ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 2, Summer 2001

DEGENERATE HOMOGENEOUS STRUCTURES OF TYPE $\mathcal{S}_1$ ON PSEUDO-RIEMANNIAN MANIFOLDS

A. MONTESINOS AMILIBIA

ABSTRACT. We obtain all of the pseudo-Riemannian manifolds endowed with homogeneous structures defined by isotropic vector fields. Thus, the (general pseudo-Riemannian) class $\mathcal{S}_1$ of homogeneous structures is fully determined.

1. Introduction. Ambrose and Singer [**1**] gave a characterization for a connected, simply connected and complete Riemannian manifold to be homogeneous, in terms of a (1,2) tensor field S on the manifold. This characterization extends the classical one given by Cartan of Riemannian symmetric spaces as the spaces of parallel curvature, which correspond to Ambrose-Singer's case $S = 0$. That characterization has also permitted Tricerri and Vanhecke [**8**] to classify those homogeneous Riemannian manifolds into eight classes which are defined by the invariant subspaces of certain space $\mathcal{S}_1 \oplus \mathcal{S}_2 \oplus \mathcal{S}_3$. In [**8**] it is proved that a connected, simply connected and complete Riemannian manifold admits a nonvanishing homogeneous structure S of type $\mathcal{S}_1$ if and only if it is isometric to the hyperbolic space.

Gadea and Oubiña [**4**] have extended the characterization in [**1**] to the pseudo-Riemannian case of any signature and proved that a connected, simply connected and complete pseudo-Riemannian manifold admits a homogeneous pseudo-Riemannian structure if and only if it is reductive homogeneous. As is well known, in the Riemannian case every homogeneous manifold is complete and reductive.

Gadea and Oubiña give in [**6**] a classification for the pseudo-Riemannian case of any signature similar to that given in [**8**] for Riemannian homogeneous structures, and they moreover characterize the three primitive classes. From now on we shall focus attention on the first class, $\mathcal{S}_1$. A connected, simply connected and complete pseudo-Riemannian manifold (M, g) of any signature (M, g) admits [**6**] a *nondegenerate*,

1991 AMS *Mathematics Subject Classification.* 53C30, 53C50, 53C15.
Key words and phrases. Homogeneous structures, pseudo-Riemannian manifolds.
Received by the editors on February 11, 1999.

see Section 2, homogeneous structure of type $\mathcal{S}_1$ if and only if g is, up to a change of sign, a metric of strictly negative constant curvature. Moreover, the nonflat pseudo-Riemannian space forms of arbitrary signature locally admit nondegenerate homogeneous pseudo-Riemannian structures of type $\mathcal{S}_1$. By the way, we note that in the complex case the situation is similar to the real one [**3**]: one of the classes in Abbena-Garbiero's classification [**2**] (of homogeneous Kahler structures) corresponds to spaces of strictly negative holomorphic curvature.

Thus, to determine the (general pseudo-Riemannian) class $\mathcal{S}_1$, it only rests upon the degenerate case which is solved in the present paper, proving the following

Theorem. *Let $(M, \bar{g})$ be an $n+2$-dimensional connected pseudo-Riemannian manifold with a degenerate homogeneous structure of type $\mathcal{S}_1$. Then $(M, \bar{g})$ is locally isometric to $\mathbf{R}^{n+2}$ with the pseudo-Riemannian metric*

$$g = du \otimes dv + dv \otimes du + \big(b(\mathbf{x}, \mathbf{x}) + 2u\big)\, dv \otimes dv + h,$$

where b and h are symmetric bilinear forms in $\mathbf{R}^n$, h is nondegenerate, $\mathbf{x}$ is the position vector in $\mathbf{R}^n$ and u, v are the coordinates in $\mathbf{R}^2$.

As we shall see in Proposition 3, $(\mathbf{R}^{n+2}, h, b)$ is isometric to the reductive homogeneous pseudo-Riemannian manifold $\mathcal{G}/\mathcal{H}$ with group $\mathcal{G}$, isotropy group $\mathcal{H}$ and Lie subspace $\mathfrak{m}$ endowed with the $\operatorname{Ad}(\mathcal{H})$-invariant inner product as defined in Section 5; but $\mathcal{G}$ is not in general the whole group of isometries of $(\mathbf{R}^{n+2}, h, b)$. Moreover, $(\mathbf{R}^{n+2}, h, b)$ is an example of a reductive homogeneous pseudo-Riemannian manifold which is noncomplete.

2. Preliminaries. Let (M, g) be a connected C^∞ pseudo-Riemannian manifold with Levi-Civita connection ∇ and curvature R. Then, in the same vein as [**8**], Gadea and Oubiña [**4**] define a *homogeneous pseudo-Riemannian structure* on (M, g) as a tensor field S of type (1,2) on M such that the connection $\tilde{\nabla} = \nabla - S$ satisfies $\tilde{\nabla} g = \tilde{\nabla} R = \tilde{\nabla} S = 0$. Such a structure is said to be of type $\mathcal{S}_1$ if there is a vector field $\xi \in \mathfrak{X}(M)$ that defines S by

$$S(X, Y) = g(X, Y)\xi - g(Y, \xi)X,$$

and it is said to be degenerate or not according to ξ being isotropic or not.

Gadea and Oubiña [**6**] have studied nondegenerate homogeneous structures of type $\mathcal{S}_1$. They turn out to be defined on certain subsets of pseudo-Riemannian manifolds of strictly negative constant curvature.

Here we treat the case of degenerate homogeneous pseudo-Riemannian structures of type $\mathcal{S}_1$.

Let (M, g) be a connected C^∞ pseudo-Riemannian manifold with $\dim M = n+2$, and let $0 \neq \xi \in \mathfrak{X}(M)$ be isotropic, that is, $g(\xi,\xi) = 0$. We assume that if we put

$$\tilde{\nabla}_X Y := \nabla_X Y - g(X,Y)\xi + g(Y,\xi)X,$$

then $\tilde{\nabla} g = \tilde{\nabla} R = \tilde{\nabla} S = 0$, where $S(X,Y) = g(X,Y)\xi - g(Y,\xi)X$. The condition $\tilde{\nabla} g = 0$ is automatically satisfied. As for $\tilde{\nabla} S$ we have

$$(\tilde{\nabla}_X S)(Y,Z) = g(Y,Z)\tilde{\nabla}_X \xi - g(Z, \tilde{\nabla}_X \xi)Y = 0.$$

