of the map f, that is, for each $x \in \Delta_m$, there exist a neighborhood U of x and a nonzero integer k so that $F_m|_U = f^k|_U$.

Definition 1.14. Let $f : \Delta \to \Delta$ be a map supported by $\mathcal{L}$. Let $\Delta' \subset \Delta$ be a finite union of closed intervals of Δ. A map $g : \Delta' \to \Delta'$ supported by $\mathcal{L}$ is said to be generated by f if there exists a finite decomposition

$$\Delta' = I_1 \cup I_2 \cup \cdots \cup I_r$$

by pairwise disjoint intervals, and nonzero integers $n_1, n_2, \dots, n_r$ such that, for each $1 \le t \le r$, we have $g|_{I_t} = f^{n_t}|_{I_t}$.

As a consequence of the proof of Theorem 1.13, we have

Corollary 1.15. *Every map $f : \Delta \to \Delta$ supported by $\mathcal{L}$ generates a homeomorphism $F : \Delta \to \Delta$ supported by $\mathcal{L}$. Moreover, if f is fixed point free, then the homeomorphism F can also be taken to be fixed point free.*

By Lemma 1.8, the intervals $I_1, I_2, \dots, I_r$ occurring in Definition 1.14 can be chosen so that for each $1 \le t \le r$, $g(I_t)$ is a closed interval isotopic to I_t. Therefore, Theorem 1.15 asserts that, given any map $f : \Delta \to \Delta$ supported by $\mathcal{L}$, there exist a decomposition

$$\Delta = I_1 \cup I_2 \cup \cdots \cup I_r$$

and nonzero integers $n_1, n_2, \dots, n_r$ so that the set $\{f^{n_1}(I_1), f^{n_2}(I_2), \dots, f^{n_r}(I_r)\}$ also constitutes a decomposition of Δ by pairwise disjoint closed intervals. Moreover, if f is fixed point free, then for each $1 \le t \le r$, the interval I_t can be chosen so that $f^{n_t}(I_t) \cap I_t = \varnothing$.

2. Irreducible elements. Let M be a compact hyperbolic surface (without boundary) and $\mathcal{L}$ a minimal geodesic perfect lamination on M having no closed leaves.

Definition 2.1. An element $f \in \mathcal{G}_{\mathcal{L}}(C)$ is said to be irreducible if there does not exist a proper nonempty subset Δ' of Δ which is invariant under f and which is a finite union of closed intervals of Δ.

We shall see later that, if $\mathcal{L}$ is orientable, that is, if it admits a nonvanishing continuous vector field ν, then the first return map with respect to ν defines an irreducible element of $\mathcal{G}_{\mathcal{L}}(C)$.

Proposition 2.2. *Let $f \in \mathcal{G}_{\mathcal{L}}(C)$. For each closed interval $I \subset \Delta$, there exist a decomposition*

$$I = I_1 \cup I_2 \cup \cdots \cup I_k$$

and positive integers $m(j)$ *for* $1 \leq j \leq k$ *such that*

$$I_1, f(I_1), \dots , f^{m(1)-1}(I_1), \dots , I_k, f(I_k), \dots , f^{m(k)-1}(I_k)$$

is a sequence of pairwise disjoint closed intervals whose union

$$\Delta(I) = \bigcup_{j=1}^{k} \bigcup_{i=0}^{m(j)-1} f^i(I_j)$$

is invariant under f.

Proof. Let $I = [x, y] \subset \Delta$. Let

$$\Delta = J_1 \cup J_2 \cup \cdots \cup J_n$$

be the decomposition of Δ given by Lemma 1.8. Let

$$S = \{e \mid e = x, y \text{ or is an endpoint of one of the intervals } J_i\}.$$

For each point $e \in S$, let $\mathcal{O}^-(e) = \{f^l(e) \mid l \leq 0\}$. In case $\mathcal{O}^-(e) \cap I^\circ \neq \varnothing$, we set $n(e)$ to be the largest nonpositive integer such that $f^{n(e)}(e) \in I^\circ$. Let

$$P = \{f^{n(e)}(e) \mid e \in S\}.$$

Then the finite set P determines a partition

$$I = I_1 \cup I_2 \cup \cdots \cup I_k$$

into pairwise disjoint closed intervals.

Let $1 \leq j \leq k$. Then there exists a positive integer m such that $f^m(I_j)^\circ \cap I \neq \varnothing$. In fact, there exists a smallest positive integer r such that $f^r(I_j)$ contains an endpoint e of one of the intervals J_i in its interior, for otherwise by Lemma 1.8,

$$I_j, f(I_j), f^2(I_j), f^3(I_j), \dots$$

would be an infinite sequence of pairwise disjoint closed intervals each isotopic to I_j, contrary to the result of Lemma 1.3. Now, since

$f^{-r}(e) \in I_j^\circ$, it follows from the definition of the intervals $I_1, I_2, \ldots, I_k$ that there exists a nonnegative integer $s < r$ so that $f^{-s}(e) \in I^\circ$. Thus, $f^{-s} \circ f^r(I_j)^\circ \cap I \neq \varnothing$.

