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GROWTH, DISTORTION AND COEFFICIENT BOUNDS FOR PLANE HARMONIC MAPPINGS CONVEX IN ONE DIRECTION

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ABSTRACT. In this paper we examine normalized harmonic functions convex in the direction of either the real or the imaginary axis. In this setting we find bounds for $|f_z(z)|$, $|f_{\bar z}(z)|$ and $|f(z)|$, as well as coefficient bounds on the series expansion of functions convex in the direction of the real axis. For the functions convex in the direction of the real axis, we provide the extremal functions for $|f_z(z)|$ and $|f_{\bar z}(z)|$.

Many important questions in the study of classes of functions relate to bounds on the modulus of the function (growth) or the modulus of the derivative (distortion). In this paper we examine both of these questions, as well as coefficient bounds, for two classes of complex-valued harmonic functions of one complex variable.

Any harmonic function in the open unit disk $\mathbf{D}$ can be written as a sum of an analytic and anti-analytic function, $f(z) = h(z) + \overline{g(z)}$. The Jacobian of the mapping f, denoted $J_f(z)$, can be computed by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. If the function f is locally univalent, as will be true of all functions in this work, then $J_f(z) \neq 0$ for $z \in \mathbf{D}$. For convenience, we will only examine *sense-preserving* functions, that is, functions for which $J_f(z) > 0$. If f has $J_f(z) < 0$, then $\bar f$ is sense-preserving. The *analytic dilatation* of a harmonic function is the quantity $\omega(z) = (g'(z)/h'(z))$. Note that if f is locally univalent and sense-preserving, $|\omega(z)| < 1$. The class of functions f in S_H is defined by

$$\begin{aligned} S_H = \{f = h + \bar g : f \text{ is univalent in } \mathbf{D} \\ \text{and satisfies } f(0) = 0, h'(0) = 1\}, \end{aligned}$$

and the compact normal family S_H^0 is defined by

$$S_H^0 = \{f = h + \bar g : f \in S_H \text{ and } g'(0) = 0\}.$$

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In this paper we examine two classes of functions convex in one direction. The first is a normalized subset of functions in S_H^0 that are convex in the direction of the real axis. The second class is comprised of functions in S_H^0 that are convex in the direction of the imaginary axis. Functions in both of these classes can be constructed by the shear construction. The shear construction is essential to the work here, because it allows us to study harmonic functions by examining their related analytic functions. We refer to work done by Hengartner and Schober [**2**], and to a generalization of their work by Royster and Ziegler [**4**], which give good bounds for the derivatives and coefficients of analytic functions. In particular, Hengartner and Schober's characterization of a class of analytic functions by the criterion $\mathrm{Re}\,\{(1-z^2)\psi'(z)\} \geq 0$ gives us a means by which to study the function-theoretic properties of the function ψ and thus the properties of related harmonic functions.

The shear construction produces a univalent harmonic functions that maps $\mathbf{D}$ to a region that is convex in the direction of the real axis. This construction relies on the following theorem of Clunie and Sheil-Small:

Theorem 0.1 [**1**]. *A harmonic $f = h + \bar{g}$ locally univalent in $\mathbf{D}$ is a univalent mapping of $\mathbf{D}$ onto a domain convex in the direction of the real axis if and only if $h - g$ is a conformal univalent mapping of $\mathbf{D}$ onto a domain convex in the direction of the real axis.*

Theorem 0.1 gives a way of constructing univalent harmonic functions with a specified analytic dilatation ω. If $\varphi = h - g$ and $\omega = (g'/h')$, we have the equations

$$h'(z) = \frac{\varphi'(z)}{1-\omega(z)} \quad \text{and} \quad g'(z) = \frac{\varphi'(z)\omega(z)}{1-\omega(z)}. \tag{1}$$

The resulting harmonic function, $f = h + \bar{g}$, is called the *horizontal shear of $\varphi(z)$ with analytic dilatation $\omega(z)$.*

Theorem 0.1 has a natural generalization when f is convex in the direction α. In that situation, $e^{-i\alpha}f$ and $\varphi = e^{-i\alpha}h - e^{i\alpha}g$ are convex in the direction of the real axis, hence the function $h - e^{i2\alpha}g$ is convex in the direction α. In particular, we can use this construction when $\alpha = (\pi/2)$ to construct function that are convex in the direction of the imaginary axis.