Let W be any nonisotropic (local) vector field. Since $\dim M \geq 2$, we can take Y nonisotropic and orthogonal to W and $Z = Y$. Then $g(Y,Y)g(\tilde{\nabla}_X \xi, W) = 0$, whence $g(\tilde{\nabla}_X \xi, W) = 0$. Since this is true for any nonisotropic W we conclude $\tilde{\nabla}\xi = 0$. Thus,

$$\nabla_X \xi = g(X,\xi)\xi,$$

or if we put $\alpha = g(\xi)$, we have

$$\nabla \xi = \alpha \otimes \xi, \qquad \nabla \alpha = \alpha \otimes \alpha.$$

Then

$$d\alpha = 0.$$

As for the curvature, we first fix the notation. We define

$$\begin{aligned} R(X,Y)Z &= \nabla_{[X,Y]} Z + \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z, \\ R(X,Y,Z,W) &= g(R(X,Y)Z, W). \end{aligned}$$

We have $\nabla_Y \nabla_X \xi = 2\alpha(Y)\alpha(X)\xi + \alpha(\nabla_Y X)\xi$, whence evidently

$$R(X,Y)\xi = 0, \qquad R(X,Y,Z,\xi) = 0.$$

By expanding the condition $\tilde{\nabla} R = 0$, we get

$$\begin{aligned}(\nabla_X R)(Y,Z,W,U) &= \alpha(Y)R(X,Z,W,U) + \alpha(Z)R(Y,X,W,U)\\ &\quad + \alpha(W)R(Y,Z,X,U) + \alpha(U)R(Y,Z,W,X).\end{aligned}$$

If we take the cyclic sum in X, Y, Z and apply Bianchi identities we get

$$\mathfrak{S}_{XYZ}\alpha(X)R(Y,Z,W,U) = 0.$$

In other terms, for every $W, U \in \mathcal{X}(M)$, we have

$$\alpha \wedge R(.,.,W,U) = 0,$$

and if we bring this into the formula for $\nabla_X R$, we get

$$\nabla_X R = 2\alpha(X)R.$$

Now since $d\alpha = 0$ for each point of M there must be a function v defined in some connected neighborhood of it that satisfies $\alpha = dv$. But then $\nabla_X(de^{-v}) = 0$. Thus, if we put $w = e^{-v}$ and restrict our study to that neighborhood, we have the following situation:

$$\begin{gathered} w \in C^\infty(M), \quad \mathbf{z} := g^{-1}(dw), \quad \mathbf{z} \neq 0, \quad g(\mathbf{z},\mathbf{z}) = 0,\\ \nabla \mathbf{z} = 0, \quad \nabla dw = 0,\\ dw \wedge R(.,.,X,Y) = 0, \quad X, Y \in \mathfrak{X}(M),\\ \nabla R = -\frac{2}{w}\, dw \otimes R. \end{gathered} \tag{2.1}$$

3. The local canonical form of the metric.

Proposition 1. *The metric on* $\mathbf{R}^{n+2}$ *has the canonical form*

$$g = du \otimes dv + dv \otimes du + \big(b(\mathbf{x},\mathbf{x}) + 2u\big)\, dv \otimes dv + \sum_{a=1}^{n} \varepsilon_a dx^a \otimes dx^a,$$

and we have

$$\xi = \partial_u, \qquad \alpha = dv.$$

Proof. Since M is connected and there is some point where dw does not vanish, we conclude that dw is everywhere nonzero because $\nabla dw = 0$. Thus, for each $t \in \mathbf{R}$ the subset $H_t = w^{-1}(\{t\})$ is a regular hypersurface of M. Let $\gamma : I \to M$ be a geodesic and put $w(t) = (w \circ \gamma)(t)$; then

$$\begin{aligned}\dot{w} &= \big((dw) \circ \gamma\big)(\dot{\gamma}),\\ \ddot{w} &= \big((\nabla_{\dot{\gamma}} dw) \circ \gamma\big)(\dot{\gamma}) + (dw \circ \gamma)(\nabla_{\dot{\gamma}}\dot{\gamma}) = 0.\end{aligned}$$

So if for some value $t \in \mathbf{R}$, $\dot{\gamma}_t$ is tangent to $H_{w(t)}$, then γ remains in that hypersurface. On the contrary, if $\dot{w}(0) = 1$, then we always have $\dot{w}(t) = 1$.

Let $e_w \in \mathfrak{X}(M)$ be such that $e_w(w) = dw(e_w) = 1$. By inner multiplication of (2.1) with e_w we have

$$R(.,., X, Y) = dw \wedge R(e_w, ., X, Y).$$

Thus $R(Z, W, X, Y) = Z(w)R(e_w, W, X, Y) - W(w)R(e_w, Z, X, Y)$. By using the symmetries of R we can write

$$\begin{aligned}R(e_w, Z, X, Y) &= R(X, Y, e_w, Z)\\ &= X(w)R(e_w, Y, e_w, Z) - Y(w)R(e_w, X, e_w, Z),\end{aligned}$$

so that if we put $m(X, Y) := R(e_w, X, e_w, Y)$, then

$$R = (dw \otimes dw) \wedge m, \tag{3.1}$$

where the wedge stands for the product of double forms.

If $X, Y \in \mathfrak{X}(H_t)$ we have $dw(\nabla_X Y) = \nabla_X(dw(Y)) - (\nabla_X dw)(Y) = 0$ because dw is parallel. Hence $\nabla_X Y \in \mathfrak{X}(H_t)$. Thus, ∇ induces on H_t a torsionless connection whose curvature, due to (3.1), vanishes. Therefore, the parallel displacement along H_t does not depend (locally) on the path. This will allow for a suitable choice of coordinates.

