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FINITE GROUPS WITH EXACTLY TWO CONJUGACY CLASSES OF THE SAME ORDER

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ABSTRACT. We show that the only finite groups having a nilpotent derived subgroup and having exactly two conjugacy classes of the same order are $\mathbf{Z}_2$, D_{10}, the dihedral group of order 10, and A_4. As a corollary, the only supersolvable finite groups having exactly two conjugacy classes of the same order are $\mathbf{Z}_2$ and D_{10}.

1. Introduction. Investigation of the S_3-conjecture, that S_3 is the only finite group whose conjugacy classes all have different orders began with Markel [**7**] in 1973 and continued through a string of papers [**1**], [**3**], [**6**], [**9**] each examining a special case. Recently, the conjecture was confirmed in the class of finite solvable groups by Zhang [**10**] and, independently, by Knörr, Lempken and Thielcke [**5**].

Weakening the hypothesis on conjugacy class orders by allowing exactly two conjugacy classes to have the same order, one obtains a larger supply of examples. For instance, in the symmetric and alternating groups, where conjugacy classes are easy to compute [**8**, 11.1.1, 11.1.5], we find S_2, S_4, S_5, A_4, A_5 and A_7 each have exactly two conjugacy classes of the same order and those are the only such groups among the symmetric and alternating groups.

Here we begin a systematic study of which finite groups satisfy the weakened hypothesis where Markel began: with supersolvable groups. Specifically, we show that the only finite groups having a nilpotent derived subgroup and having exactly two conjugacy classes of the same order are $\mathbf{Z}_2$, D_{10} (the dihedral group of order 10) and A_4. As a corollary, the only supersolvable finite groups having exactly two conjugacy classes of the same order are $\mathbf{Z}_2$ and D_{10}.

The proof uses elementary techniques. However, the fact that a group satisfying our weakened hyypothesis is not necessarily a rational group

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(defined below) results in a number of cases, each requiring detailed analysis.

2. Some preliminaries. All groups considered herein are finite. For any element g of a group G, $[g]$ denotes the conjugacy class of g in G and $N_G(g)$ abbreviates $N_G(\langle g \rangle)$. Other notation is standard, see [**2**].

Recall a finite group G is rational if every complex character of G is rational valued. The following characterization is fundamental.

Proposition 2.1 [**4**, Proposition 9]. *G is a rational group if and only if for all $y \in G$ and every positive integer k, $\langle y \rangle = \langle y^k \rangle$ implies that $[y] = [y^k]$, that is, $\mathrm{Aut}(\langle y \rangle) \cong N_G(y)/C_G(y)$.*

Examining the proof of Proposition 2.1, we see that the following element-wise version holds.

Proposition 2.2. *Let G be a finite group, and let t be an element of G such that $\langle t \rangle = \langle t^k \rangle$ implies $[t] = [t^k]$ for every positive integer k, that is, $N_G(t)/C_G(t) \cong \mathrm{Aut}(\langle t \rangle)$, then $\chi(t)$ is rational for every complex character χ of G.*

We need three facts about rational groups.

Proposition 2.3 [**7**, Proposition 1]. *If G is a rational group and N is a normal subgroup of G, then G/N is rational.*

Proposition 2.4 [**7**, Corollary 2]. *If G is an abelian rational group, then G is an elementary abelian 2-group.*

Proposition 2.5 [**7**, Proposition 1.31]. *If G is a rational group with G' nilpotent, then G is a $\{2, 3\}$-group.*

The next result will be useful in locating distinct conjugacy classes having the same order.

Proposition 2.6. *Let N be a normal subgroup of a group G such that G/N is cyclic of order $n > 2$. If $G/N = \langle Nt \rangle$, then there is a positive integer k such that $\langle t \rangle = \langle t^k \rangle$ and $[t] \neq [t^k]$. Suppose further that there is a positive integer m such that $n > 2m$ and $\langle t \rangle \neq \langle t^m \rangle$. Then there are positive integers k and l such that $\langle t \rangle = \langle t^k \rangle$ and $\langle t^m \rangle = \langle t^{ml} \rangle$, but $[t]$ $[t^k]$, $[t^m]$ and $[t^{ml}]$ are all distinct. Moreover, $|[t]| = |[t^k]|$ and $|[t^m]| = |[t^{ml}]|$.*

Proof. Let ρ denote the canonical homomorphism $G \to G/N$. Let $\Psi : G/N \to \mathbf{C}$ be the natural isomorphism from G/N to $\langle e^{(2\pi i)/n} \rangle$ defined by $\Psi(Nt^k) = e^{(2\pi k i)/n}$ for every positive integer k. Let $\Phi = \Psi \circ \rho$, then Φ is a linear character of G. Since $n > 2$, $\Phi(t) = e^{(2\pi i)/n}$ is not a rational number. Thus, Proposition 2.2 implies that there exists a positive integer k such that $\langle t \rangle = \langle t^k \rangle$ and $[t] \neq [t^k]$.