1. Growth and distortion theorems for a specific class of harmonic maps. Hengartner and Schober [**2**] studied analytic functions ψ that are convex in the direction of the imaginary axis. They used a normalization which requires, in essence, that the right and left extremes of $\psi(\mathbf{D})$ be the images of 1 and -1. This normalization is: there exist points z_n' converging to $z=1$ and z_n'' converging to $z=-1$ such that

$$\lim_{n\to\infty} \operatorname{Re}\{\psi(z_n')\} = \sup_{|z|<1} \operatorname{Re}\{\psi(z)\} \tag{2}$$

and

$$\lim_{n\to\infty} \operatorname{Re}\{\psi(z_n'')\} = \inf_{|z|<1} \operatorname{Re}\{\psi(z)\}.$$

If CIA is the class of domains, D, that are convex in the direction of the imaginary axis and admit a mapping ψ so that $\psi(\mathbf{D}) = D$ and ψ satisfies the normalization (2), then we have the following result:

Theorem 1.1 [**2**]. *Suppose ψ is analytic and nonconstant for $|z|<1$. Then we have $\operatorname{Re}\{(1-z^2)\psi'(z)\} \geq 0$ for $|z|<1$ if and only if*

1. *ψ is univalent on $\mathbf{D}$,*
2. *$\psi(\mathbf{D}) \in CIA$, and*
3. *ψ is normalized by* (2).

Using this characterization of functions, Hengartner and Schober then proved the following:

Theorem 1.2 [**2**]. *If ψ is analytic for $|z|<1$ and satisfies $\operatorname{Re}\{(1-z^2)\psi'(z)\} \geq 0$, then for $|z| \leq r < 1$,*

$$\frac{(1-r)|\psi'(0)|}{(1+r)(1+r^2)} \leq |\psi'(z)| \leq \frac{|\psi'(0)|}{(1-r)^2}. \tag{3}$$

The upper bound is sharp for $\psi(z) = (z/(1-z))$, which maps $\mathbf{D}$ onto the right half-plane $\operatorname{Re}\{z\} > -1/2$, and the lower bound is sharp for

$\psi(z) = (i/2)\log[(1-iz)^2/(1-z^2)]$, which maps $\mathbf{D}$ onto a vertical strip, slit from $i\log\sqrt{2}$ to infinity along the positive imaginary axis.

To be able to use this result for functions that are convex in the direction of the real axis, let us consider the following situation. Suppose that $\varphi(z)$ is a function that is analytic and convex in the direction of the real axis. Furthermore, suppose that the φ is normalized by the following. Let there exist points z_n' converging to $z = e^{i\alpha}$ and z_n'' converging to $z = e^{i(\alpha+\pi)}$ such that

$$\lim_{n\to\infty} \operatorname{Im}\{\varphi(z_n')\} = \sup_{|z|<1} \operatorname{Im}\{\varphi(z)\} \tag{4}$$

and

$$\lim_{n\to\infty} \operatorname{Im}\{\varphi(z_n'')\} = \inf_{|z|<1} \operatorname{Im}\{\varphi(z)\}.$$

Consequently, if $\psi(z)$ satisfies (2), then

$$\varphi(z) = i\psi(e^{-i\alpha}z) \tag{5}$$

satisfies (4). Knowing this, we can apply Theorem 1.2 to $\varphi(z)$ and see that the result still holds, with ψ replaced by φ. In this situation $\operatorname{Re}\{(-i)(e^{i\alpha} - e^{-i\alpha}z^2)\varphi'(z)\} \geq 0$. We can now prove derivative bounds for harmonic functions convex in the direction of the real axis.

Theorem 1.3. *Let $f = h + \bar{g}$ be convex in the direction of the real axis, $f \in S_H^0$, and let $\varphi' = h' - g'$ and $\omega = (g'/h')$. Furthermore, let φ satisfy normalization* (4)*. Then, for $|z| \leq r$, we have*

$$\frac{1-r}{(1+r)^2(1+r^2)} \leq |f_z(z)| \leq \frac{1}{(1-r)^3} \tag{6}$$

and

$$\frac{|\omega(z)|(1-r)}{(1+r)^2(1+r^2)} \leq |f_{\bar{z}}(z)| \leq \frac{r}{(1-r)^3}. \tag{7}$$

Equality occurs for both upper bounds when $\varphi(z) = (z/(1+iz))$ and $\omega(z) = -iz$, and for the lower bounds when $\varphi(z) = (1/2)\log((1+z^2)/(1-z)^2)$ and $\omega(z) = z$.