Assume that $H_0 \neq \varnothing$ and that $p \in H_0$. Since $\mathbf{z}_p \in T_pH_0$ because $\mathbf{z}_p(w) = (dw)_p(\mathbf{z}_p) = g(\mathbf{z}_p, \mathbf{z}_p) = 0$, and $\mathbf{z}_p$ is orthogonal to the whole T_pH_0, we can take vectors $e_1(0), \ldots, e_n(0)$ of T_pH_0 such that if we call

$$e_z(0) := \mathbf{z}_p,$$

we have

$$\begin{aligned}
&g(e_z(0), e_z(0)) = 0, \\
&g(e_z(0), e_a(0)) = 0, \quad a = 1, \dots, n, \\
&g(e_a(0), e_b(0)) = \varepsilon_a \delta_{ab}, \quad a, b = 1, \dots, n, \quad \varepsilon_a = \pm 1.
\end{aligned}$$

The two-dimensional subspace of T_pM which is orthogonal to that generated by the vectors $e_1(0), \dots, e_n(0)$ has a nondegenerate metric induced by g and contains $e_z(0)$ which is an isotropic vector. Therefore, a vector $e_w(0) \in T_pM$ exists such that

$$\begin{aligned}
g(e_w(0), e_z(0)) &= 1, \\
g(e_w(0), e_w(0)) &= 0, \\
g(e_w(0), e_a(0)) &= 0, \quad a = 1, \dots, n.
\end{aligned}$$

Now we consider the geodesic γ in M with initial condition $\gamma_0 = p$, $\dot\gamma_0 = e_w(0)$. By parallel displacement of the basis $\{e_z(0), e_w(0), e_a(0)\}$ along γ we obtain the basis $\{e_z(t), e_w(t), e_a(t)\}$ of $T_{\gamma(t)}M$ and we have $\gamma_t \in H_t$ because $\dot\gamma_0(w) = dw(e_w(0)) = g(\mathbf{z}_p, e_w(0)) = g(e_z(0), e_w(0)) = 1$, whence $w(t) = t$. Starting from the point γ_t we make the parallel displacement of $e_a(t)$ along H_t. Since initially $(dw)_p(e_a(0)) = 0$ and dw is parallel, we have that $e_a(t) \in T_{\gamma_t}H_t$. Therefore, that parallel displacement does not depend on the path. Thus, there is a neighborhood V of p on which we have vector fields $e_z = \mathbf{z}$, $e_a \in \mathfrak{X}(V)$ such that $\nabla e_z = 0$, $\nabla_{e_a} e_b = 0$, $\nabla_{e_z} e_a = 0$, $a, b = 1, \dots, n$. Therefore, $[e_z, e_a] = [e_a, e_b] = 0$, $a, b = 1, \dots, n$. Consequently, the flows ϕ^a_s, ϕ^z_s of these vector fields commute. Thus, there is a neighborhood of $0 \in \mathbf{R}^{n+2}$ where there is a well-defined map

$$\psi(z, w, x^1, \dots, x^n) = (\phi^z_z \circ \phi^1_{x^1} \circ \cdots \circ \phi^n_{x^n})(\gamma_w)$$

which is the inverse of a chart for M with coordinates (z, w, x^a) such that the coordinate w is the function w, and such that

$$\partial_z := \frac{\partial}{\partial z} = e_z = \mathbf{z}, \qquad \partial_a := \frac{\partial}{\partial x^a} = e_a, \quad a = 1, \dots, n,$$
$$\partial_w(w) = \frac{\partial}{\partial w}(w) = 1.$$

We put

$$b := g(\partial_w, \partial_w), \qquad s_a := g(\partial_w, \partial_a).$$

Then, having in mind that $g(\mathbf{z}) = g(\partial_z) = dw$, that $g(\partial_z, \partial_z) = g(\mathbf{z}, \mathbf{z}) = 0$ and that $g(\partial_a, \partial_b) = \varepsilon_a \delta_{ab}$, we have

$$\begin{aligned} g = {} & dz \otimes dw + dw \otimes dz + b dw \otimes dw \\ & + \sum_{a=1}^{n} s_a (dw \otimes dx^a + dx^a \otimes dw) + \sum_{a=1}^{n} \varepsilon_a dx^a \otimes dx^a. \end{aligned}$$

Also we have the initial conditions

$$b_{(z=0, x^a=0)} = g(\dot\gamma_w, \dot\gamma_w) = g(\dot\gamma_0, \dot\gamma_0) = g(e_w(0), e_w(0)) = 0, \tag{3.2}$$

$$s_{a(z=0, x^a=0)} = g(e_w(w), e_a(w)) = 0. \tag{3.3}$$

Since $\nabla \mathbf{z} = \nabla \partial_z = 0$, we have immediately $\mathcal{L}_{\partial_z} g = 0$, that is,

$$\frac{\partial b}{\partial z} = \frac{\partial s_a}{\partial z} = 0.$$

Conversely, these conditions guarantee that $\nabla \mathbf{z} = 0$ and also that $\nabla_{\partial z} \partial_{e_a} = 0$, as is easily verified. Now we need to impose the condition $\nabla_{\partial_a} \partial_{e_b} = 0$. The usual formula for Christoffel symbols gives

$$\nabla_{\partial_a} \partial_{e_b} = \frac{1}{2} \left(\frac{\partial s_a}{\partial x^b} + \frac{\partial s_b}{\partial x^a} \right) \partial_z,$$

and we conclude that

$$\frac{\partial s_a}{\partial x^b} + \frac{\partial s_b}{\partial x^a} = 0.$$

But then, if we call

$$s_{abc} := \frac{\partial^2 s_a}{\partial x^b \partial x^c}$$

we have $s_{abc} = s_{acb}$ and $s_{abc} = -s_{bac}$. Thus, as is well known, $s_{abc} \equiv 0$. Since, by (3.3), we have $s_{a(z=0, x^a=0)} = 0$, we conclude that there are some functions $S_{ab}(w)$ such that

$$\begin{aligned} S_{ab}(w) + S_{ba}(w) &= 0, \\ s_a &= \sum_{b=1}^{n} S_{ab}(w) x^b. \end{aligned}$$

From this, we get $\Gamma^b_{wa} = \varepsilon_b S_{ba}(w)$. But along γ we have $(\nabla_{\partial_w}\partial_a)\circ\gamma = \nabla_{\dot\gamma} e_a = 0$; this implies that $\Gamma^b_{wa} = 0$ at the points where $z = x^1 = \cdots = x^n = 0$. And this leads to $S_{ab}(w) = 0$, $\Gamma^b_{wa} \equiv 0$ and $s_a \equiv 0$.

