ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 2, Summer 2001

CONSTRUCTION OF WEIGHT TWO EIGENFORMS VIA THE GENERALIZED DEDEKIND ETA FUNCTION

DONALD L. VESTAL, JR.

ABSTRACT. The generalized Dedekind eta function has been used in various ways to construct modular functions of different weights. In this paper we give a way to construct modular forms of weight two for the modular groups $\Gamma_0(N)$ which, in some cases, turn out to be Hecke eigenforms (though never cusp forms).

1. The generalized Dedekind eta function. Let $\mathfrak{h}$ denote the upper half plane (so $\mathfrak{h} = \{\tau \mid \operatorname{Im}\tau > 0\}$), and let $P_2(x) = \{x\}^2 - \{x\} + (1/6)$ denote the second Bernoulli polynomial, defined on the fractional part of x, $\{x\} = x - \lfloor x \rfloor$. For integers g and δ, with $\delta > 0$, we define the generalized Dedekind eta function as

$$(1) \quad \eta_{\delta,g}(\tau) = e^{\pi i \delta P_2(g/\delta)\tau} \prod_{\substack{m \equiv g \pmod{\delta} \\ m>0}} (1-q^m) \prod_{\substack{m \equiv -g \pmod{\delta} \\ m>0}} (1-q^m)$$

where $\tau \in \mathfrak{h}$ and $q = e^{2\pi i\tau}$. These functions are a variation of the eta functions defined by Schoeneberg in [**5**] and can be used to create modular functions in various ways (see [**4**] and [**6**]). For example, from [**6**], we have

Theorem. *Let N be a positive integer, and let*

$$f(\tau) = \prod_{\substack{\delta \mid N \\ 0 \le g < \delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau),$$

where $r_{\delta,g} \in Z$ and $r_{\delta,ag} = r_{\delta,g}$ for all a relatively prime to N. Set

Received by the editors on January 14, 2000, and in revised form on February 15, 2000.

$k = \sum_{\delta|N} r_{\delta,0}$. If

$$\sum_{\substack{\delta|N \\ 0\le g<\delta}} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} \equiv 0 \pmod 2,$$

$$\sum_{\substack{\delta|N \\ 0\le g<\delta}} \frac{N}{6\delta}\, r_{\delta,g} \equiv 0 \pmod 2,$$

and if k is an even integer, then f is a modular function of weight k on $\Gamma_0(N)$.

2. The functions $H_{\delta,g}(\tau)$. In this paper we will consider a class of modular forms, reminiscent of the Eisenstein series, of weight two, derived from the generalized Dedekind eta function.

First recall that, for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\delta)$ and $g \not\equiv 0 \pmod \delta$,

$$\eta_{\delta,g}(A\tau) = \nu_{\delta,g}(A)\eta_{\delta,ag}(\tau)$$

where

$$\nu_{\delta,g}(A) = \begin{cases} \exp\left(\pi i\left[\frac{a}{c}\,\delta P_2(g/\delta) + \frac{d}{c}\,\delta P_2\left(\frac{ag}{\delta}\right)\right.\right. & \\ \qquad\left.\left. -2\operatorname{sgn} c \cdot s(a, c/\delta; 0, g/\delta)\right]\right) & \text{if } c \neq 0 \\ \exp\left(\pi i \frac{b}{d}\,\delta P_2(g/\delta)\right) & \text{if } c = 0, \end{cases}$$

and $s(h,k;x,y)$ is the generalized Dedekind sum (see [**4**] and [**5**]). Let

$$H_{\delta,g}(\tau) = \frac{1}{2\pi i}\,\frac{\eta'_{\delta,g}(\tau)}{\eta_{\delta,g}(\tau)}. \tag{2}$$