For each $1 \leq j \leq k$, let $m(j)$ denote the smallest positive integer less than or equal to r such that $f^{m(j)}(I_j)^\circ \cap I \neq \varnothing$. It follows from the minimality if $m(j)$ that

$$I_j, f(I_j), f^2(I_j), \ldots, f^{m(j)-1}(I_j)$$

is a sequence of pairwise disjoint closed intervals, each isotopic to I_j, and that $f^{m(j)}(I_j)$ is a closed interval isotopic to I_j and contained in I. In fact, the interval $f^{m(j)}(I_j)$ meets I and does not contain the points x or y in its interior. It follows that

$$I_1, f(I_1), \ldots, f^{m(1)-1}(I_1), \ldots, I_k, f(I_k), \ldots, f^{m(k)-1}(I_k)$$

is a sequence of pairwise disjoint closed intervals whose union

$$\Delta(I) = \bigcup_{j=1}^{k} \bigcup_{i=0}^{m(j)-1} f^i(I_j)$$

is invariant under f. □

The following is an immediate consequence of Proposition 2.2.

Corollary 2.3. *Let I be a closed interval contained in Δ. If $f \in \mathcal{G}_{\mathcal{L}}(C)$ is irreducible, then $\Delta(I) = \Delta$.*

Definition 2.4. An element $f \in \mathcal{G}_{\mathcal{L}}(C)$ is said to be minimal if the set

$$\mathcal{O}(x) = \{f^n(x) \mid n \in \mathbf{Z}\}$$

is dense in Δ for each point $x \in \Delta$.

Corollary 2.5. *Let $f \in \mathcal{G}_{\mathcal{L}}(C)$. The following are equivalent:*

(1) *f is irreducible,*

(2) *f is minimal,*

(3) *There exists a point $x \in \Delta$ for which the set $\mathcal{O}(x)$ is dense in Δ.*

Proof. First suppose that f is irreducible. Then, for each closed interval $I \subset \Delta$, we have that $\Delta(I) = \Delta$. This implies that, for each point $x \in \Delta$, there exists an integer m, dependent on I, such that $f^m(x) \in I^\circ$. This implies that f is minimal.

Clearly (2) implies (3). Finally, to see that (3) implies (1), suppose that for some point $x \in \Delta$ the set $\mathcal{O}(x)$ is dense in Δ. We suppose to the contrary that f is not irreducible. Then there exists a nonempty proper subset Δ' of Δ which is invariant under f and which is a finite union of closed intervals. Since $\mathcal{O}(x)$ meets Δ' and Δ' is invariant under f, it follows that $\mathcal{O}(x) \subset \Delta'$. This contradicts our assumption that $\mathcal{O}(x)$ is dense in Δ. □

Corollary 2.6. *Let $f : \Delta \to \Delta$ be an irreducible element of $\mathcal{G}_{\mathcal{L}}(C)$. Let I be any closed interval contained in C. Then f generates a fixed point free homeomorphism F in $\mathcal{G}_{\mathcal{L}}(I)$, i.e., setting $\Delta' = I \cap \mathcal{L}$, there exists a decomposition*

$$\Delta' = I_1 \cup I_2 \cup \cdots \cup I_r$$

by pairwise disjoint closed intervals and nonzero integers $n_1, n_2, \dots, n_r$ such that

$$\{f^{n_1}(I_1), f^{n_2}(I_2), \dots, f^{n_r}(I_r)\}$$

also constitutes a decomposition of Δ' by a pairwise disjoint closed interval such that for each $1 \le t \le r$ we have $f^{n_t}(I_t) \cap I_t = \varnothing$ and $f^{n_t}(I_t)$ is isotopic to I_t.

Proof. By Corollary 2.5, it follows that there exists a fixed point free map $g : \Delta' \to \Delta'$ such that for each point $x \in \Delta'$ there exist a neighborhood U of x and a nonzero integer k such that $g|_U = f^k|_U$. The rest now follows immediately from Corollary 1.15. □

Theorem 2.7. *Let M be a closed hyperbolic surface. Let $\mathcal{L}$ be a minimal geodesic lamination having no closed leaves. Let $C \in \mathcal{T}_{\mathcal{L}}$, and let $\Delta = C \cap \mathcal{L}$. Suppose that $\mathcal{L}$ is orientable, and let ν be a continuous nonvanishing vector field defined on $\mathcal{L}$. Then the map $f : \Delta \to \Delta$*

defined to be the first return map with respect to ν defines an irreducible element of the group $\mathcal{G}_{\mathcal{L}}(C)$.

