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THE WORPITZKY-PRINGSHEIM THEOREM ON CONTINUED FRACTIONS

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ABSTRACT. We compare the classical convergence theorems of Worpitzky and Pringsheim in the theory of continued fractions. We give an extension of Worpitzky's theorem, and we also discuss inequalities concerning the limit point-limit circle dichotomy.

1. Introduction. Given complex numbers a_n and b_n, $n = 1, 2, \dots$, where $a_n \neq 0$ for every n, the continued fraction

$$\mathbf{K}(a_n \mid b_n) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \tag{1.1}$$

converges to the value k if $T_n(0) \to k$ as $n \to \infty$, where

$$t_n(z) = a_n/(z + b_n), \quad T_n = t_1 \circ t_2 \circ \cdots \circ t_n. \tag{1.2}$$

Throughout this paper t_n and T_n will be defined by (1.2).

In 1865 Worpitzky proved that if $|a_n| \leq 1/4$ for all n, then $\mathbf{K}(a_n \mid 1)$ converges, and after rescaling the complex plane by a factor 2 we can express this as follows.

Worpitzky's theorem. *If* $|a_n| \leq 1$ *for all* n, *then* $\mathbf{K}(a_n \mid 2)$ *converges.*

Worpitzky's theorem was later generalized by Pringsheim who, in 1889, proved the following result, see [**4**, pp. 92–94], [**5**, pp. 30–35], [**6**, p. 58] and [**8**, p. 42].

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Pringsheim's theorem. *If* $|b_n| \geq 1+|a_n|$ *for all* n, *then* $\mathbf{K}(a_n \mid b_n)$ *converges.*

Even though these results are classical, there is still much that can be said about them, and here we explore the difference between them in terms of the geometry of Möbius maps. From now on, we shall assume that for all n,

$$|b_n| \geq 1 + |a_n|; \tag{1.3}$$

this is Pringsheim's hypothesis and it is equivalent to the geometric assumption that, for all n, $t_n(\overline{\mathbf{D}}) \subset \overline{\mathbf{D}}$ where $\mathbf{D}$ is the open unit disk in $\mathbf{C}$ and $\overline{\mathbf{D}}$ is its closure. It follows that all of the continued fractions in this paper are convergent and satisfy

$$T_n(\overline{\mathbf{D}}) \subset T_{n-1}(\overline{\mathbf{D}}) \subset \cdots \subset T_1(\overline{\mathbf{D}}) \subset \overline{\mathbf{D}}. \tag{1.4}$$

Now let r_n denote the radius of the disk $T_n(\overline{\mathbf{D}})$; then $r_1 \geq r_2 \geq \cdots \geq r_n \to \tilde{r}$ for some $\tilde{r}$ with $0 \leq \tilde{r} \leq 1$. The continued fraction (1.1) is said to be of the *limit point type* when $\tilde{r} = 0$, and of the *limit circle type* when $\tilde{r} > 0$.

Later we shall make the alternative assumption that, for all n,

$$|b_n| \geq 1 + \sqrt{|a_n|}, \tag{1.5}$$

and also the stronger assumption that, for all n,

$$|b_n| \geq 1 + \max\{|a_n|, \sqrt{|a_n|}\}, \tag{1.6}$$

which is just the amalgamation of (1.3) and (1.5). Of course, the relationships between (1.3), (1.5) and (1.6) depend on whether $|a_n| \leq 1$ or $|a_n| \geq 1$, so these inequalities will also appear as assumptions from time to time.

The ideas in this paper are based on the geometry of the isometric circle of a Möbius map, and we shall see that this geometry brings the difference between Pringsheim's theorem and Worpitzky's theorem into sharp focus. First, we define the isometric circle of a Möbius map. Suppose that

$$g(z) = \frac{az+b}{cz+d}, \quad c \neq 0, \quad ad - bc = 1;$$

then the *isometric circle* C_g of g is defined by

$$C_g = \{z : |g'(z)| = 1\} = \{z : |cz + d| = 1\},$$

(see, for example, [**3**, p. 25] for more details). Of course, C_g is precisely the set of points where g acts (infinitesimally) as an Euclidean isometry and this is the reason for this terminology. Note that $|g'(z)| > 1$ inside C_g and that $|g'(z)| < 1$ outside C_g so the location of C_g determines where g acts as a (local) contraction or expansion. To be more explicit, g acts as an inversion in C_g followed by an Euclidean isometry, so that if we denote inversion in C_g by I_g then, for any disk Δ,

$$\text{radius}\,[g(\Delta)] = \text{radius}\,[I_g(\Delta)].$$

This is usually the easiest way to compute the radius of $g(\Delta)$. Another important geometric fact is that g maps the exterior of C_g onto the interior of $C_{g^{-1}}$; thus, if C_g and $C_{g^{-1}}$ are exterior to each other, then g maps the interior of $C_{g^{-1}}$ into itself and is a local contraction at each point interior to this circle. We shall now see what these facts imply in the context of the Worpitzky and Pringsheim's theorems.