Since $\eta_{\delta,g}(\tau)$ is holomorphic and nonzero on $\mathfrak{h}$, $H_{\delta,g}(\tau)$ is holomorphic on $\mathfrak{h}$. We now consider what happens at the cusps. The function $\eta_{\delta,g}(\tau)$ is meromorphic at any cusp γ of $\mathfrak{h}$ (see [**5**]), so

$$\eta_{\delta,g}(A\tau) = \sum_{n=M}^{\infty} a_n q_\delta^n,$$

where $M \in \mathbf{Z}$ and $q_\delta = e^{2\pi i\tau/\delta}$. Differentiating both sides of the above equation with respect to τ yields

$$(c\tau+d)^{-2}\,\eta'_{\delta,g}(A\tau) = \sum_{n=M}^{\infty} a_n\left(\frac{2\pi i}{\delta}\right) n q_\delta^n;$$

in particular, if $\eta_{\delta,g}(\tau)$ has a pole of order M at γ, then $\eta'_{\delta,g}(\tau)$ also has a pole of order M at γ. Similarly, if $\eta_{\delta,g}(\tau)$ has a zero of order M at γ, then $\eta'_{\delta,g}(\tau)$ also has a zero of order M at γ. Consequently, $H_{\delta,g}(\tau)$ will be holomorphic at γ.

Of special interest is the expansion of $H_{\delta,g}(\tau)$ at infinity. We find this by looking at the logarithmic derivative of the expansion of $\eta_{\delta,g}(\tau)$ at infinity: starting with (1) we have

(3)

$$\begin{aligned}\log\eta_{\delta,g}(\tau) &= \pi i\delta P_2(g/\delta)\tau + \sum_{\substack{m\equiv g \pmod{\delta}\\ m>0}} \log(1-q^m) + \sum_{\substack{m\equiv -g \pmod{\delta}\\ m>0}} \log(1-q^m)\\ &= \pi i\delta P_2(g/\delta)\tau - \sum_{\substack{m\equiv g \pmod{\delta}\\ m>0}} \sum_{n=1}^{\infty} \frac{q^{mn}}{n} - \sum_{\substack{m\equiv -g \pmod{\delta}\\ m>0}} \sum_{n=1}^{\infty} \frac{q^{mn}}{n}.\end{aligned}$$

Differentiating (3) with respect to τ yields

$$\begin{aligned}\frac{\eta'_{\delta,g}}{\eta_{\delta,g}}(\tau) &= \pi i\delta P_2\left(\frac{g}{\delta}\right) - \sum_{\substack{m\equiv g \pmod{\delta}\\ m>0}} \sum_{n=1}^{\infty} 2\pi i m q^{mn}\\ &\qquad - \sum_{\substack{m\equiv -g \pmod{\delta}\\ m>0}} \sum_{n=1}^{\infty} 2\pi i m q^{mn}\\ &= \pi i\delta P_2\left(\frac{g}{\delta}\right) - 2\pi i\sum_{n=1}^{\infty}\Bigg(\sum_{\substack{m\equiv g \pmod{\delta}\\ m>0}} m + \sum_{\substack{m\equiv -g \pmod{\delta}\\ m>0}} m\Bigg) q^{mn}\\ &= \pi i\delta P_2\left(\frac{g}{\delta}\right) - 2\pi i\sum_{N=1}^{\infty}\Bigg(\sum_{\substack{m\equiv g \pmod{\delta}\\ m>0}} m + \sum_{\substack{m\equiv -g \pmod{\delta}\\ m>n}} m\Bigg) q^{N}.\end{aligned}$$

Let

$$\sigma_{\delta,g}(N)=\sum_{\substack{d|N\\ d\equiv g \pmod{\delta}}} d+\sum_{\substack{d|N\\ d\equiv -g \pmod{\delta}}} d;$$

then the expansion of $H_{\delta,g}(\tau)$ at infinity can be written as

$$H_{\delta,g}(\tau)=\frac{1}{2\pi i}\frac{\eta'_{\delta,g}}{\eta_{\delta,g}}(\tau)=\frac{1}{2}\,\delta P_2(g/\delta)-\sum_{N=1}^{\infty}\sigma_{\delta,g}(N)q^N. \tag{4}$$