Suppose, in addition, that there is a positive integer m such that $n > 2m$ and $\langle t \rangle \neq \langle t^m \rangle$. Then $\Phi(t^m) = e^{(2\pi m i)/n}$ is also not rational so applying Proposition 2.2 again we get that there exists an integer l such that $\langle t^m \rangle = \langle t^{ml} \rangle$ and $[t^m] \neq [t^{ml}]$. To see that $[t]$, $[t^k]$, $[t^m]$ and $[t^{ml}]$ are all distinct it suffices to note that t and t^m do not have the same order.

Inasmuch as $\langle t \rangle = \langle t^k \rangle$, $C_G(t) = C_G(t^k)$. Hence, $|[t]| = |[t^k]|$. Similarly, $|[t^m]| = |[t^{ml}]|$.

Lemma 2.7. *Suppose $[x]$ and $[w]$ are the only two distinct conjugacy classes of a group G having the same order and $z \notin [x] \cup [w]$, then $N_G(z)/C_G(z) \cong \mathrm{Aut}\,(\langle z \rangle)$.*

Proof. It suffices to show any two generators of $\langle z \rangle$ are conjugate in G. Assume $\langle z \rangle = \langle z^k \rangle$, then $C_G(z) = C_G(z^k)$. Therefore, $|[z]| = |[z^k]|$. By hypothesis, $[z] = [z^k]$, otherwise, we would have a second pair of distinct conjugacy classes having the same order.

Finally, we record a well-known lemma for reference.

Lemma 2.8. *If H is a subgroup of G and $H \cap G' = 1$, then no two distinct elements of H are conjugate in G.*

Proof. If h, $h^g \in H$, then $[h, g] \in H \cap G' = 1$.

3. Main theorem.

Theorem 3.1. *G is a finite group with exactly two conjugacy classes of the same order whose commutator subgroup G' is nilpotent if and only if G is isomorphic to either $\mathbf{Z}_2$, A_4 or D_{10}, the dihedral group of order* 10.

Corollary 3.2. *G is a supersolvable finite group with exactly two conjugacy classes of the same order if and only if G is isomorphic to either $\mathbf{Z}_2$ or D_{10}.*

Before proving the theorem, we show how Corollary 3.2 follows from Theorem 3.1. If G satisfies the hypotheses of the corollary, then G' is nilpotent [**8**, 7.2.13]. To verify the corollary, it is sufficient to note that $\mathbf{Z}_2$ and D_{10} are supersolvable whereas A_4 is not supersolvable.

In the proof of the theorem, we repeatedly apply the same general strategy:

1. Determine which primes may divide the order of a group that satisfies the hypotheses in each case.

2. Eliminate the possibility of certain large conjugacy class orders.

3. Using our hypothesis that there can be at most one duplicate conjugacy class order, obtain an upper bound on the order of the group from the class equation.

4. Examine the resulting groups.

Proof of Theorem 3.1. Obviously $\mathbf{Z}_2$ is a group with exactly two conjugacy classes of the same order with a nilpotent commutator subgroup. Also, the commutator subgroup of A_4 is abelian and the conjugacy classes are [1], [(12)(34)], [(123)], [(132)] having orders 1, 3, 4, 4, respectively. For D_{10}, the commutator subgroup is cyclic of order 5. The involutions form a conjugacy class of order 5 while the elements of order 5 form two classes, each of order 2. The conjugacy class orders of D_{10}, therefore, are 1, 2, 2 and 5.

We prove the converse by proving four lemmas corresponding to four possible group configurations. We show no group having the first configuration can satisfy our hypothesis on conjugacy class orders while the other three each lead to one of the groups in the conclusion.

Lemma 3.3. *Assume G is a rational group with G' nilpotent and $Z(G) = 1$. Then G cannot have exactly two conjugacy classes of the same order.*

Proof. In this case we essentially follow the proof of Markel. Assume G does have exactly two conjugacy classes of the same order.

By Proposition 2.5, G is a $\{2,3\}$-group, so $|G| = 2^a 3^b$ for some nonnegative integers a and b. Since $Z(G) = 1$, a and b are both positive. Let S be an arbitrary Sylow 2-subgroup of G. We first derive some preliminary results that will prove useful in determining the possible conjugacy class orders of G in this case.

$$G = G'S. \tag{3.1}$$

By Propositions 2.3 and 2.4, G/G' is an elementary abelian 2-group. Hence, $G = G'S$.

$$S \cap G' = 1, \quad \begin{aligned}&S \text{ is an elementary abelian and}\\ &G' \text{ is a Sylow 3-subgroup of } G.\end{aligned} \tag{3.2}$$

If $S \cap G' > 1$, then let z be any nonidentity element of $Z(S) \cap G' \leq Z(G')$, utilizing the nilpotency of G'. Thus, $G = G'S \leq C_G(z)$, contrary to the hypothesis. Therefore, $S \cap G' = 1$. The rest now follows from (3.1).

$$C_G(G') \leq G'. \tag{3.3}$$

Since S is abelian, $S \cap C_G(G') \leq Z(G) = 1$. Therefore, $C_G(G')$ is a 3-group and the result follows.

$$C_G(S) = S. \tag{3.4}$$

By (3.1) and (3.2), $C_G(S) = S(G' \cap C_G(S))$. Assume $G' \cap C_G(S) > 1$, and let y be an element of order three in $G' \cap C_G(S)$. Then $|N_G(y)/C_G(y)|$ is odd. However, Proposition 2.1 implies that $|N_G(y)/C_G(y)| = 2$.