After computation, we get the Christoffel symbols that are not identically zero:

$$\Gamma^z_{ww} = \frac{1}{2}\,\frac{\partial b}{\partial w}, \qquad \Gamma^a_{ww} = -\,\frac{1}{2}\,\varepsilon_a\,\frac{\partial b}{\partial x^a}, \qquad \Gamma^z_{wa} = \frac{1}{2}\,\frac{\partial b}{\partial x^a}.$$

With the aid of these formulae, we can easily compute the components $m_{ab} = R(\partial_w, \partial_a, \partial_w, \partial_b)$ which completely determine the curvature. We get

$$m_{ab} = -\,\frac{1}{2}\,\frac{\partial^2 b}{\partial x^a \partial x^b}. \tag{3.4}$$

Thus the curvature is given by $R = \sum_{a,b=1}^n m_{ab}(dw\wedge dx^a)\otimes(dw\wedge dx^b)$. Then

$$\nabla_{\partial_w} R = \sum_{a,b=1}^n \frac{\partial m_{ab}}{\partial w}\,(dw\wedge dx^\alpha)\otimes(dw\wedge dx^b),$$

$$\nabla_{\partial_c} R = \sum_{a,b=1}^n \frac{\partial m_{ab}}{\partial x^c}\,(dw\wedge dx^a)\otimes(dw\wedge dx^b).$$

Hence, the condition $\nabla R + (2/w)\,dw\otimes R = 0$ requires that

$$\frac{\partial m_{ab}}{\partial w} + \frac{2m_{ab}}{w} = \frac{1}{w^2}\,\frac{\partial w^2 m_{ab}}{\partial w} = 0, \qquad \frac{\partial m_{ab}}{\partial x^c} = 0, \qquad a, b, c, = 1, \dots, n. \tag{3.5}$$

If we bring (3.4) to the last formula, we have

$$\frac{\partial^3 b}{\partial x^a \partial x^b \partial x^c} = 0, \quad a, b, c = 1, \dots, n.$$

Now at the points on which $z = x^1 = \cdots = x^n = 0$ we have $\Gamma^z_{wa} = (1/2)(\partial b/\partial x^a) = 0$. Since, by (3.2), b vanishes at those points, we conclude that $b = \sum_{a,b=1}^n B_{ab}(w)x^a x^b$ and $m_{ab} = -B_{ab}(w)$. The first formula of (3.5) now implies that

$$B_{ab}(w) = \frac{1}{w^2}\,b_{ab}, \quad b_{ab} \in \mathbf{R}.$$

Thus we get the following form of the metric

$$g = dz \otimes dw + dw \otimes dz + \frac{1}{w^2}\, b(\mathbf{x},\mathbf{x})\, dw \otimes dw + \sum_{a=1}^{n} \varepsilon_a\, dx^a \otimes dx^a,$$

where we have put $b(\mathbf{x},\mathbf{x}) = \sum_{a,b=1}^{n} b_{ab}x^a x^b$ and $\mathbf{x} = (x^1, \dots, x^n)$ is the position vector in $\mathbf{R}^n$. Now we substitute $w = e^{-v}$, $u := -e^{-v}z$ and finally obtain the canonical form of the metric given in the statement, which is a pseudo-Riemannian metric defined in all of $\mathbf{R}^{n+2}$. □

4. The curvature and geodesics of this metric.

Proposition 2. *Let $(\mathbf{R}^{n+2}, h, b)$ denote $\mathbf{R}^{n+2}$ equipped with the metric*

$$g = du \otimes dv + dv \otimes du + \big(b(\mathbf{x},\mathbf{x}) + 2u\big)\, dv \otimes dv + \sum_{a,b=1}^{n} h_{ab} dx^a \otimes dx^b.$$

Then its curvature and its Ricci tensor are given by

$$\begin{aligned} R &= -\sum_{a,b=1}^{n} b_{ab}(dv \wedge dx^a) \otimes (dv \wedge dx^b), \\ \mathrm{Ricci} &= -\,tr\,(h^{-1} \cdot b)\, dv \otimes dv. \end{aligned}$$

Except for the straight lines in the hyperplane $v = v_0$, the geodesics are not defined for all t. Thus $(\mathbf{R}^{n+2}, h, b)$ is a connected, simply connected and noncomplete pseudo-Riemannian manifold.

Proof. For brevity, we shall put $h = \sum_{a,b=1}^{n} h_{ab} dx^a \otimes dx^b$ with $\det h_{ab} \neq 0$ so that h is a constant nondegenerate pseudo-Riemannian metric on $\mathbf{R}^n$. Then the metric of M is locally given by

$$\begin{aligned} g &= du \otimes dv + dv \otimes du + \big(b(\mathbf{x},\mathbf{x}) + 2u\big)\, dv \otimes dv + h, \\ g^{-1} &= \partial_u \otimes \partial_v + \partial_v \otimes \partial_u - \big(b(\mathbf{x},\mathbf{x}) + 2u\big)\partial_u \otimes \partial_u + h^{-1}, \end{aligned} \tag{4.1}$$

where $h^{-1} = h^{ab}\partial_a \otimes \partial_b$ and the matrix (h^{ab}) is the inverse of (h_{ab}).

The nonvanishing Christoffel symbols are

$$\Gamma^u_{vv} = b(\mathbf{x},\mathbf{x}) + 2u, \qquad \Gamma^u_{vu} = 1, \qquad \Gamma^u_{va} = \sum_{b=1}^{n} b_{ab}x^b,$$
$$\Gamma^v_{vv} = -1, \qquad \Gamma^a_{vv} = -\sum_{b,c=1}^{n} h^{ab}b_{bc}x^c.$$

It is very easy to verify that this metric satisfies our requirements, i.e., that ∂_u is a degenerate homogeneous pseudo-Riemannian structure in $\mathbf{R}^{n+2}$ with the metric (4.1). The curvature of $(\mathbf{R}^{n+2}, h, b)$ is given by

$$R = -\sum_{a,b=1}^{n} b_{ab}(dv \wedge dx^a) \otimes (dv \wedge dx^b).$$

The Ricci tensor is

$$\mathrm{Ricci} = -\operatorname{tr}(h^{-1}\cdot b)\, dv \otimes dv,$$

and the scalar curvature vanishes. Therefore, for the right choice of dimension and signature, (4.1) is a solution of Einstein's general relativity equations for a universe filled with a swarm of photons, see [**7**, p. 579].