The coefficients of this expansion can be used to derive combinatorial results. For example, the fourth power of the classical theta function $\big(\theta(\tau)=\sum_{n\in\mathbf{Z}}q^{n^2}\big)$ can be written as $1+\sum_{N\geq 1}s_4(N)q^N$, where $s_4(N)$ denotes the number of ways of writing N as a sum of four squares. One can show that $\theta^4(\tau)=(1/3)H_{4,1}(\tau)+(2/3)H_{4,2}(\tau)$, which gives the formula for $s_4(N)$:

$$s_4(N)=8\sigma_{4,1}(N)+4\sigma_{4,2}(N).$$

Similarly, one can find a formula for the number of ways a positive integer N can be written as a sum of four triangular numbers:

$$t_4(N)=\sigma_{4,1}(2N+1)=\sigma(2N+1),$$

where $t_4(N)$ denotes the number of ways of writing N as a sum of four triangular numbers (see [**6**] for the details of these derivations).

We can use the transformation formula of $\eta_{\delta,g}(\tau)$ to find a transformation formula for $H_{\delta,g}(\tau)$: if $A\in\Gamma_0(\delta)$, then

$$\eta_{\delta,g}(A\tau)=\nu_{\delta,g}(A)\eta_{\delta,ag}(\tau);$$

this implies (after differentiating by τ) that

$$\eta'_{\delta,g}(A\tau)=\nu_{\delta,g}(A)(c\tau+d)^2\eta'_{\delta,ag}(\tau).$$

Therefore,

$$\begin{aligned}H_{\delta,g}(A\tau)&=\frac{1}{2\pi i}\,\frac{\eta'_{\delta,g}(A\tau)}{\eta_{\delta,g}(\tau)}\\&=(c\tau+d)^2\,\frac{1}{2\pi i}\,\frac{\eta'_{\delta,ag}}{\eta_{\delta,ag}}(\tau)\\&=(c\tau+d)^2\,H_{\delta,ag}(\tau).\end{aligned}$$

Suppose that $(\delta/g) = 2, 3, 4$ or 6. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\delta)$, then $(a, \delta) = 1$ and hence $ag \equiv \pm g \pmod{\delta}$. So

$$H_{\delta,g}(A\tau) = (c\tau + d)^2 \, H_{\delta,ag}(\tau) = (c\tau + d)^2 \, H_{\delta,g}(\tau),$$

which implies that $H_{\delta,g}(\tau)$ is a modular form of weight two on $\Gamma_0(\delta)$.

As an example, consider $H_{2,1}(\tau)$. This is a modular form of weight two on $\Gamma_0(2)$. Using the formulas of Shimura and Gunning (see [**6**]), we find that the space of modular forms of weight two on $\Gamma_0(2)$ has dimension one, so that, in fact, $H_{2,1}(\tau)$ is the modular form of weight two on $\Gamma_0(2)$. In particular, it is the eigenform (with respect to the Hecke transform) for $\Gamma_0(2)$. We will discuss eigenforms more in Section 4.

When (δ/g) is not 2, 3, 4, or 6, then the function $H_{\delta,g}(\tau)$ may not be a modular form on $\Gamma_0(\delta)$. For example, $H_{5,1}(\tau)$ is not a modular form for $\Gamma_0(5)$:

$$H_{5,1}\left(\frac{2\tau + 1}{5\tau + 3}\right) = (5\tau + 3)^2 \, H_{5,2}(\tau) \neq (5\tau + 3)^2 \, H_{5,1}(\tau).$$

However, note that for $A \in \Gamma_1(\delta)$, we always have $H_{\delta,g}(A\tau) = (c\tau + d)^2 H_{\delta,g}(\tau)$, since $a \equiv 1 \pmod{\delta}$; hence, $H_{\delta,g}(\tau)$ is always a modular form of weight two on $\Gamma_1(\delta)$.