With the preliminaries complete, we'll count the elements of G according to their order. We begin with the 3-elements. Since G' is a normal Sylow 3-subgroup of G, $|G'| = 3^b$ is the number of 3-elements.

Every conjugacy class containing 2-elements contains a representative in S by the Sylow theorems. By (3.2) and Lemma 2.8 the number of nonidentity 2-elements in G is given by $\sum_{s \in S^\sharp} |[s]|$, $S^\sharp$ denoting the set of nonidentity elements of S. S is abelian so, for every $s \in S^\sharp$, $|[s]| = [G : C_G(s)] = 3^j$ for some integer j such that $1 \leq j \leq b$.

Next, let y be any *mixed* element of G, that is, an element whose order is divisible by 6, then $|[y]|$ divides $2^{a-1}3^{b-1}$.

Suppose 3^b divides $|C_G(y)|$. Then $C_G(y)$ contains G'. That means $y \in C_G(G') \leq G'$, which contradicts that y is a mixed element. Thus 3 divides $|[y]|$.

Now suppose 2^a divides $|C_G(y)|$; then we may assume $S \leq C_G(y)$. That means $y \in C_G(S) = S$, a contradiction. Therefore, 2 divides $|[y]|$. It follows that $|[y]| = 2^i 3^j$ with $1 \leq i \leq a-1$ and $1 \leq j \leq b-1$.

Let $[w]$ be one of the two conjugacy classes of equal order. Counting the elements of G according to their order (3-elements, 2-elements not in $[w]$, mixed elements not in $[w]$, then if necessary, the elements in $[w]$), we obtain an upper bound for the order of G from the class equation:

$$|G| = 2^a 3^b \leq 3^b + \sum_{j=1}^{b} 3^j + \sum_{i=1}^{a-1} \sum_{j=1}^{b-1} 2^i 3^j + d$$

for some integer d. If w is a 3-element, we may take $d = 0$ for 3^b is the total number of 3-elements. If w is a 2-element or a mixed element, take $d = 3^b$ or $2^{a-1}3^{b-1}$, respectively, because $|[w]| \leq d$ in each case.

If $d = 0$, the inequality is equivalent to

$$2^a - 1 \leq 3^{b-1}(3 - 2^a),$$

which implies $a = 1$. Hence, if the duplicate ordered conjugacy class contains 3-elements, then $a = 1$.

If $d = 3^b$ or $2^{a-1}3^{b-1}$, the inequality is, respectively, equivalent to

$$2^a - 1 \le 3^{b-2}(15 - 3 \cdot 2^a)$$
$$2^a - 1 \le 3^{b-2}(9 - 2^{a+1})$$

either of which implies $a \le 2$.

Suppose that $a = 1$, that is, $|S| = 2$. Let $S = \langle s \rangle$, and let τ_s be the automorphism of G' induced by conjugation by s. Suppose that $y \in G'$ is a fixed point of τ_s. Then $y \in C_G(s) \cap G' = C_G(S) \cap G' = S \cap G' = 1$. Hence, τ_s is a fixed point free automorphism of order 2 and [**2**, 10.1.4] implies G' is abelian. Therefore, $|[y]| \le 2$, for all $y \in G'$.

Since there can be at most one repeated conjugacy class order for 3-elements, $|G'| \le 1 + 2 + 2 = 5$. As G' is a 3-group, $|G'| = 3$, hence $|G| = 6$. However, the only two nonisomorphic groups of order 6, S_3 and $\mathbf{Z}_6$, do not satisfy the hypothesis of the theorem. Hence, $a \ne 1$ and, furthermore, no conjugacy class of repeated order contains 3-elements. (We showed that can happen only when $a = 1$).

Assume that $a = 2$. Further suppose that there is an element s in S such that $|[s]| = 3^b$. Then $S = C_G(s)$. Arguing exactly as above, G' is abelian and so, for any element y in G', $|[y]|$ divides $|S| = 4$. Therefore, $|G'| \le 1 + 2 + 4 = 7$. Once again, $|G'| = 3$. Thus, for all t in S, $[t]$ has order either 1 or 3. By (3.2) and Lemma 2.8, no two elements of S are conjugate, contradicting our hypothesis on the number of repeated conjugacy class orders.