As

$$\sup_{z\in\mathbf{D}} |t_n'(z)| = \sup_{z\in\mathbf{D}} \frac{|a_n|}{|b_n + z|^2} = \frac{|a_n|}{(|b_n| - 1)^2},$$

the significance of (1.5) is that it holds if and only if $|t_n'(z)| \leq 1$ throughout $\overline{\mathbf{D}}$ or, equivalently, if and only if the isometric circle of t_n does not meet $\mathbf{D}$. Thus, if (1.5) holds for all n, then every t_n is *contracting at every point of* $\mathbf{D}$, and this is a powerful condition. However, (1.5) by itself does not imply that t_n maps $\mathbf{D}$ into itself, and this is the reason for introducing (1.6).

Suppose now that we have Pringsheim's assumption (1.3). As the isometric circles of t_n and t_n^{-1} are $|z + b_n| = \sqrt{|a_n|}$ and $|z| = \sqrt{|a_n|}$, respectively, and as

$$|b_n| \geq 1 + |a_n| \geq 2\sqrt{|a_n|},$$

these two isometric circles are exterior (or externally tangent) to each other. Now the isometric circle of t_n is $|z + b_n| = \sqrt{|a_n|}$, and the center $-b_n$ of this lies outside $\overline{\mathbf{D}}$. However, this isometric circle will be disjoint from $\mathbf{D}$ if and only if (1.5) holds, and this is not guaranteed by (1.3) unless $|a_n| \geq 1$. If we now assume that (1.6) holds, then we have the stronger statement that the isometric circle of t_n is exterior (or externally tangent) to both the isometric circle of t_n^{-1} and $\partial\mathbf{D}$.

In conclusion, both Worpitzky's hypothesis and Pringsheim's hypothesis guarantee that the isometric circles of t_n and t_n^{-1} are exterior to each other. In Worpitzky's theorem the isometric circle of t_n is exterior to the unit circle $\partial\mathbf{D}$, whereas in Pringsheim's theorem these two circles may intersect. As each t_n is locally an expansion at any point inside its isometric circle and a contraction at any point outside its isometric circle, one may reasonably suppose that it is this difference that accounts for the known fact that $r_n \to 0$ in Worpitzky's theorem but not necessarily in Pringsheim's theorem. The additional assumption $|b_n| \geq 1 + \sqrt{|a_n|}$ prevents the isometric circle of t_n from intersecting the unit circle $\partial\mathbf{D}$, and this is the underlying reason why we will be able to obtain a useful estimate of r_n in this case. We shall now discuss the results we are able to obtain by using these methods, and our aim is to obtain estimates on the radii r_n and so obtain *necessary*, or *sufficient*, conditions for either the limit point or the limit circle case. Again we emphasize that all of our results here will contain Pringsheim's hypothesis (1.3) so that convergence is not an issue.

It is well known that, under Worpitzky's hypothesis (that is, when $0 < |a_n| \leq 1$ and $b_n = 2$ for all n),

$$r_n \leq \frac{1}{2n+1}, \tag{1.7}$$

this inequality is best possible. We note in passing that the author [**1**] has recently given a more delicate inequality in these circumstances, namely, that

$$\begin{aligned} r_n &\leq \frac{1}{((2/|a_1|) + (2/|a_1 a_2|) + \cdots + (2/|a_1 \cdots a_{n-1}|) + (3/|a_1 \cdots a_n|))} \\ &\leq \frac{1}{2n+1}. \end{aligned}$$

Our first result lies somewhere between Worpitzky's theorem and Pringsheim's theorem in the sense that it contains Worpitzky's theorem but has a stronger hypothesis than Pringsheim's theorem.

Theorem 1.1 *Suppose that* $|a_n| \leq 1$ *and* $|b_n| \geq 1 + \sqrt{|a_n|}$ *for all* n. *Then* $r_n \leq 1/(2n+1)$, *and the continued fraction* $\mathbf{K}(a_n \mid b_n)$ *is of the limit point case.*

The convergence here is a consequence of Pringsheim's theorem, and the real content of this result is the bound (1.7) in circumstances which are more general than those covered by Worpitzky's theorem. Further, Theorem 1.1 is in some sense best possible for, if the numbers a and b satisfy $-1 < a < 0$ and $1 + |a| = b$, so that $|a| < 1$ and $|b| < 1 + \sqrt{|a|}$, and if the continued fraction $\mathbf{K}(a_n \mid b_n)$ is defined by $a_n = a$ and $b_b = b$ for all n, then $\tilde{r} > 0$ ([**7**, p. 121]) so that r_n does not tend to 0.