Proof. First of all, by Lemma 1.6 it follows that the first return map f on Δ is a map supported by $\mathcal{L}$. We next claim that the map f is minimal, i.e., that the orbit of each point $x \in \Delta$ is dense in Δ. To see this we observe that since $\mathcal{L}$ contains no closed leaves, no point of Δ is periodic. This, together with the fact that $\mathcal{L}$ is minimal, implies that the map f is minimal.

All that remains is to show that f is a homeomorphism. It will then follow from Corollary 2.5 that the map f is also irreducible. We start by showing that f is injective. If f were not injective, there would exist distinct points x_1 and x_n in Δ such that $y = f(x_1) = f(x_2)$. But the $f(y) \in \{x_1, x_2\}$ which implies that y is a point of period two contradicting the minimality of f. To see that f is onto, let $z \in \Delta$. Then there exists a sequence $\{y_n\}_{n\geqq 0} \subset \{f^m(z)\}_{m\geqq 0}$ tending to the point z. For each n, let $x_n = f^{-1}(y_n)$, and let x be a limit point of the sequence $\{x_n\}$. Then, since Δ is compact, $x \in \Delta$. Moreover,

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} y_n = z.$$

This concludes the proof of Theorem 2.7. □

3. Invariant measures. Throughout this section, all measures are assumed to be probability measures. Let $f \in \mathcal{G}_{\mathcal{L}}(C)$ be an irreducible element. By Corollary 2.5, the map f is minimal and therefore any f-invariant measure μ_0 on Δ necessarily has full support on Δ (cf. [**3**]).

We saw in Corollary 2.6 that, for each closed interval $\Delta' \subset \Delta$, there exist a decomposition

$$\Delta' = I_1 \cup I_2 \cup \cdots \cup I_r$$

by pairwise disjoint closed intervals and nonzero integers $n_1, n_2, \ldots, n_r$ such that

$$\{f^{n_1}(I_1), f^{n_2}(I_2), \ldots, f^{n_r}(I_r)\}$$

also constitutes a decomposition by pairwise disjoint closed intervals with $f^{n_t}(I_t) \cap I_t = \varnothing$ for each $1 \leq t \leq r$.

Definition 3.1. Let μ_0 be an f-invariant measure on Δ. We say that the pair (f, μ_0) carries discrete dynamics on Δ if, given any two nondegenerate closed intervals I and J with $\mu_0(I) = \mu_0(J)$, there exist decompositions

$$I = I_1 \cup I_2 \cup \cdots \cup I_r$$

and

$$J = J_1 \cup J_2 \cup \cdots \cup J_r$$

and nonzero integers $n_1, n_2, \ldots, n_r$ such that for each $1 \leq t \leq r$ we have $f^{n_t}(I_t) = J_t$.

In the above definition it is understood that the $\{I_t\}$ consists of mutually pairwise disjoint closed intervals. Similarly for the collection $\{J_t\}$.

Lemma 3.2. *Let μ_0 be an f-invariant measure on Δ such that the pair (f, μ_0) carries discrete dynamics on Δ. Then, for any f-invariant measure ν_0 on Δ, and for all closed intervals I and J contained in Δ, we have that $\mu_0(I) = \mu_0(J)$ if and only if $\nu_0(I) = \nu_0(J)$. In particular, the pair (f, ν_0) also carries discrete dynamics on Δ.*

Proof. Let I and J be two closed intervals of equal μ_0 measure, and let

$$I = I_1 \cup I_2 \cup \cdots \cup I_r$$

and

$$J = J_1 \cup J_2 \cup \cdots \cup J_r$$

be the decompositions given by Definition 3.1. Then

$$\nu_0(J) = \sum_{i=1}^{r} \nu_0(J_i) = \sum_{i=1}^{r} \nu_0(f^{n_i}(I_i)) = \sum_{i=1}^{r} \nu_0(I_i) = \nu_0(I).$$

Conversely, suppose that I and J are closed intervals of equal ν_0 measure, and suppose to the contrary that $\mu_0(I) \neq \mu_0(J)$. Without loss of generality, we can assume that $\mu_0(J) > \mu_0(I) > 0$. Recall that μ_0 has full support on Δ. Let $J' \subset J$ be a closed interval with $\mu_0(J') = \mu_0(I)$. Then it follows from the above calculation that $\nu_0(J') = \nu_0(I)$. But this is a contradiction since $\nu_0(J') < \nu_0(J) = \nu_0(I)$. □

Any f-invariant measure μ_0 on Δ defines a measure on the curve C simply by intersecting any Borel subset of C with the lamination $\mathcal{L}$ and measuring this intersection with μ_0. By abuse of notation we shall denote the resulting measure on C by μ_0. We now show that if (f, μ_0) carries discrete dynamics on Δ, then the resulting measure μ_0 on C is a transverse measure, i.e., is invariant under isotopy.