Proof. Since $\varphi' = h' - g'$ and $g' = \omega h'$, we have $f_z(z) = h'(z) = (\varphi'(z)/(1-\omega(z)))$ and $\overline{f_{\bar{z}}(z)} = g'(z) = (\omega(z)\varphi'(z)/(1-\omega(z)))$. Now $\omega(z)$ is a Schwarz function; therefore,

$$|f_z(z)| = \left|\frac{\varphi'(z)}{1-\omega(z)}\right| \leq \frac{|\varphi'(z)|}{1-|\omega(z)|} \leq \frac{|\varphi'(z)|}{1-|z|}.$$

Furthermore,

$$|f_z(z)| \geq \frac{|\varphi'(z)|}{1+|\omega(z)|} \geq \frac{|\varphi'(z)|}{1+|z|}.$$

Using Theorem 1.2 gives inequality (6).

Similarly,

$$|f_{\bar{z}}(z)| \leq \frac{|\varphi'(z)||\omega(z)|}{1-|\omega(z)|} \leq |\varphi'(z)|\frac{|z|}{1-|z|}$$

and

$$|f_{\bar{z}}(z)| \geq \frac{|\varphi'(z)||\omega(z)|}{1+|\omega(z)|} \geq |\varphi'(z)|\frac{|\omega(z)|}{1+|z|}.$$

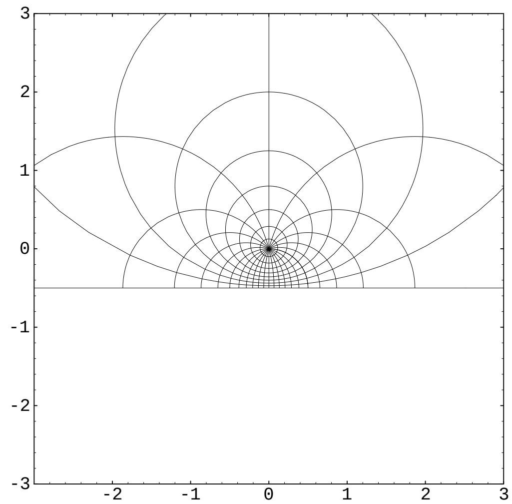
Applying Theorem 1.2 again yields (7).

The sharpness functions come from examining the sharpness functions for Theorem 1.2. Let $\varphi(z) = i\psi(e^{-i(\pi/2)}z)$, and wisely choose the analytic dilatation $\omega(z)$. The mapping properties of these functions are shown in Figures 1 and 2. The figures show the images of concentric circles and equally spaced rays. $\square$

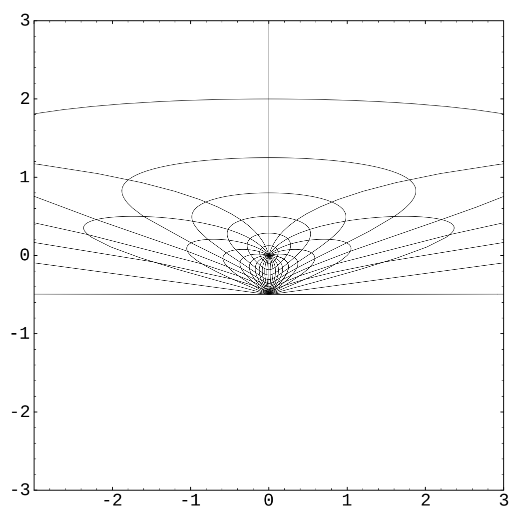
By applying the inequalities from Theorem 1.3 to $f(z) = h(z) + \overline{g(z)}$, we can find an upper bound for $|f(z)|$.