Therefore, there are no conjugacy classes of order 3^b. Hence, the order of the duplicate conjugacy class cannot exceed $2 \cdot 3^{b-1}$. Counting the elements of G according to their order once again, we have

$$\begin{aligned} |G| = 2^2 3^b &\le 3^b + \sum_{j=1}^{b-1} 3^j + \sum_{j=1}^{b-1} 2 \cdot 3^j + 2 \cdot 3^{b-1} \\ &= 3^b + \left(\frac{3}{2}\right)(3^b - 3) + 2 \cdot 3^{b-1}. \end{aligned}$$

Dividing both sides of the inequality by $2^2 3^b$, we get

$$1 \le \frac{1}{4} + \frac{3}{8}\left(1 - \frac{3}{3^b}\right) + \frac{1}{6} < 1.$$

That contradiction completes the proof of Lemma 3.3.

Lemma 3.4. *Assume G is a rational group with G' nilpotent and $Z(G) > 1$. If G has exactly two conjugacy classes of the same order, then $G \cong \mathbf{Z}_2$.*

Proof. If $|Z(G)| > 2$, there are at least three distinct conjugacy classes of order 1. Hence $|Z(G)| = 2$. Thus, for every $y \in G$, 2^a does not divide $|[y]|$.

Now suppose there exists $y \in G$ such that $|[y]| = 2^{a-1}3^b$. That means $|C_G(y)| = 2$, so $C_G(y) = Z(G)$, which implies $C_G(y) = G$ and G is isomorphic to $\mathbf{Z}_2$. Hence we may assume there are no conjugacy classes of order $2^{a-1}3^b$. We can obtain an upper bound on the order of G by summing the possible orders of its conjugacy classes, adding 1 twice for the classes of the two central elements and subtracting the impossible class order:

$$
\begin{aligned}
|G| = 2^a 3^b &\leq 1 + \sum_{i=0}^{a-1}\sum_{j=0}^{b} 2^i 3^j - 2^{a-1}3^b \\
&= 1 + (2^a - 1)\frac{3^{b+1}-1}{2} - 2^{a-1}3^b.
\end{aligned}
\tag{3.5}
$$

Dividing both sides of inequality (3.5) by $2^a 3^b$, we get a contradiction, thereby completing the proof of Lemma 3.4.

Henceforth, assume G is not rational; G has exactly two conjugacy classes of the same order and G' is nilpotent. We establish some notation and a preliminary result for use in the final two lemmas. Proposition 2.1 implies that there exist an element x in G and a positive integer m such that $\langle x\rangle = \langle x^m\rangle$ but $[x] \neq [x^m]$. Since $C_G(x) = C_G(x^m)$, $[x]$ and $[x^m]$ are the two conjugacy classes that have the same order.

Next we show $Z(G) = 1$. For, suppose $Z(G) > 1$, then as in Lemma 3.4, $Z(G)$ has order 2. Furthermore, $x \in Z(G)$, for otherwise we would have two pairs of conjugacy classes of the same order. Thus, $\langle x^m\rangle = \langle x\rangle \leq Z(G)$ which implies $x = x^m$, a contradiction.

Lemma 3.5. *Assume G is not rational; G' is nilpotent and $G = G'S$ where S is a Sylow 2-subgroup of G. If G has exactly two conjugacy classes of the same order, then $G \cong D_{10}$.*

Proof.

$$S \cap G' = 1. \tag{3.6}$$

By the nilpotence of G', $Z(S) \cap G' \leq Z(S) \cap Z(G') \leq Z(G) = 1$.

$$S \cong G/G' \quad \text{and, therefore, } S \text{ is abelian.} \tag{3.7}$$

That follows immediately from our hypotheses and (3.6).

(3.8) If z is an element of prime order in $Z(G')$, then $N_G(z)/C_G(z)$ is cyclic and isomorphic to a factor group of a subgroup of S.

Suppose z has prime order q. Then $N_G(z)/C_G(z)$ is isomorphic to a subgroup of $\mathrm{Aut}\,(\langle z\rangle) \cong \mathbf{Z}_{q-1}$. Since $G' \leq C_G(z)$, $N_G(z)/C_G(z)$ is isomorphic to a factor group of a subgroup of $G/G' \cong S$.

$$\text{If } S \text{ is elementary abelian, then } G \cong D_{10}. \tag{3.9}$$

Let q be any prime dividing the order of G'. We know q is odd by (3.6). Let z be an element of order q in $Z(G')$. By (3.8), $N_G(z)/C_G(z)$ is either trivial or isomorphic to $\mathbf{Z}_2$.

Assume $N_G(z)/C_G(z)$ is trivial. Then $N_G(z) = C_G(z)$. It follows that no two distinct elements in $\langle z\rangle$ are conjugate. If $q > 3$, then z, z^2 and z^3 are all generators for $\langle z\rangle$, and $[z], [z^2]$ and $[z^3]$ are pairwise disjoint conjugacy classes of equal order, which contradicts our hypothesis on the number of repeated conjugacy class orders. Therefore, in case $N_G(z)/C_G(z) = 1$, $q = 3$.