The equations of geodesics are

$$\begin{aligned}
&\ddot{u} + \big(b(\mathbf{x},\mathbf{x}) + 2u\big)\dot{v}^2 + 2b(\mathbf{x},\dot{\mathbf{x}})\dot{v} + 2\dot{u}\dot{v} = 0,\\
&\ddot{v} - \dot{v}^2 = 0,\\
&\ddot{\mathbf{x}} - \dot{v}^2 h^{-1}\cdot b\cdot \mathbf{x} = 0.
\end{aligned}$$

Then $v = v_0 - \ln(1-\dot{v}_0 t)$. If we put primes to represent differentiation with respect to v, then the third equation becomes $\mathbf{x}'' = -\mathbf{x}' + h^{-1}\cdot b\cdot\mathbf{x}$, whose solution is

$$\begin{pmatrix} \mathbf{x}(v) \\ \mathbf{x}'(v) \end{pmatrix} = \exp\begin{pmatrix} 0 & (v-v_0)I \\ (v-v_0)h^{-1}\cdot b & -(v-v_0)I \end{pmatrix}\begin{pmatrix} \mathbf{x}(v_0) \\ \mathbf{x}'(v_0) \end{pmatrix},$$

or undoing the change:

$$\begin{pmatrix} \mathbf{x}(t) \\ (1-\dot{v}_0 t)/\dot{v}_0\, \dot{\mathbf{x}}(t) \end{pmatrix} = \exp\begin{pmatrix} 0 & -\ln(1-\dot{v}_0 t)I \\ -\ln(1-\dot{v}_0 t)h^{-1}\cdot b & \ln(1-\dot{v}_0 t)I \end{pmatrix}\begin{pmatrix} \mathbf{x}_0 \\ (\dot{\mathbf{x}}_0/\dot{v}_0) \end{pmatrix}.$$

With the same change, the first equation is now

$$u'' + 3u' + 2u = -\,b(\mathbf{x}, \mathbf{x}) - 2b(\mathbf{x}, \mathbf{x}').$$

A particular solution is $f(v) = -(1/2)h(\mathbf{x}(v), \mathbf{x}'(v))$. The general solution of $u'' + 3u' + 2u = 0$ is $u = e^{v_0 - v}(A + Be^{v_0 - v})$. Thus,

$$u(t) = (1 - \dot{v}_0 t)\left(A + B(1 - \dot{v}_0 t) - \frac{1}{2\dot{v}_0}\, h(\mathbf{x}(t), \dot{\mathbf{x}}(t))\right),$$

and the constants A, B must be determined from the initial conditions. Of course, these formulae hold when $\dot{v}_0 \neq 0$. If $\dot{v}_0 = 0$, we simply have the equations of straight lines in the hyperplane $v = v_0$, that is, $v(t) = v_0$, $u(t) = \dot{u}_0 t + u_0$, $\mathbf{x}(t) = \dot{\mathbf{x}}_0 t + \mathbf{x}_0$. With the exception of this case, the geodesics are not defined for all t. $\square$

5. The homogeneous pseudo-Riemannian space $(\mathbf{R}^{n+2}, h, b)$. Let $\mathcal{G}$ be $\mathbf{R}^{2n+2}$ with the product

$$\begin{aligned}(5.1)\quad & (a_1, t_1, s_1) \cdot (a_2, t_2, s_2) \\ &\quad = \left(a_1 + a_2 e^{-t_1} + \tau(s_1) \cdot J_{t_1} \cdot s_2,\ t_1 + t_2,\ J_{t_1} \cdot s_2 + s_1\right),\end{aligned}$$

where

$$a_i, t_i \in \mathbf{R}, \quad s_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathbf{R}^n \times \mathbf{R}^n, \quad i = 1, 2,$$
$$\tau\begin{pmatrix} x \\ y \end{pmatrix} = (y \cdot h, -x \cdot h), \quad J_t = \exp\begin{pmatrix} 0 & tI \\ th^{-1} \cdot b & -tI \end{pmatrix}.$$

Let

$$\mathcal{H} = \left\{\left(0, 0, \begin{pmatrix} 0 \\ y \end{pmatrix}\right) : y \in \mathbf{R}^n\right\} \subset \mathcal{G} \tag{5.2}$$

and

$$\mathfrak{m} = \left\{\left(a, t, \begin{pmatrix} x \\ -x \end{pmatrix}\right) : a, t \in \mathbf{R},\ x \in \mathbf{R}^n\right\} \subset \mathfrak{g}. \tag{5.3}$$

Then $\mathcal{H} \cong (\mathbf{R}^n, +)$ is a closed subgroup of $\mathcal{G}$, $\mathfrak{m}$ is an $\mathrm{Ad}\,(\mathcal{H})$-invariant subspace of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ and the inner product $\langle\ ,\ \rangle$ in $\mathfrak{m}$ defined by

$$\left\langle \left(a, t, \begin{pmatrix} x \\ -x \end{pmatrix}\right), \left(a, t, \begin{pmatrix} x \\ -x \end{pmatrix}\right) \right\rangle = at + h(x, x) \tag{5.4}$$

is $\mathrm{Ad}\,(\mathcal{H})$-invariant.

Let $(\mathbf{R}^{n+2}, h, b)$ be as in Proposition 2 and $\mathcal{G}$, $\mathcal{H}$, $\mathfrak{m}$ and $\langle\ ,\ \rangle$ as in (5.1), (5.2), (5.3) and (5.4) above. Then we have

Proposition 3. *$(\mathbf{R}^{n+2}, h, b)$ is isometric to the reductive homogeneous pseudo-Riemannian manifold $\mathcal{G}/\mathcal{H}$ with Lie subspace $\mathfrak{m}$, endowed with the $\mathrm{Ad}\,(\mathcal{H})$-invariant inner product $\langle\ ,\ \rangle$.*

As a first step towards this description of $(\mathbf{R}^{n+2}, h, b)$ as a homogeneous space, we compute the algebra $\mathfrak{k}$ of Killing vector fields of (4.1).

Lemma 4. *The vector field $Z \in \mathfrak{X}(\mathbf{R}^{n+2}, h, b)$ belongs to $\mathfrak{k}$ if and only if it can be written as*

$$\begin{aligned} Z = &\big((p(\mathbf{x}) - k)e^v u - h(q', \mathbf{x}) + ae^{-v}\big)\partial_u \\ &+ (ke^v + l - h(p, \mathbf{x})e^v)\partial_v + B \cdot \mathbf{x} + ue^v p + q. \end{aligned} \tag{5.5}$$

where $a, k, l \in \mathbf{R}$; $p = p^c\partial_c$ with $p^c \in \mathbf{R}$ for $c = 1, \dots, n$; $q = q^a(v)\partial_a$ satisfies $q'' = -q' + h^{-1} \cdot b \cdot q$; $B \in \mathfrak{o}(h) \cap \mathfrak{o}(b)$; and, if $b \neq 0$, then $k = 0$ and $p = 0$.