Theorem 3.3. *Let f be an irreducible element of $\mathcal{G}_\mathcal{L}(C)$, and let μ_0 be an f-invariant measure such that the pair (f, μ_0) carries discrete dynamics on $\Delta = C \cap \mathcal{L}$. Let $\mathcal{T}_\mathcal{L}(C) = \{X \in \mathcal{T}_\mathcal{L} \mid X \subset C\}$. Then the measure μ_0 on C defines a transverse measure on C, i.e.,*

(1) *μ_0 has full support on $\mathcal{L}$.*

(2) *If $C' \in \mathcal{T}_\mathcal{L}(C)$ and $C' \cup_{i\geq 0} C_i$ where $C_i \in \mathcal{T}_\mathcal{L}(C)$ and $C_i \cap C_j = \partial C_i \cap \partial C_j$ for all $i \neq j$, then $\mu_0(C') = \sum_{i\geq 0} \mu_0(C_i)$.*

(3) *If C_1 and C_2 are elements of $\mathcal{T}_\mathcal{L}(C)$ which are isotopic through elements of $\mathcal{T}_\mathcal{L}$, then $\mu_0(C_1) = \mu_0(C_2)$.*

Proof. Conditions (1) and (2) are immediate since μ_0 is a measure with full support on $\Delta = C \cap \mathcal{L}$. Thus, it suffices to verify condition (3). Also, in view of (2), without loss of generality, we can take C_1 and C_2 to be intervals contained in C. Let $I = C_1 \cap \mathcal{L}$ and $J = C_2 \cap \mathcal{L}$. We must show that $\mu_0(I) = \mu_0(J)$. Suppose to the contrary. Without loss of generality we can assume that $\mu_0(I) < \mu_0(J)$. Let J' be a proper subinterval of J with $\mu_0(J') = \mu_0(I)$. Let I' be the corresponding subinterval I which is isotopic to J'. Since the pair (f, μ_0) carries discrete dynamics on Δ, it follows that there exists a map $g : I \to J'$ which is supported by $\mathcal{L}$ and which is both injective and surjective. Composing the map g with the isotopy from J' to I', we obtain a new map g' also supported by $\mathcal{L}$ which maps I one-to-one onto I'. But this now contradicts the result of Corollary 1.11 since the interval I' is a proper subinterval of I. This concludes the proof of Theorem 3.3. □

Let f be an irreducible element of $\mathcal{G}_{\mathcal{L}}(C)$, and let μ_0 be an f-invariant measure. Then the essence of Theorem 3.3 is that if the pair (f, μ_0) carries discrete dynamics on Δ, then the measure μ_0 descends to a measure on the family of isotopy classes of $\mathcal{T}_{\mathcal{L}}(C)$. Thus, the transverse measure μ_0 on C extends to a transverse measure μ_D for each $D \in \mathcal{T}_{\mathcal{L}}$. In fact, each such D admits a decomposition

$$D = D_1 \cup D_2 \cup \cdots \cup D_n$$

where each D_j is a closed interval isotopic to a closed interval I_j contained in C. Hence we set

$$\mu_D(D) = \sum_{j=1}^{n} \mu_0(I_j).$$

In view of Theorem 3.3, this is well defined and depends only on the isotopy class of D. Thus, the measure μ_0 determines a transverse measure μ on the lamination $\mathcal{L}$, i.e., it gives $\mathcal{L}$ the structure of a measured lamination.

Conversely, if we are given a transverse measure μ on $\mathcal{L}$ then the restriction of μ on C, denoted μ_C, is necessarily an f-invariant measure on C with full support on $\mathcal{L}$ (cf. Lemma 1.8).

In summary:

Theorem 3.4. *Let f be an irreducible element of $\mathcal{G}_{\mathcal{L}}(C)$, and let μ_0 be an f-invariant measure such that the pair (f, μ_0) carries discrete dynamics on $\Delta = C \cap \mathcal{L}$. Then μ_0 defines a transverse measure μ on $\mathcal{L}$.*

Acknowledgments. I wish to thank Mariusz Urbański with whom I discussed the content of this paper in great depth, and who provided me with many useful ideas. I also wish to thank the referees for their time and effort and for the many valuable comments and suggestions on improving the original version of the paper.

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