3. Constructing modular forms. We now focus exclusively on $\Gamma_0(\delta)$. Let $M_k(\Gamma')$ denote the vector space of modular forms of weight k on Γ'. Based on the previous section, we have results such as

$$\begin{aligned} H_{2,1}(\tau) &= \frac{1}{2}\,(2)P_2\left(\frac{1}{2}\right) - \sum_{N=1}^{\infty} \sigma_{2,1}(N) q^N \\ &= -\,\frac{1}{12} - \sum_{N=1}^{\infty} \sigma_{2,1}(N) q^N \in M_2(\Gamma_0(2)), \\ H_{6,1}(\tau) &= \frac{1}{2}\,(6)P_2\left(\frac{1}{6}\right) - \sum_{N=1}^{\infty} \sigma_{6,1}(N) q^N \\ &= \frac{1}{12} - \sum_{N=1}^{\infty} \sigma_{6,1}(N) q^N \in M_2(\Gamma_0(6)), \end{aligned}$$

and

$$H_{20,5}(\tau) = \frac{1}{2}\,(20)P_2\left(\frac{5}{20}\right) - \sum_{N=1}^{\infty} \sigma_{20,5}(N)q^N$$
$$= -\,\frac{5}{24} - \sum_{N=1}^{\infty} \sigma_{20,5}(N)q^N \in M_2(\Gamma_0(20)).$$

Although $H_{5,1}(\tau)$ and $H_{5,2}(\tau)$ are not modular on $\Gamma_0(5)$, they can be used to construct modular forms for $\Gamma_0(5)$, in some cases with a character as a multiplier. For example, let $F(\tau) = H_{5,1}(\tau) + H_{5,2}(\tau) + H_{5,3}(\tau) + H_{5,4}(\tau)$. Then, for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(5)$, we have

$$\begin{aligned} F(A\tau) &= H_{5,1}(A\tau) + H_{5,2}(A\tau) + H_{5,3}(A\tau) + H_{5,4}(A\tau) \\ &= (c\tau + d)^2\big(H_{5,a}(\tau) + H_{5,2a}(\tau) + H_{5,3a}(\tau) + H_{5,4a}(\tau)\big) \\ &= (c\tau + d)^2 F(\tau), \end{aligned}$$

and thus $F(\tau) \in M_2(\Gamma_0(5))$. Similarly, suppose χ is the quadratic character defined by the Legendre symbol modulo 5: $\chi(a) = (a/5)$. Now let $G(\tau) = H_{5,1}(\tau) - H_{5,2}(\tau) - H_{5,3}(\tau) + H_{5,4}(\tau)$. Then

$$\begin{aligned} G(A\tau) &= H_{5,1}(A\tau) - H_{5,2}(A\tau) - H_{5,3}(A\tau) + H_{5,4}(A\tau) \\ &= (c\tau + d)^2\big(H_{5,a}(\tau) - H_{5,2a}(\tau) - H_{5,3a}(\tau) + H_{5,4a}(\tau)\big). \end{aligned}$$

If $a \equiv \pm 1 \pmod 5$, then $G(A\tau) = (c\tau + d)^2 G(\tau)$. If $a \equiv \pm 2 \pmod 5$, then $G(A\tau) = (c\tau + d)^2\big(H_{5,2}(\tau) - H_{5,4}(\tau) - H_{5,1}(\tau) + H_{5,3}(\tau)\big) = -(c\tau + d)^2 G(\tau)$. We summarize this by writing

$$G(A\tau) = \left(\frac{a}{5}\right)(c\tau + d)^2\, G(\tau).$$

Since $(a/5) = (d/5)$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(5)$, we have shown that $G(A\tau) = \chi(d)(c\tau + d)^2 G(\tau)$, and hence G is a modular form of weight two on $\Gamma_0(5)$ with χ as a multiplier.

We generalize the last two examples with the following.