Assume $N_G(z)/C_G(z) \cong \mathbf{Z}_2$. Let y be an element of $N_G(z)$ that does not centralize z. Then y induces a fixed point free automorphism of order 2 on $\langle z\rangle$. By [**2**, 10.1.4], $(z^k)^y = (z^k)^{-1}$ for every $z^k \in \langle z\rangle$. Hence, two distinct elements of $\langle z\rangle$ are conjugate if and only if they are inverses. If $q > 5$, then $[z], [z^2]$ and $[z^3]$ are pairwise disjoint conjugacy classes of the same order, a contradiction. Thus, if $N_G(z)/C_G(z) \cong \mathbf{Z}_2$, q is either 3 or 5.

In either case, G' is a $\{3,5\}$-group.

Suppose that both 3 and 5 divide the order of G'. There exists elements u and t in $Z(G')$ of order 5 and 15, respectively, so u and t cannot both be conjugate to a generator of $\langle x\rangle$.

If u is not conjugate to any generator of $\langle x\rangle$, then by Lemma 2.7 $N_G(u)/C_G(u) \cong \operatorname{Aut}(\langle u\rangle) \cong \mathbf{Z}_4$, a contradiction since we are assuming S is elementary abelian. If t is not conjugate to any generator of $\langle x\rangle$, we obtain a similar contradiction.

G is not a 2-group, since $Z(G) = 1$. Hence either 3 or 5 divides the order of G. In either case we will see that one can mimic the proof of Lemma 3.3 to derive the desired conclusions. Since S is abelian and $G = G'S$, we get that $S \cap C_G(G') = 1$ and therefore $C_G(G') \leq G'$ by the same argument given in Lemma 3.3. In order to obtain all of the preliminary results utilized in the proof of Lemma 3.3 we must also show that $C_G(S) = S$.

Suppose that $G' \cap C_G(S) > 1$, and let y be a nonidentity element of prime order in $G' \cap C_G(S)$. Then y has order either 3 or 5. Let t be any involution in S. Since y commutes with t, ty has order either 6 or 10. It follows that y and ty cannot both be conjugate to a generator of $\langle x\rangle$.

If y is not conjugate to any generator of $\langle x\rangle$, then by Lemma 2.7, once again, $N_G(y)/C_G(y) \cong \operatorname{Aut}(\langle y\rangle)$, which has order either 2 or 4. However, $S \leq C_G(y)$, so $|N_G(y)/C_G(y)|$ is odd, a contradiction. If ty is not conjugate to any generator of $\langle x\rangle$, we obtain a contradiction by the same argument. Hence, $G' \cap C_G(S) = 1$ and so $C_G(S) = S$.

The proof of Lemma 3.3 shows G cannot be a $\{2,3\}$-group. Therefore, $|G| = 2^a5^b$ for some positive integers a and b.

By imitating the proof of Lemma 3.3, we bound the order of G:

$$|G| = 2^a5^b \leq 5^b + \sum_{j=1}^{b} 5^j + \sum_{i=1}^{a-1}\sum_{j=1}^{b-1} 2^i5^j + d$$

where $d = 0$, 5^b or $2^{a-1}5^{b-1}$. In all three cases, simple algebra shows $a = 1$.

Still following the proof of 3.3, we can deduce $|G'| = 5$ and $|G| = 10$. Therefore, G is isomorphic to either $\mathbf{Z}_{10}$ or D_{10}. As $\mathbf{Z}_{10}$ does not satisfy our hypotheses, $G \cong D_{10}$.

We complete the proof by establishing that S is elementary abelian.

Assume S is not elementary abelian. Then $S \cong \mathbf{Z}_{2^{r_1}} \times \mathbf{Z}_{2^{r_2}} \times \cdots \times \mathbf{Z}_{2^{r_n}}$ for some positive integers $r_1, \ldots, r_n$, where, without loss of generality, $r_1 > 1$. By (3.7) there exists a normal subgroup N of G such that $N/G' \cong \mathbf{Z}_{2^{r_2}} \times \cdots \times \mathbf{Z}_{2^{r_n}}$ and $G/N \cong \mathbf{Z}_{2^{r_1}}$. That means there exists an element t in S such that Nt generates G/N. Proposition 2.6 implies that a positive integer k exists such that $\langle t \rangle = \langle t^k \rangle$ but $[t] \neq [t^k]$. Thus, by our hypothesis on the number of conjugacy classes of repeated order, x must be conjugate to some power of t. Hence x is a 2-element and we may assume $x \in S$.