To prove Theorem 1.1 we only have to show that $r_n \leq 1/(2n+1)$. This, however, is a special case of the following result.

Theorem 1.2. *Suppose that for all* n, $|b_n| \geq 1 + \max\{|a_n|, \sqrt{|a_n|}\}$. *Then*

$$r_n \leq \frac{1}{1 + 2((1/\sqrt{|a_1|}) + \cdots + (1/\sqrt{|a_n|}))}. \tag{1.8}$$

Notice that this holds if $|b_n| \geq 1 + |a_n|$ *and* $|a_n| \geq 1$ *for all* n. *If, in addition,* $\sum_n |a_n|^{-1/2}$ *diverges, then* $\tilde{r} = 0$ *and the continued fraction is in the limit point case.*

We shall prove Theorem 1.2 in Section 2. In Section 3 we shall comment on some other results that are concerned with the distinction between the limit point and limit circle case.

2. The proof of Theorem 1.2. This section contains a lemma and its proof followed by the proof of Theorem 1.2.

Lemma 2.1. *Suppose that* Σ *and* Δ *are disjoint open disks of radius* R *and* r, *respectively, and let* I *denote inversion in* $\partial\Sigma$. *Then*

$$\text{radius}\,[I(\Delta)] \leq \frac{rR}{R + 2r}.$$

Proof. Let d be the distance between the centers of Σ and Δ. Then

$$\text{radius}\,[I(\Delta)] = \frac{1}{2}\left(\frac{R^2}{d-r} - \frac{R^2}{d+r}\right) = \frac{R^2 r}{d^2 - r^2},$$

and the result follows as $d \geq r + R$.

Proof of Theorem 1.2. We introduce the auxiliary Möbius mappings

$$h_n(t) = \frac{R_n t}{2t + R_n}, \quad R_n = \sqrt{|a_n|}.$$

Note that R_n is the radius of the isometric circle of t_n and that each $h_n(t)$ is an increasing function of t for $t > 0$. Now take any positive integer n. The unit disk $\mathbf{D}$ lies outside the isometric circle of t_n, so its image $t_n(\mathbf{D})$ lies in $\mathbf{D}$ and has the same radius as the image of $\mathbf{D}$ under inversion in the isometric circle of t_n. We deduce that

$$\text{radius}\,[t_n(\mathbf{D})] \leq h_n(1).$$

As $t_n(\mathbf{D})$ lies inside $\mathbf{D}$, it is exterior to the isometric circle of t_{n-1} so, as before, we find that

$$\begin{aligned} r_n &= \text{radius}\,[t_{n-1} \circ t_n(\mathbf{D})] \\ &\leq h_{n-1}(\text{radius}\,(t_n(\mathbf{D})) \\ &\leq h_{n-1} \circ h_n(1), \end{aligned}$$

and continuing in this way we find that

$$r_n = \text{radius}\,[t_1 \circ \cdots \circ t_n(\mathbf{D})] \leq h_1 \circ \cdots \circ h_n(1).$$

If we express the auxiliary mappings h_n in terms of matrices we find (by induction) that for $n \geq 2$,

$$\begin{pmatrix} R_1 & 0 \\ 2 & R_1 \end{pmatrix} \cdots \begin{pmatrix} R_n & 0 \\ 2 & R_n \end{pmatrix} = \begin{pmatrix} R_1 \cdots R_n & 0 \\ 2T_n & R_1 \cdots R_n \end{pmatrix},$$

where

$$T_n = R_1 \cdots R_n \left(\frac{1}{R_1} + \cdots + \frac{1}{R_n}\right).$$

It follows that

$$\begin{aligned} r_n &\le h_1 \circ \cdots \circ h_n(1) \\ &= \frac{R_1 \cdots R_n}{2T_n + R_1 \cdots R_n} \\ &= \frac{1}{1 + 2W_n}, \end{aligned}$$

where

$$W_n = \frac{1}{R_1} + \cdots + \frac{1}{R_n} = \frac{1}{\sqrt{|a_1|}} + \cdots + \frac{1}{\sqrt{|a_n|}},$$

and the proof of Theorem 1.2 is complete.