Theorem 1.4. *Let $f = h + \bar{g}$ be convex in the direction of the real axis, $f \in S_H^0$, and let $\varphi = h - g$ satisfy normalization* (4). *Then, for $|z| \leq r$, we have*

$$|f(z)| \leq \frac{r}{(1-r)^2}. \tag{8}$$

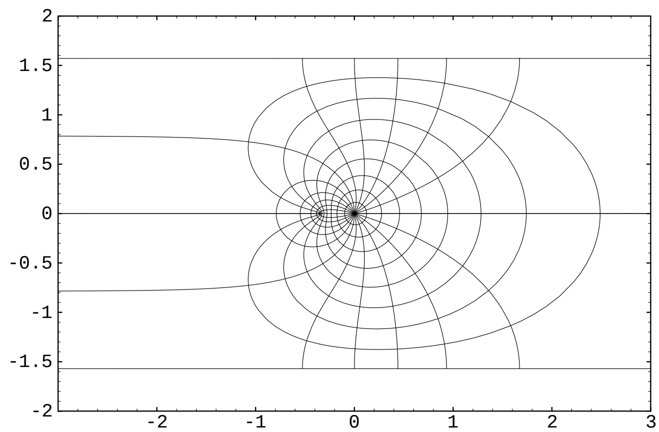


(a) $\varphi(z) = (z/(1+iz))$

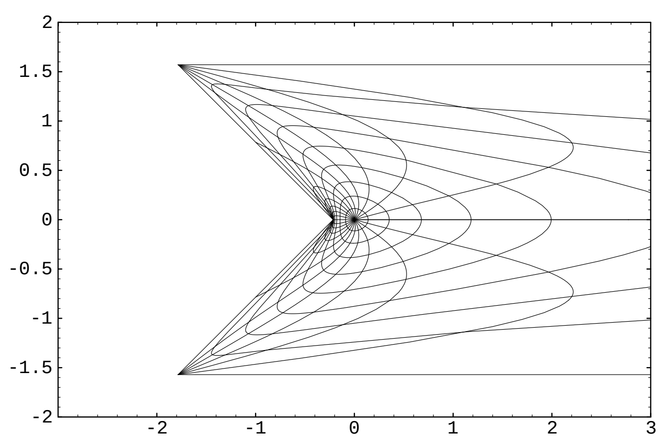


(b) shear of $\varphi(z)$

FIGURE 1. The shear of $\varphi(z) = (z/(1+iz))$ with dilatation $\omega(z) = -iz$.



(a) $\varphi(z) = (1/2)\log((1+z^2)/(1-z)^2)$



(b) shear of $\varphi(z)$

FIGURE 2. The shear of $\varphi(z) = (1/2)\log((1+z^2)/(1-z)^2)$ with dilatation $\omega(z) = z$.

Proof. Since $f(z) = h(z) + \overline{g(z)}$, we have the following equalities:

$$\begin{aligned} f(z) &= h(z) + \overline{g(z)} \\ &= \int_0^r h'(\rho e^{i\theta})e^{i\theta}\,d\rho + \overline{\int_0^r g'(\rho e^{i\theta})e^{i\theta}\,d\rho} \\ &= \int_0^r h'(\rho e^{i\theta})e^{i\theta}\,d\rho + \int_0^r \overline{g'(\rho e^{i\theta})}e^{-i\theta}\,d\rho \\ &= \int_0^r f_z(\rho e^{i\theta})e^{i\theta}\,d\rho + \int_0^r f_{\bar z}(\rho e^{i\theta})e^{-i\theta}\,d\rho, \end{aligned}$$

hence

$$\begin{aligned} |f(z)| &= |h(z) + \overline{g(z)}| \le |h(z)| + |g(z)| \\ &\le \int_0^r |f_z(\rho e^{i\theta})|\,d\rho + \int_0^r |f_{\bar z}(\rho e^{i\theta})|\,d\rho. \end{aligned}$$

Applying inequalities (6) and (7) to the above yields

$$\begin{aligned} |f(z)| &\le \int_0^r \frac{1}{(1-\rho)^3}\,d\rho + \int_0^r \frac{\rho}{(1-\rho)^3}\,d\rho \\ &= \frac{r}{(1-r)^2}. \qquad \square \end{aligned}$$

2. Generalized growth and distortion bounds. Royster and Ziegler [**4**] generalized the work done by Hengartner and Schober. Instead of requiring that ends of a diameter map to the extremes of $\varphi(\mathbf{D})$, they worked in the generality of having $e^{i(\mu-\nu)}$ and $e^{i(\mu+\nu)}$ map to the right and left extremes of $\varphi(\mathbf{D})$. They proved the following theorem.