Now suppose that $r_j > 2$ for some $1 \leq j \leq n$. Without loss of generality we may assume $r_1 > 2$. With N and t as above, Proposition 2.6 (with $m = 2$) implies that there exist positive integers k and l such that $\langle t \rangle = \langle t^k \rangle$ and $\langle t^2 \rangle = \langle t^{2l} \rangle$, but $[t] \neq [t^k]$ and $[t^2] \neq [t^{2l}]$, a contradiction. Hence, S contains no elements of order greater than 4.

With that structure, if S is not isomorphic to $\mathbf{Z}_4$, then it is easy to find elements r and s in S of order 4 such that $\langle r \rangle \neq \langle s \rangle$. By Lemma 2.8, $[r], [r^{-1}], [s]$ and $[s^{-1}]$ are all different, a contradiction again. Therefore, $S \cong \mathbf{Z}_4$, from which it follows that $S = \langle x \rangle$ and $[x]$ and $[x^{-1}]$ are the only distinct conjugacy classes having the same order.

Let q be any prime dividing the order of G' and z an element of order q in $Z(G')$. By (3.6) q is odd. Thus z is not conjugate to any power of x. Therefore, $N_G(z)/C_G(z) \cong \mathrm{Aut}(\langle z \rangle) \cong \mathbf{Z}_{q-1}$ by Lemma 2.7. By (3.8) $N_G(z)/C_G(z)$ is isomorphic to either $\mathbf{Z}_2$ or $\mathbf{Z}_4$, so q is either 3 or 5. Hence, G' is a $\{3, 5\}$-group.

Suppose both 3 and 5 divide $|G'|$, and let z be an element of order 15 in $Z(G')$. Since z is not a 2-element, $N_G(z)/C_G(z) \cong \mathrm{Aut}(\langle z \rangle) \cong \mathbf{Z}_2 \times \mathbf{Z}_4$, contradicting $S \cong \mathbf{Z}_4$.

Suppose G' is a 5-group. Let r be any nonidentity element of G'. Then $|N_G(r)/C_G(r)| = |\mathrm{Aut}(\langle r \rangle)|$ is divisible by 4. It follows that $|C_G(r)|$ is odd and, consequently, $C_G(x) = C_G(x^2) = C_G(x^{-1}) = S$. Thus, $|[x]| = |[x^2]| = |[x^{-1}]|$, a contradiction by Lemma 2.8.

Suppose G' is a 3-group. Then $|G| = 2^2 3^b$ for some positive integer b. We first show $b > 1$. If $b = 1$, then $G/C_G(G')$ embeds in $\mathrm{Aut}\,(G') \cong \mathbf{Z}_2$. Thus, $x^2 \in C_G(G') \cap S \leq Z(G)$, a contradiction.

Next we'll show G' has a small center. Let $z \in Z(G')$, then $|[z]|$ divides 4. The orders of the conjugacy classes of elements of G' are all distinct, so the only possibility is $|Z(G')| = 1{+}2$. Furthermore, if $z \neq 1$, then $x^2 \in C_G(z)$ and $z^x = z^{-1}$ since $|G/C_G(z)| = |N_G(z)/C_G(z)| = 2$.

Following our standard strategy, we now show that large conjugacy classes cannot exist. Suppose $|[y]| = 2^2 3^{b-1}$. Then $C_G(y)$ has order 3. Hence, $Z(G') = C_G(y)$ and $G' \leq C_G(y)$, contradicting the fact that $b > 1$. Suppose $|[y]| = 2 \cdot 3^b$. By a similar argument, $C_G(y)$ has order 2 and $S \leq C_G(y)$, a contradiction.

It will take longer to prove that there are no conjugacy classes of order 3^b, but assuming we have shown that, we can obtain a familiar contradiction. The remaining possible conjugacy class orders are 3^i where $0 \leq i \leq b-1$, $2 \cdot 3^i$ where $0 \leq i \leq b-1$ and $2^2 3^i$ where $0 \leq i \leq b-2$. Let $2^k 3^j$ be the repeated conjugacy class order. We see that $j < b$. Thus, from the class equation,

$$\begin{aligned}
2^2 3^b &\leq \sum_{i=0}^{b-1} 3^i + \sum_{i=0}^{b-1} 2 \cdot 3^i + \sum_{i=0}^{b-2} 2^2 3^i + 2^k 3^j \\
&= \frac{3^b - 1}{2} + (3^b - 1) + 2(3^{b-1} - 1) + 2^k 3^j \\
&< \frac{3^b}{2} + 3^b + 2 \cdot 3^{b-1} + 2^2 3^j.
\end{aligned}$$

Dividing both sides by $2^2 3^b$,

$$1 < \frac{1}{8} + \frac{1}{4} + \frac{1}{6} + \frac{1}{3^{b-j}} \leq \frac{13}{24} + \frac{1}{3} < 1.$$

Therefore, to complete the proof, it suffices to show that there are no conjugacy classes of order 3^b. Suppose $|[y]| = 3^b$. Then by conjugating if necessary, $C_G(y) = S = \langle x \rangle$. As noted earlier, $Z(G') \leq C_G(x^2)$ and so $y \neq x^2$. Thus, $y = x$ or $y = x^{-1}$ and the two distinct conjugacy classes of the same order, $[x]$ and $[x^{-1}]$ each have order 3^a.