Proof. Let $Z = U\partial_u + V\partial_v + X^a\partial_a \in \mathfrak{X}(M)$. We shall use the summation convention over the indexes $a, b, c, \dots = 1, \dots, n$. Then

$Z \in \mathfrak{k}$ if and only if the following equations hold:

$$U + \frac{\partial U}{\partial v} + b(\mathbf{x}, X) + \big(b(\mathbf{x}, \mathbf{x}) + 2u\big)\,\frac{\partial V}{\partial v} = 0, \tag{5.6}$$

$$\frac{\partial V}{\partial v} + \frac{\partial U}{\partial u} + \big(b(\mathbf{x}, \mathbf{x}) + 2u\big)\,\frac{\partial V}{\partial u} = 0, \tag{5.7}$$

$$\frac{\partial V}{\partial u} = 0, \tag{5.8}$$

$$\frac{\partial U}{\partial x^a} + \big(b(\mathbf{x}, \mathbf{x}) + 2u\big)\,\frac{\partial V}{\partial x^a} + h_{ab}\,\frac{\partial X^b}{\partial v} = 0, \tag{5.9}$$

$$\frac{\partial V}{\partial x^a} + h_{ab}\,\frac{\partial X^b}{\partial u} = 0, \tag{5.10}$$

$$h_{ac}\,\frac{\partial X^c}{\partial x^b} + h_{bc}\,\frac{\partial X^c}{\partial x^a} = 0. \tag{5.11}$$

From (5.8) we have $V = V(u, \mathbf{x})$. If we bring this to (5.7) and differentiate with respect to u, we get $\partial^2 U/\partial u^2 = 0$. Therefore, taking account of (5.7), we conclude that

$$U = -\,\frac{\partial V(v, \mathbf{x})}{\partial v}\,u + B(v, \mathbf{x}),$$

for some function $B(v, \mathbf{x})$. For brevity we put $X_a := h_{ab}X^b$. Then (5.11) reads $\partial X_a/\partial x^b + \partial X_b/\partial x^a = 0$. So there are functions $A_{ab}(u, v)$, $C_a(u, v)$ with $A_{ab}(u, v) + A_{ba}(u, v) = 0$ such that

$$X_a = A_{ab}(u, v)x^b + C_a(u, v).$$

From (5.10), we have

$$\frac{\partial V(v, \mathbf{x})}{\partial x^a} = -\,\frac{\partial X_a}{\partial u} = -\,\frac{\partial A_{ab}(u, v)}{\partial u}\,x^b - \frac{\partial C_a(u, v)}{\partial u}.$$

By anti-differentiation and having in mind that $A_{ab} + A_{ba} = 0$, we have

$$\begin{gathered} V(v, x) = -\,\frac{\partial C_a(u, v)}{\partial u}\,x^a + C(u, v), \\ \frac{\partial A_{ab}(u, v)}{\partial u} = 0. \end{gathered} \tag{5.12}$$

By differentiation of (5.12) with respect to u, we get $C_a(u,v) = p_a(v)u + q_a(v)$ for some functions $p_a(v), q_a(v)$. Also $\partial C(u,v)/\partial u = 0$. Therefore, the situation is now as follows

$$\begin{aligned} X_a &= A_{ab}(v)x^b + p_a(v)u + q_a(v), \\ V &= -\,p_a(v)x^a + C(v), \\ U &= \big(p'_a(v)x^a - C'(v)\big)u + B(v,\mathbf{x}). \end{aligned}$$

We substitute this in (5.9):

$$\begin{aligned} p'_a(v)u &+ \frac{\partial B(v,\mathbf{x})}{\partial x^a} \\ &- \big(b(\mathbf{x},\mathbf{x}) + 2u\big)\, p_a(v) + A'_{ab}(v)x^b + p'_a(v)u + q'_a(v) = 0. \end{aligned} \tag{5.13}$$

From the coefficient in u we see that $p_a(v) = p_a e^v$ and the numbers p_a can be regarded as the components of a form $p \in (\mathbf{R}^n)^*$. By differentiation of the whole formula with respect to x^b, we get

$$\frac{\partial^2 B(v,\mathbf{x})}{\partial x^a \partial x^b} + A'_{ab}(v) - 2p_a e^v b_{bc}x^c = 0.$$

By interchanging indexes a and b and subtracting, we have

$$A'_{ab}(v) - e^v(p_a b_{bc} - p_b b_{ac})x^c = 0,$$

whence A_{ab} is constant (take values for $x^a = 0$) and $p_b b_{ac} = p_a b_{bc}$. Assume that some of the p_a are not zero, for instance, $p_1 \neq 0$. Then $p_1 b_{ac} = p_a b_{1c}$, whence $b_{ac} = (p_a/p_1)b_{1c}$, and further $b_{1c} = b_{c1} = (p_c/p_1)b_{11}$. Thus,

$$b_{ac} = \frac{p_a p_c}{p_1^2}\, b_{11}.$$

In other words, b is decomposable, and if we call $r := b_{11}/p_1^2$, we have $b(\mathbf{x},\mathbf{x}) = rp(\mathbf{x})^2$ where $p(\mathbf{x}) = p_a x^a$. Substituting in (5.13), we have

$$\begin{aligned} \frac{\partial B(v,\mathbf{x})}{\partial x^a} &= rp_a e^v p(\mathbf{x})^2 - q'_a(v) \\ &= \frac{re^v}{3}\,\frac{\partial p(\mathbf{x})^3}{\partial x^a} - q'_a(v), \\ B(v,\mathbf{x}) &= \frac{re^v}{3}\, p(\mathbf{x})^3 - q'_a(v)x^a + a(v) \\ &= \frac{e^v}{3}\, b(\mathbf{x},\mathbf{x})p(\mathbf{x}) - q'_a(v)x^a + a(v), \end{aligned}$$

for some function $a(v)$. After substitution in (5.6), we have

$$\begin{aligned}(5.14)\quad &\left(b_{ab}x^a h^{bc} p_c e^v - C''(v) + C'(v)\right) u \\ &- \frac{e^v}{3}\, b(\mathbf{x},\mathbf{x}) p(\mathbf{x}) + a'(v) + a(v) - \left(q''_a(v) + q'_a(v)\right) x^a \\ &\qquad + b_{ab}x^a h^{bc}\left(A_{cd}x^d + q_c(v)\right) + b(\mathbf{x},\mathbf{x})C'(v) = 0.\end{aligned}$$