Theorem 1. *Let χ be a Dirichlet character modulo N (N a positive integer), and set*

$$f(\tau) = \sum_{k=1}^{N} \chi(k)\, H_{N,k}(\tau).$$

Then f is a modular form of weight two on $\Gamma_0(N)$ with multiplier χ.

Proof. Each $H_{N,k}(\tau)$ is holomorphic at the cusps of $\Gamma_0(N)$, so we need only check the transformation formula. Taking $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, we find

$$\begin{aligned} f(A\tau) &= \sum_{k=1}^{N} \chi(k)\, H_{N,k}(A\tau) \\ &= \sum_{k=1}^{N} \chi(k)(c\tau+d)^2\, H_{N,ak}(\tau) \\ &= \bar{\chi}(a) \sum_{k=1}^{N} \chi(ak)(c\tau+d)^2\, H_{N,ak}(\tau) \\ &= \bar{\chi}(a)(c\tau+d)^2 \sum_{k=1}^{N} \chi(ak)\, H_{N,ak}(\tau) \\ &= \bar{\chi}(a)(c\tau+d)^2 f(\tau). \end{aligned}$$

Since $ad \equiv 1 \pmod N$, we can replace $\bar{\chi}(a)$ with $\chi(d)$. This gives

$$f(A\tau) = \chi(d)(c\tau+d)^2 f(\tau),$$

which implies that f is a modular form of weight 2 on $\Gamma_0(N)$ with multiplier χ. □

Note that in the case where $\chi(-1) = -1$, the function f is the zero function. If $N = 5$ and $\chi(n)$ is the trivial Dirichlet character modulo 5, then $f(\tau)$ is the function $F(\tau)$ defined above. Similarly, if $N = 5$ and $\chi(n) = (n/5)$, then $f(\tau) = G(\tau)$.

4. Eigenforms. According to Theorem 1, the functions $F(\tau)$ and $G(\tau)$ are modular forms of weight two with the corresponding characters as multipliers. In fact, they are eigenforms with respect to the Hecke operator. For example, consider $F(\tau)$. For any positive integer m, let $\lambda_m = \sum_{d|m, 5\nmid d} d = \sum_{d|m} \chi(d)d$, where χ denotes the trivial character modulo 5. If T_m denotes the Hecke operator, then $T_m(F) = \lambda_m F$. To show this, we need the following.

Lemma. *Let χ be a Dirichlet character. Then, for any positive integers m and n,*

$$\left(\sum_{d|m}\chi(d)d\right)\left(\sum_{d|n}\chi(d)d\right) = \sum_{d|\gcd(m,n)}\left(\chi(d)d\sum_{e|(mn/d^2)}\chi(e)e\right).$$

Proof. According to Theorem 1.12 of [**3**], the following are equivalent:

(1) The arithmetic function g is the convolution of two completely multiplicative functions.

(2) There is a completely multiplicative function B such that for all positive integers m and n,

$$g(m)g(n) = \sum_{d|\gcd(m,n)} B(d)g\left(\frac{mn}{d^2}\right);$$

in particular, the function B is determined by $B(p) = g(p)^2 - g(p^2)$ for any prime p.

We apply this result to the arithmetic function $g(n) = \sum_{d|n}\chi(d)d$. Since g is the convolution of $x(n)n$ and 1, both of which are completely multiplicative, we can write

$$\left(\sum_{d|m}\chi(d)d\right)\left(\sum_{d|n}\chi(d)d\right) = \sum_{d|\gcd(m,n)} B(d)\left(\sum_{e|(mn/d^2)}\chi(e)e\right),$$

where B is some completely multiplicative function. In particular, B is determined by the relation $B(p) = g(p)^2 - g(p^2) = (1+\chi(p)p)^2 - (1+\chi(p)p+\chi(p^2)p^2) = \chi(p)p$, which gives the desired result. $\square$