Now suppose there exists $w \in G$ with $|[w]| = 3^j$ for some j, $1 \leq j \leq b$. Then without loss of generality, $S \leq C_G(w)$. That means

$w \in C_G(S) = S$. We conclude $w = x^2$. Thus, the only conjugacy classes of order a power of 3 are $[1], [x^2], [x]$ and $[x^{-1}]$ of respective orders 1, 3^j, 3^b and 3^b.

Assume $j = b-1$, then $|C_G(x^2)| = 2^2 3$. Consequently, $C_G(x^2) = Z(G')S$. Assume further that there exists an element u with $|[u]| = 2 \cdot 3^k$, $1 \leq k \leq b-1$. Then u is not a 2-element nor is u in $Z(G')$. However, by conjugating if necessary, we may assume $x^2 \in C_G(u)$ so $u \in C_G(x^2) = Z(G')S$. It follows that $\langle u \rangle = Z(G')(\langle u \rangle \cap S) = Z(G')\langle x^2 \rangle$. In particular, $C_G(u) \leq C_G(x^2)$, but $C_G(u) \neq C_G(x^2)$ under our hypotheses on repeated conjugacy class orders. We conclude $|C_G(u)| = 6$ and $|[u]| = 2 \cdot 3^{b-1}$. Thus, under the assumption that $j = b-1$, any conjugacy class having order 2 times a positive power of 3 actually has order $2 \cdot 3^{b-1}$. Summing the possible class orders in this case, we obtain from the class equation

$$\begin{aligned} 2^2 3^b &\leq 1 + 3^{b-1} + 3^b + 3^b + 2 + 2 \cdot 3^{b-1} + \sum_{i=0}^{b-2} 2^2 3^i \\ &< 1 + 3^{b-1} + 3^b + 3^b + 2 + 2 \cdot 3^{b-1} + 2 \cdot 3^{b-1}. \end{aligned}$$

Dividing both sides by $2^2 3^b$, we have a contradiction (barely).

In the preceding two paragraphs we have shown that if there is a conjugacy class of order 3^j, $1 \leq j < b$, then there is only one such class, namely, $[x^2]$, and j must be less than $b-1$. Now the remaining possible conjugacy class orders are 1, 3^j, 3^b, 3^b, $2 \cdot 3^i$ for $0 \leq i \leq b-1$ and $2^2 3^i$ for $0 \leq i \leq b-2$. Again, the class equation and a little arithmetic provide a contradiction.

Lemma 3.6. *Assume G is not rational; G' is nilpotent and $G'S < G$ where S is a Sylow 2-subgroup of G. If G has exactly two conjugacy classes of the same order, then $G \cong A_4$.*

Proof.

(3.10) $G/(G'\langle x \rangle)$ is an elementary abelian 2-group and $G = SG'\langle x \rangle$.

Hence, x is not an element of $G'S$.

For every $y \notin G'\langle x \rangle$, y is not conjugate to x nor x^m. Thus, if $\langle y \rangle = \langle y^k \rangle$, then $[y] = [y^k]$ or else we would have a second pair of conjugacy classes of the same order. Let χ be any complex character of $G/(G'\langle x\rangle)$, and let $\hat{\chi}$ be the character of G defined by $\hat{\chi}(g) = \chi(G'\langle x\rangle g)$ for $g \in G$. Then $\chi(G'\langle x\rangle y) = \hat{\chi}(y)$ is rational by Proposition 2.2. Therefore, $G/(G'\langle x\rangle)$ is a rational group. The result follows from Proposition 2.4 and our hypotheses.

(3.11) $[G : G'S] = p$ and x is a p-element for some odd prime p.

By hypothesis, $[G : G'S] = n$ for some odd integer $n > 1$. From (3.10), $G/G'S$ is cyclic.

Let p be any prime dividing n, and suppose that $p \neq n$. Then $n = pm$ for some $m > 2$. Proposition 2.6 provides a quick contradiction. Hence, $[G : G'S] = p$.

Let $(G'S)t$ generate $G/G'S$ where t is a p-element. By Proposition 2.6, t is contained in a conjugacy class of repeated order. It follows that x is a p-element.

(3.12) G/G' is isomorphic to a subgroup of $\mathbf{Z}_p \times \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$.

Note $(G'S \cap G'\langle x\rangle)/G' \leq G'S/G'$, a 2-group. On the other hand, $(G'S \cap G'\langle x\rangle)/G' \leq (G'\langle x\rangle)/G' \cong \langle x\rangle/(G' \cap \langle x\rangle)$, a p-group. Hence, $G'S \cap G'\langle x\rangle = G'$ and G/G' is isomorphic to a subgroup of $G/G'S \times G/(G'\langle x\rangle)$. Result (3.12) now follows from (3.10) and (3.11).