The term of third degree in $\mathbf{x}$ must be zero, that is, $p = 0$ or otherwise $b = 0$. From the factor in u we get $C(v) = ke^v + l$. By differentiation of (5.14) with respect to x^a and taking $x^a = 0$, we see that

$$q'' = -\,q' + q \cdot h^{-1} \cdot b,$$

where $q : \mathbf{R} \to (\mathbf{R}^n)^*$ is given by $q(v) = q_a(v)\, dx^a$. Note that we usually shall consider a bilinear form b as a map $b : \mathbf{R}^n \to (\mathbf{R}^n)^*$. In this same spirit, h^{-1} is conceived as a map $h^{-1} : (\mathbf{R}^n)^* \to \mathbf{R}^n$. By evaluating (5.14) at $x^a = 0$ we see that $a(v) = ae^{-v}$. So (5.14) becomes $b_{ab}h^{bc}A_{cd}x^a x^d + b(\mathbf{x},\mathbf{x})ke^v = 0$ so that $k = 0$ if $b \neq 0$ and $b_{ac}h^{cd}A_{db} + b_{bc}h^{cd}A_{da} = 0$, or equivalently

$$\operatorname{sym}(b \cdot h^{-1} \cdot A) = 0.$$

Let us interpret the conditions upon A. If we put $B := h^{-1} \cdot A \in \mathfrak{gl}(n;\mathbf{R})$, the condition for B to belong to the Lie algebra of the group $O(h)$, that is, the group of linear h-isometries of $\mathbf{R}^n$, is $h(B(\mathbf{v}),\mathbf{w}) + h(\mathbf{v}, B(\mathbf{w})) = h(h^{-1}(A(\mathbf{v})),\mathbf{w}) + h(\mathbf{v}, h^{-1}(A(\mathbf{w}))) = A(\mathbf{v},\mathbf{w}) + A(\mathbf{w},\mathbf{v}) = 0$, and this is the condition of A being skew-symmetric. The condition for B to belong to the Lie algebra of the group $O(b)$ is that $b(B(\mathbf{v}),\mathbf{w}) + b(\mathbf{v},B(\mathbf{w})) = 0$ for all $\mathbf{v},\mathbf{w} \in \mathbf{R}^n$. But the lefthand side is

$$\begin{aligned}b(B(\mathbf{v}),\mathbf{w}) + b(\mathbf{v},B(\mathbf{w})) &= b\big((h^{-1}\cdot A)(\mathbf{v}),\mathbf{w}\big) + b\big(\mathbf{v},(h^{-1}\cdot A)(\mathbf{w})\big) \\ &= (b\cdot h^{-1}\cdot A)(\mathbf{v},\mathbf{w}) + (b\cdot h^{-1}\cdot A)(\mathbf{w},\mathbf{v}) = 0.\end{aligned}$$

Thus the conditions upon A can be expressed as $B \in \mathfrak{o}(h) \cap \mathfrak{o}(b)$.

We now change the notation putting $q := q \cdot h^{-1}$, $p := p \cdot h^{-1}$, $\mathbf{x} := x^a\partial_a$ and considering p, q as vector fields in $\mathbf{R}^{n+2}$ given by $q(u,v,x^1,\dots,x^n) = q^a(v)\partial_a$, $p(u,v,x^1,\dots,x^n) = p^a\partial_a$. Then we get the expression stated in the lemma. $\square$

Proof of Proposition 3. We consider the subspaces of $\mathfrak{k}$ given by the vector fields as (5.5) with the following additional conditions:

$$\begin{aligned}
\mathfrak{g} &= \{Z \in \mathfrak{k} : k = 0,\ B = 0,\ p = 0\},\\
\mathfrak{h} &= \{Z \in \mathfrak{g} : a = l = 0,\ q(0) = 0\},\\
\mathfrak{m} &= \{Z \in \mathfrak{g} : q(0) + q'(0) = 0\}.
\end{aligned}$$

Thus a vector $G \in \mathfrak{g}$ is given as $G = (-h(q', \mathbf{x}) + ae^{-v})\partial_u + l\partial_v + q$, where $a, l \in \mathbf{R}$ and $q = q^a(v)\partial_a$ satisfies $q'' = -q' + h^{-1} \cdot b \cdot q$.

Let $\varkappa : \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n \to \mathfrak{g}$ be the map given by

$$\varkappa(a, l, x, y) = (-h(q', \mathbf{x}) + (1/2)ae^{-v})\partial_u + l\partial_v + q,$$

where $q : \mathbf{R} \to \mathbf{R}^n$ is the solution of the differential equation $q'' = -q' + K \cdot q$, $q(0) = x$, $q'(0) = y$, where we have put $K := h^{-1} \cdot b$. After calculation, we have

$$\begin{aligned}
&[\varkappa(a_1,l_1,x_1,y_1),\ \varkappa(a_2,l_2,x_2,y_2)]\\
&\qquad = \varkappa\big(a_1l_2 - a_2l_1 + 2(h(y_1,x_2) - h(y_2,x_1)),\ 0,\ l_1y_2 - l_2y_1,\\
&\qquad\qquad\qquad l_1y_2 + l_2y_1 + K(l_1x_2 - l_2x_1)\big).
\end{aligned} \tag{5.15}$$

In the course of the computation one encounters the expression

$$\big(h(q_1', q_2) - h(q_2', q_1)\big)\partial_u.$$

But we have

$$\begin{aligned}
\frac{d}{dv}\big(h(q_1', q_2) - h(q_2', q_1)\big) &= h(-q_1' + h^{-1} \cdot b \cdot q_1, q_2)\\
&\quad - h(-q_2' + h^{-1} \cdot b \cdot q_2, q_1)\\
&= - h(q_1', q_2) + h(q_2', q_1)\\
&\quad + b(q_1, q_2) - b(q_2, q_1)\\
&= - h(q_1', q_2) + h(q_2', q_1).
\end{aligned}$$

Hence $(h(q_1', q_2) - h(q_2', q_1))\partial_u = (h(y_1, x_2) - h(y_2, x_1))e^{-v}\partial_u$.

From (5.15), we easily see that $\mathfrak{g}$ and $\mathfrak{h}$ are subalgebras of $\mathfrak{k}$, that $\mathfrak{h} = \varkappa(0, 0, 0, \mathbf{R}^n)$ is abelian and that $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ so that the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is reductive. Also, $\mathfrak{g}^{(3)} = 0$, whence $\mathfrak{g}$ is solvable.

We endow $\mathfrak{m}$ with the inner product $\langle\ ,\ \rangle$ given by g at the origin of $\mathbf{R}^{n+2}$. Thus, if $Y = \varkappa(a, l, x, -x) \in \mathfrak{m}$, we have

$$\langle Y, Y\rangle = al + h(x, x). \tag{5.16}$$

Then

$$\begin{aligned}\langle[\varkappa(0,0,0,y), Y], Y\rangle &= \langle\varkappa(2h(y,x), 0, -y, y), Y\rangle \\ &= lh(y,x) - lh(y,x) = 0.\end{aligned}$$

Therefore, $\langle\ ,\ \rangle$ is $\operatorname{ad}\mathfrak{h}$-invariant.