Theorem 2.1 [**4**]. *Let $\varphi(z)$ be a nonconstant function regular in $\mathbf{D}$. The function $\varphi(z)$ maps $\mathbf{D}$ univalently onto a domain Ω convex in the direction of the imaginary axis if and only if there are numbers μ and ν, $0 \le \mu < 2\pi$ and $0 \le \nu < \pi$, such that*

$$\mathrm{Re}\,\{-ie^{-i\mu}(1 - 2\cos\nu e^{-i\mu}z + e^{-2i\mu}z^2)\varphi'(z)\} \ge 0, \quad z \in \mathbf{D}. \tag{9}$$

Furthermore, $\varphi(e^{i(\mu-\nu)})$ and $\varphi(e^{i(\mu+\nu)})$ are the right and left extremes, respectively, of Ω.

Using the normalization (9), Royster and Ziegler made the following definition.

Definition 2.2. Let Γ be the class of all normalized analytic univalent functions which map $\mathbf{D}$ onto domains convex in the direction of the imaginary axis. That is, $\varphi \in \Gamma$ if and only if $\varphi(0) = 0$, $\varphi'(0) = 1$, and $\varphi(z)$ satisfies (9) for some choice of μ and ν. We indicate dependence on the parameters ν and μ in (9) by letting $\Gamma(\nu, \mu)$ denote the set of all functions in Γ which satisfy (9) for a given pair μ and ν and $\Gamma(\nu) = \cup_\mu \Gamma(\nu, \mu)$.

We are finally prepared to state a theorem that gives bounds for the modulus of the derivative of $\varphi(z)$.

Theorem 2.3 [**4**]. *If $\varphi(z)$ is in $\Gamma(\nu)$, then*

$$\frac{1-r}{(1+r)(1+2r|\cos\nu|+r^2)} \leq |\varphi'(z)| \tag{10}$$

and

$$|\varphi'(z)| \leq \begin{cases} \dfrac{1+r}{(1-r)(1-2r|\cos\nu|+r^2)} & \text{for } r < \dfrac{1-\sin\nu}{|\cos\nu|} \\ \dfrac{1}{\sin\nu(1-r)^2} & \text{for } \dfrac{1-\sin\nu}{|\cos\nu|} \leq r < 1. \end{cases} \tag{11}$$

Inequality (11) *should be interpreted to mean that the top inequality is used for all r when ν is* 0 *or π, and the bottom inequality is used for all r when $\nu = \pi/2$.*

Using Royster and Ziegler's result, we are able to obtain the following bounds for $|f_z(z)|$ and $|f_{\bar{z}}(z)|$:

Theorem 2.4. *Let $f = h + \bar{g} \in S_H^0$ be convex in the direction of the imaginary axis. Let $\varphi = h + g$ and $\omega = (g'/h')$ so that f is the shear in the direction of the imaginary axis of φ with dilatation ω, and $\varphi(z) \in \Gamma(\nu)$. Then the following hold for $|z| \leq r$:*

$$\frac{1-r}{(1+r)^2(1+2r|\cos\nu|+r^2)} \le |f_z(z)| \tag{12}$$

(13)
$$|f_z(z)| \le \begin{cases} \dfrac{1+r}{(1-r)^2(1-2r|\cos\nu|+r^2)} & \textit{for } r < \dfrac{1-\sin\nu}{|\cos\nu|} \\ \dfrac{1}{\sin\nu(1-r)^3} & \textit{for } \dfrac{1-\sin\nu}{|\cos\nu|} \le r < 1, \end{cases}$$

$$\frac{|\omega(z)|(1-r)}{(1+r)^2(1+2r|\cos\nu|+r^2)} \le f_{\bar z}(z)| \tag{14}$$

(15)
$$|f_{\bar z}(z)| \le \begin{cases} \dfrac{r(1+r)}{(1-r)^2(1-2r|\cos\nu|+r^2)} & \textit{for } r < \dfrac{1-\sin\nu}{|\cos\nu|} \\ \dfrac{r}{\sin\nu(1-r)^3} & \textit{for } \dfrac{1-\sin\nu}{|\cos\nu|} \le r < 1, \end{cases}$$

where inequalities (13) *and* (15) *are interpreted in the same way as* (11).