(3.13) $p = 3$ and G is a $\{2, 3, 7\}$-group.

Let $q \neq p$ be any prime dividing the order of G'. We repeat our standard argument. Let z be an element of order q in $Z(G')$. Then z and x have distinct orders. Lemma 2.7 implies $N_G(z)/C_G(z) \cong \mathbf{Z}_{q-1}$. Moreover, $z \in Z(G')$, so $N_G(z)/C_G(z) \cong (N_G(z)/G')/(C_G(z)/G')$. From (3.12) $N_G(z)/C_G(z)$ is isomorphic to either $\mathbf{Z}_2, \mathbf{Z}_p$ or $\mathbf{Z}_2 \times \mathbf{Z}_p$. Thus, q is equal to either 3, $p + 1$, or $2p + 1$. But $q = p + 1$ implies $p = 2$. Hence, we have shown that G is a $\{2, 3, p, 2p + 1\}$-group.

Suppose p divides $|G'|$. Then there is an element z of order p in $Z(G') \cap Z(P)$ where P is a Sylow p-subgroup of G. No power of x is conjugate to z since $x \notin G'$. Thus, as usual, $N_G(z)/C_G(z) \cong \mathbf{Z}_{p-1}$. By the choice of z, $G'P \leq C_G(z)$. From (3.12), $N_G(z)/C_G(z)$ must be an elementary abelian 2-group. Thus, $p = 3$ provided $p \mid |G'|$.

Suppose $p \neq 3$, then $\langle x \rangle \cap G' = 1$. Since $p \geq 5$, $\langle x \rangle$ has at least four generators which give rise to four distinct conjugate classes of the same order, a contradiction. Thus, $p = 3$ and G is a $\{2, 3, 7\}$-group.

$$|G| = 2^a \cdot 3. \tag{3.14}$$

Assume 2 divides $|G/G'|$. Let $G'y$ have order 6 in G/G'. Then y is not conjugate to any power of x. Hence, y and y^{-1} must be conjugate or else we would get another repeated conjugacy class order. That implies $G'y = G'y^{-1}$, contradicting the order of $G'y$ being 6. Thus, $|G/G'| = 3$.

If 3 divides $|G'|$, then using the nilpotence of G' we can find an element of order 3 in $Z(G)$, a contradiction.

Assume 7 divides $|G|$. Then G' contains a Sylow 7-subgroup of G. Take $z \in Z(G')$ having order 7. Then $|N_G(z)/C_G(z)|$ divides $|G/G'| = 3$ so $N_G(z)/C_G(z) \neq \mathrm{Aut}\,(\langle z \rangle)$. Using Proposition 2.1, we obtain a second pair of repeated conjugacy class orders, a contradiction.

Therefore, G' is a Sylow 2-subgroup of G and $|G| = 2^a \cdot 3$. Moreover, the two conjugacy classes of the same order are $[x]$ and $[x^{-1}]$ and $|[x]|$ divides 2^a. We now can analyze possible conjugacy class orders.

Suppose G has a conjugacy class $[y]$ of order $2^{a-1} \cdot 3$. Then $|C_G(y)| = 2$. It follows readily that $y \in Z(G')$ and $|G| = 6$, which is not possible.

It must be that G has a conjugacy class of order 2^a. For, if not, the only possible conjugacy class orders are 2^j with $0 \leq j \leq a-1$, $2^j 3$ with $0 \leq j \leq a-2$, and the repeated class order is 2^j for some $i < a$. The class equation gives a contradiction.

A representative of a class of order 2^a has order 3. Therefore, the class is $[x]$ or $[x^{-1}]$. Either way, $|[x]| = |[x^{-1}]| = 2^a$ and $\langle x \rangle$ is self-centralizing. As a consequence, $|Z(G')| > 2$ for otherwise x would centralize $Z(G')$. Furthermore, there is no conjugacy class of order 2^j with $0 < j < a$ for some representative of such a class would centralize x.

Assume that there is an element t with $|[t]| = 2^{a-2}3$. Then $Z(G') \leq C_G(t)$, the latter having order 4. Thus, $Z(G') = C_G(t)$, which implies $C_G(t) = G'$. Therefore, $|G| = 12$. The nonabelian groups of order 12 are A_4, D_{12} and $\langle a, b : a^3 = 1, b^4 = 1, a^b = a^{-1} \rangle$. The latter two groups do not have the necessary structure. For example, both have a nontrivial center, not to mention too many repeated conjugacy class orders. Therefore, $G \cong A_4$.

If there is no conjugacy class of order $2^{a-2} \cdot 3$, then the only possible class orders are 1, 2^a and $2^j \cdot 3$ with $0 \leq j \leq a-3$. The repeated class order is 2^a. One last time, the class equation gives a contradiction.

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