Since

$$\begin{aligned} F(\tau) &= H_{5,1}(\tau) + H_{5,2}(\tau) + H_{5,3}(\tau) + H_{5,4}(\tau) \\ &= -\frac{1}{3} - \sum_{N=1}^{\infty}\big(\sigma_{5,1}(N) + \sigma_{5,2}(N) + \sigma_{5,3}(N) + \sigma_{5,4}(N)\big)q^N, \end{aligned}$$

we can write the Fourier expansion of $F(\tau)$ as $\sum a_n q^n$ where $a_0 = -1/3$ and $a_n = \sigma_{5,1}(n) + \sigma_{5,2}(n) + \sigma_{5,3}(n) + \sigma_{5,4}(n) = \sum_{d|n}\chi(d)d$. Then

$T_mF = \sum b_n q^n$ where

$$b_n = \sum_{d|\gcd(m,n)} \chi(d)\, d a_{mn/d^2}$$

(see [**1**, Proposition 39]). Now

$$\lambda_m a_n = \Bigl(\sum_{d|m} \chi(d)d\Bigr)\Bigl(\sum_{d|n} \chi(d)d\Bigr),$$

and

$$\begin{aligned} b_n &= \sum_{d|\gcd(m,n)} \chi(d)\, d a_{mn/d^2} \\ &= \sum_{d|\gcd(m,n)} \Bigl(\chi(d)d \sum_{e|(mn/d^2)} \chi(e)e\Bigr). \end{aligned}$$

By the lemma, $\lambda_m a_n = b_n$, and thus $T_mF = \lambda_m F$. A similar argument shows that $G(\tau)$ is an eigenform with respect to the quadratic character $(\cdot/d)$.

The function defined in Theorem 1 is always a modular form of weight two on $\Gamma_0(N)$ with χ as a multiplier. We conclude by showing that, if the function is not the zero function, then it turns out to be an eigenform:

Theorem 2. *Let χ be a Dirichlet character modulo N (N a positive integer), with $\chi(-1) = 1$, and set*

$$f(\tau) = \sum_{k=1}^{N} \chi(k)\, H_{N,k}(\tau).$$

Then f is an eigenform of weight two on $\Gamma_0(N)$ with multiplier χ.

Proof. Let m be a positive integer, and write $f(\tau)$ as $\sum a_n q^n$ and $T_m f$ as $\sum b_n q^n$. Then for $n > 0$, $a_n = \sum_{d|n} \chi(d)d$. Let $\lambda_m = \sum_{d|m} \chi(d)d$. Then

$$\lambda_m a_n = \Bigl(\sum_{d|m} \chi(d)d\Bigr)\Bigl(\sum_{d|n} \chi(d)d\Bigr),$$

and

$$\begin{aligned} b_n &= \sum_{d \mid \gcd(m,n)} \chi(d)\, d a_{mn/d^2} \\ &= \sum_{d \mid \gcd(m,n)} \biggl(\chi(d) d \sum_{e \mid (mn/d^2)} \chi(e) e \biggr). \end{aligned}$$

By the lemma, $\lambda_m a_n = b_n$, and hence f is an eigenform. □

Acknowledgments. The author wishes to thank Harold Stark, Sinai Robins and Eric Stade for their helpful comments. The author also thanks the referees for their suggestions.

REFERENCES

1. N. Kogblitz, *Introduction to elliptic curves and modular forms*, 2nd ed., Springer-Verlag, Berlin, New York, 1993.

2. D.S. Kubert and S. Lang, *Modular units*, Springer-Verlag, Berlin, New York, 1981.

3. P.J. McCarthy, *Introduction to arithmetical functions*, Springer-Verlag, Berlin, New York, 1986.

4. S. Robins, *Generalized Dedekind η-products*, Contemp. Math. **166**, Amer. Math. Soc. (1994), 119–128.

5. S. Schoeneberg, *Elliptic modular functions*, Springer-Verlag, Berlin, New York, *1974.

6. D. Vestal, *Generalized Dedekind eta functions with applications to additive number theory*, Ph.D. Thesis, 1998.

Missouri Western State College, St. Joseph, Missouri 64507
E-mail address: vestal@griffon.mwsc.edu