Proof. Since $f = h + \bar g$ and $\omega h' = g'$, we have

$$f_z(z) = h'(z) = \frac{\varphi'(z)}{1+\omega(z)}$$

and

$$\overline{f_{\bar z}(z)} = g'(z) = \frac{\omega(z)\varphi'(z)}{1+\omega(z)}.$$

Then, by the same method used to prove Theorem 1.3, and using Theorem 2.3, the proof of the theorem is completed. □

Note that in the special case of $\nu = \pi/2$, the result is the same as for the case of convex in the direction of the real axis stated in Theorem 1.3. This is expected, because $\nu = \pi/2$ corresponds to the case where the left and right extremes of $\varphi(\mathbf{D})$ are the images of ends of a diameter. In the case when $\nu = 0$ or $\nu = \pi$, which corresponds to the extremes of $\varphi(\mathbf{D})$ being in the cluster set of one point (as is the case with the analytic Koebe function), Theorem 2.4 reduces to

$$\frac{1-r}{(1+r)^4} \le |f_z(z)| \le \frac{1+r}{(1-r)^4} \tag{16}$$

and

$$\frac{|\omega(z)|(1-r)}{(1+r)^4} \le |f_{\bar z}(z)| \le \frac{r(1+r)}{(1-r)^4}. \tag{17}$$

Using inequalities (16) and (17), we can prove the following result for $|f(z)|$.

Theorem 2.5. *Let f satisfy the hypotheses of Theorem* 2.4, *and let $|z| \le r$. Then, for $\nu = 0$ or π,*

$$|f(z)| \le \frac{r(3+r^2)}{3(1-r)^3}.$$

If $\nu = \pi/2$, then

$$|f(z)| \le \frac{r}{(1-r)^2}.$$

Proof. As in the proof of Theorem 1.4, we can write

$$f(z) = \int_0^r f_z(\rho e^{i\theta})e^{i\theta}\,d\rho + \int_0^r f_{\bar z}(\rho e^{i\theta})e^{-i\theta}\,d\rho.$$

Thus, if $\nu = 0$ or π,

$$\begin{aligned} |f(z)| &\le \int_0^r |f_z(\rho e^{i\theta})|\,d\rho + \int_0^r |f_{\bar z}(\rho e^{i\theta})|\,d\rho \\ &\le \int_0^r \frac{1+\rho}{(1-\rho)^4} + \int_0^r \frac{\rho(1+\rho)}{(1-\rho)^4}\,d\rho \\ &= \frac{r(3+r^2)}{3(1-r)^3}. \end{aligned}$$

The proof for the case $\nu = \pi/2$ is identical to the proof of Theorem 1.4. □

3. Bounds on $|a_n|$ and $|b_n|$. Sheil-Small [**5**] proved that if $f \in S_H^0$ and $f(\mathbf{D})$ is convex in one direction, then the following hold for the coefficients:

$$|a_n| \le \frac{(n+1)(2n+1)}{6} \quad \text{and} \quad |b_n| \le \frac{(n-1)(2n-1)}{6},$$

where $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n}$.

In previous sections, we saw how the geometry of the related analytic function $\varphi(z)$ affects bounds of a harmonic function and its derivatives. The following theorem shows how the geometry of $\varphi(z)$ affects the coefficients $|a_n|$ and $|b_n|$. We begin by looking at Hengartner and Schober's work [**2**].

Theorem 3.1 [**2**]. *If* $\psi(z) = a_0 + (\alpha + i\beta)z + \sum_{n=2}^{\infty} a_n z^n$ *is analytic in* $\mathbf{D}$ *and satisfies* $\mathrm{Re}\,\{(1-z^2)\psi'(z)\} \geq 0$*, then*

$$|a_n| \leq \alpha \quad \text{for } n = 2, 4, 6, \ldots \tag{18}$$

and

$$|a_n| \leq \left(1 - \frac{1}{n}\right)\alpha + \frac{1}{n}|\alpha + i\beta| \quad \text{for } n = 1, 3, 5, \ldots . \tag{19}$$

Consequently,

$$|a_n| \leq |\psi'(0)| \quad \text{for } n = 1, 2, 3, 4, \ldots . \tag{20}$$

Equality is obtained in all three inequalities by $\psi(z) = (1/(1-z))$. Furthermore, among bounds which depend on both α and β, (18) is sharp for the function

$$\psi(z) = \frac{\alpha}{1-z} + \frac{i\beta}{2}\log\frac{1+z}{1-z}, \quad \alpha > 0.$$

To apply Theorem 3.1 to the situation where the function $\varphi(z)$ is convex in the direction of the real axis and normalized by equation (4), we again observe that if $\psi(z)$ satisfies normalization (2), then

$$\varphi(z) = i\psi(e^{-i\alpha}z),$$

thus inequality (20) holds for such φ.