For describing $\mathfrak{g}$ as a matrix subalgebra, we need some notation. First we put

$$J := \begin{pmatrix} 0 & I \\ K & -I \end{pmatrix}, \qquad J_t := \exp(tJ),$$

where I is the identity in $\mathbf{R}^n$. Now if $s = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbf{R}^n \times \mathbf{R}^n$, we shall write $\tau(s) = (y \cdot h, -x \cdot h)$. Evidently, $J_{t_1} \cdot J_{t_2} = J_{t_1+t_2}$. From the easily verified fact that $h \cdot K = K \cdot h$, it can be directly proved that

$$\tau(s) \cdot J = -\,\tau(J \cdot s + s). \tag{5.17}$$

Now we consider the subspace of $(2n + 2) \times (2n + 2)$ matrices of the form

$$M(a, l, x, y) = \begin{pmatrix} -l & \tau(s) & a \\ 0 & lJ & s \\ 0 & 0 & 0 \end{pmatrix},$$

where $s = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. Then it can be shown at once with the aid of (5.17) that the map

$$\varkappa(a, l, x, y) \longmapsto M(a, l, x, y)$$

is a Lie algebra isomorphism. Accordingly, we shall identify $\mathfrak{g}$ with this matrix Lie algebra and $\mathfrak{m}$ with the subspace

$$\{M(a, l, x, -x) : (a, l, x) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n\}.$$

We consider the subset $\mathcal{G}$ of $(2n + 2) \times (2n + 2)$ real matrices of the following form

$$N(a, t, s) = \begin{pmatrix} e^{-t} & \tau(s) \cdot J_t & a \\ 0 & J_t & s \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$N(a_1,t_1,s_1)\cdot N(a_2,t_2,s_2)$$
$$=\begin{pmatrix} e^{-(t_1+t_2)} & e^{-t_1}\tau(s_2)\cdot J_{t_2}+\tau(s_1)\cdot J_{t_1+t_2} & a_2e^{-t_1}+\tau(s_1)\cdot J_{t_1}\cdot s_2+a_1 \\ 0 & J_{t_1+t_2} & J_{t_1}\cdot s_2+s_1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now

$$\tau(J_{t_1}\cdot s_2+s_1)\cdot J_{t_1+t_2}-e^{-t_1}\tau(s_2)\cdot J_{t_2}-\tau(s_1)\cdot J_{t_1+t_2}$$
$$=\big(\tau(J_{t_1}\cdot s_2)\cdot J_{t_1}-e^{-t_1}\tau(s_2)\big)\cdot J_{t_2}.$$

Hence $\mathcal{G}$ is a group if and only if for every $t\in\mathbf{R}$ and $s\in\mathbf{R}^{2n}$ we have $\tau(J_t\cdot s)\cdot J_t=e^{-t}\tau(s)$. But with the aid of (5.17) we get

$$\frac{d}{dt}\left(\tau(J_t\cdot s)\cdot J_t\right)=\tau(J\cdot J_t\cdot s)\cdot J_t+\tau(J_t\cdot s)\cdot J\cdot J_t=-\,\tau(J_t\cdot s)\cdot J_t.$$

Therefore, the condition holds and $\mathcal{G}$ is a Lie group whose Lie algebra is $\mathfrak{g}$. It is clear that $\mathcal{G}$ is diffeomorphic to $\mathbf{R}^{2n+2}$. Also, let $\mathcal{H}$ be the subgroup of $\mathcal{G}$ given by the matrices

$$N\left(0,0,\begin{pmatrix}0\\z\end{pmatrix}\right),$$

whose Lie algebra is $\mathfrak{h}$. Then $\mathcal{H}\cong(\mathbf{R}^n,+)$. Since

$$N\left(0,0,\begin{pmatrix}0\\z\end{pmatrix}\right)\cdot N\left(a,t,\begin{pmatrix}x\\y\end{pmatrix}\right)=N\left(a+h(x,z),t,\begin{pmatrix}x\\y+z\end{pmatrix}\right),$$

the orbits for the left action of $\mathcal{H}$ on $\mathcal{G}$ can be parametrized by the elements $N\big(a,t,\left(\begin{smallmatrix}x\\-x\end{smallmatrix}\right)\big)$. Since $\mathcal{G}\setminus\mathcal{H}$ is diffeomorphic to $\mathcal{G}/\mathcal{H}$, we see that $\mathcal{G}/\mathcal{H}$ is diffeomorphic to $\mathbf{R}^{n+2}$. Finally it can be proved at once that the metric (5.16) for $\mathfrak{m}$ is $\mathrm{Ad}\,(\mathcal{H})$-invariant. □

Remark. We have dropped a direct summand from $\mathfrak{k}$ consisting of vector fields of the form

$$X=\big(p(\mathbf{x})-k\big)e^v u\partial_u+\big(k-h(p,\mathbf{x})\big)e^v\partial_v+B\cdot\mathbf{x}+ue^vp,$$

with the conditions stated in Proposition 4. Thus, in general, $\mathcal{G}$ is not the whole group of isometries of $(\mathbf{R}^{n+2}, h, b)$.

REFERENCES

1. W. Ambrose and I.M. Singer, *On homogeneous Riemannian manifolds*, Duke Math. J. **25** (1958), 647–669.

2. E. Abbena and S. Garbiero, *Almost Hermitian homogeneous structures*, Proc. Edinburgh Math. Soc. (2) **31** (1988), 375–395.

3. P.M. Gadea, A. Montesinos Amilibia and J. Muñoz Masqué, *Characterizing the complex hyperbolic space by Kähler homogeneous structures*, Math. Proc. Cambridge Philos. Soc., to appear.

4. P.M. Gadea and J.A. Oubiña, *Homogeneous pseudo-Riemannian structures and homogeneous almost para-Hermitian structures*, Houston J. Math. **18** (1993), 449–465.

5. ———, *Homogeneous almost para-Hermitian structures*, Indian J. Pure Appl. Math. **26** (1995), 351–362.

6. ———, *Reductive homogeneous pseudo-Riemannian manifolds*, Monatsh. Math. **124** (1997), 17–34.

7. C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, W.H. Freeman and Company, New York 1973.

8. F. Tricerri and L. Vanhecke, *Homogeneous structures on Reimannian manifolds*, London Math. Soc. Lecture Note Ser. **83**, Cambridge Univ. Press, Cambridge, 1983.

Departamento de Geometría y Topología, Universidad de Valencia, 46100 Burjasot (Valencia), Spain
E-mail address: montesinos@iec.csic.es