Theorem 3.2. *Let* $f \in S_H^0$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=2}^{\infty} b_n z^n}$, *and let* $f(\mathbf{D})$ *be convex in the direction of the real axis. Also let* $\varphi(z) = h(z) - g(z)$ *satisfy normalization* (4)*. Then*

$$|a_n| \leq \frac{n+1}{2} \quad \text{and} \quad |b_n| \leq \frac{n-1}{2} \quad \text{for } n \geq 2.$$

Proof. We start with the integral representations

$$h(z)=\int_0^z \frac{\varphi'(\zeta)}{1-\omega(\zeta)}\,d\zeta \quad \text{and} \quad g(z)=\int_0^z \frac{\varphi'(\zeta)\omega(\zeta)}{1-\omega(\zeta)}\,d\zeta,$$

where $\omega(z)=(g'(z)/h'(z))$. Let

$$\varphi(z)=\sum_{n=1}^{\infty}\phi_n z^n$$

and

$$\frac{\omega(z)}{1-\omega(z)}=\sum_{n=1}^{\infty} w_n z^n.$$

Now

$$\begin{aligned}
&g(z)\\
&=\int_0^z [\phi_1+2\phi_2\zeta+3\phi_3\zeta^2+\cdots][w_1\zeta+w_2\zeta^2+w_3\zeta^3+\cdots]\,d\zeta\\
&=\int_0^z [\phi_1w_1\zeta+(\phi_1w_2+2\phi_2w_1)\zeta^2+(\phi_1w_3+2\phi_2w_2+3\phi_3w_1)\zeta^3+\cdots]\,d\zeta\\
&=\frac{1}{2}\,(\phi_1w_1)z^2+\frac{1}{3}\,(\phi_1w_2+2\phi_2w_1)z^3+\frac{1}{4}\,(\phi_1w_3+2\phi_2w_2+3\phi_3w_1)z^4+\cdots.
\end{aligned}$$

We have

$$\begin{aligned}
b_1&=0\\
b_2&=\frac{1}{2}\,(\phi_1w_1)\\
b_3&=\frac{1}{3}\,(\phi_1w_2+2\phi_2w_1)\\
&\vdots\\
b_n&=\frac{1}{n}\sum_{k=1}^{n-1}k\phi_k w_{n-k} \quad \text{for } n\geq 2.
\end{aligned}$$

Now for $h(z)$, we have

$$\begin{aligned}
h(z)&=\int_0^z \varphi'(\zeta)\frac{1}{1-\omega(\zeta)}\,d\zeta\\
&=\int_0^z \varphi'(\zeta)\left(\frac{\omega(\zeta)}{1-\omega(\zeta)}+1\right)d\zeta,
\end{aligned}$$

hence $a_n = b_n + \phi_n = \phi_n + (1/n)\sum_{k=1}^{n-1} k\phi_k w_{n-k}$. Since $(\omega(z)/(1-\omega(z)))$ is subordinate to $(z/(1-z))$, we know that $|w_n| \leq 1$ for all n (see, for example, [**3**, p. 238]). Also, by the discussion which follows Theorem 3.1, we know that $|\phi_k| \leq |\varphi'(0)| = 1$. Using these estimates, we get

$$\begin{aligned} |b_n| &= \frac{1}{n}\sum_{k=1}^{n-1} k\phi_k w_{n-k} \\ &\leq \frac{1}{n}\sum_{k=1}^{n-1} k \\ &= \frac{n-1}{2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |a_n| &\leq |\phi_n| + \frac{1}{n}\sum_{k=1}^{n-1} k|\phi_k| \\ &= \frac{1}{n}\sum_{k=1}^{n} k|\phi_k| \\ &\leq \frac{1}{n}\sum_{k=1}^{n} k \\ &= \frac{n+1}{2}. \qquad \square \end{aligned}$$

Note that the bounds in Theorem 3.2 are smaller than the bounds for the whole class of functions convex in one direction.

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