3. Further results. Because of (1.7), every continued fraction satisfying Worpitzky's criterion is of the limit point type, but, by contrast, continued fractions satisfying Pringsheim's criterion may be of the limit circle type. The following results give sufficient conditions for a continued fraction satisfying Pringsheim's hypothesis to be of limit point type. The first of these has the virtue of being in terms of the transformations t_n; the remaining results are given in terms of the T_n and, although these occur in [**7**], we shall have something to say about their proofs.

Theorem 3.1. *Suppose that* $|b_n| \ge 1 + |a_n|$ *for all* n; *then*

$$r_n \le \prod_{k=1}^{n} \frac{|a_k|}{(|b_k| - 1)^2}.$$

In particular, if the corresponding infinite product diverges to zero, then the continued fraction $\mathbf{K}(a_n \mid b_n)$ *is of the limit point type.*

This is trivial, for if $g(z) = a/(b + z)$ then

$$\sup_{z \in \mathbf{D}} |g'(z)| = \frac{|a|}{(|b| - 1)^2}.$$

As $t_n(\overline{\mathbf{D}}) \subset \overline{\mathbf{D}}$ for all n, Theorem 3.1 follows immediately from the Chain Rule for derivatives.

Theorem 3.2 [**7**]. *In any Pringsheim fraction, we have*

$$r_n \leq \frac{1}{|T_1^{-1}(\infty)\cdots T_{n-1}^{-1}(\infty)|(1+|T_n^{-1}(\infty)|)}. \tag{3.1}$$

We know that $|T_n^{-1}(\infty)| > 1$ because $T_n(\mathbf{D}) \subset \mathbf{D}$. If we replace $1+|T_n^{-1}(\infty)|$ in (3.1) by 2, we obtain the next corollary.

Corollary 3.3. *For the continued fraction* (1.1), *where* $|b_n| \geq 1+|a_n|$,

$$2\tilde{r} \leq \left(\prod_{n=1}^{\infty} |T_n^{-1}(\infty)|\right)^{-1}. \tag{3.2}$$

In particular, if $\tilde{r} > 0$, *then* $|T_n^{-1}(\infty)| \to 1$.

Finally we consider the *tangency case*; this is the situation in which the disks $T_n(\mathbf{D})$ are all tangent to each other and to the disk $\mathbf{D}$ at a point ζ, say, on the unit circle. In this case we can actually give an explicit expression for the value of $\tilde{r}$.

Theorem 3.4 [**7**]. *Suppose that the Pringsheim continued fraction* (1.1) *is the tangency case. Then equality holds in* (3.1), *and*

$$\tilde{r} = \frac{1}{2\prod_{k=1}^{\infty} |T_k^{-1}(\infty)|}. \tag{3.3}$$

Theorems 3.2 and 3.4 occur in [**7**] (in equation (7.8) and an unnumbered formula in the middle of page 121), although they are not stated quite so explicitly there. On page 118 of [**7**], Thron states that his proof of Pringsheim's theorem is approximately the same length as that in Perron's text, but gives greater insight. This is certainly true, but even the exposition in [**7**] can be made considerably shorter, and geometrically more transparent, if we focus attention on the mappings t_n and T_n rather than their coefficients. For example, the formula (7.6) in [**7**], which follows from a complicated expression for t_n, is nothing more than the formula $T_n(\infty) = T_{n-1}(0)$. Similarly, the formula $b_n = 1/a_n + G_{n-1}a_n$ on page 120 of [**7**] (which should read $b_n = 1/a_n - G_{n-1}a_n$) is derived from an expression for $t_n(z)$ that takes up no less than eight lines of text but is nothing more than $t_n T_n^{-1}(\infty) = T_{n-1}^{-1}(\infty)$. Other simplifications along these lines are possible, but rather than discuss these we prefer to give brief commentary on similar, but more geometric, proofs of these results. In these we prefer not to use various intermediate variables (as is done in [**7**]) for these tend to conceal the underlying geometry. For an entirely geometric treatment (and proof) of Pringsheim's theorem, see [**2**]; for more details on Worpitzky's theorem, see [**1**].

Proof of Theorem 3.2. The proof of Theorem 3.2 is based on the two equations

$$|T_n(\infty) - c_n| = r_n|T_n^{-1}(\infty)|, \quad |T_n(0) - c_n| = \frac{r_n}{|T_n^{-1}(\infty)|}, \tag{3.4}$$

where c_n is the center, and r_n the radius, of $T_n(\partial\mathbf{D})$. These formulae are easily derived (by algebra) from Thron's proof, but they have a geometric origin which we shall now describe. As inverse points are preserved under Möbius maps, $T_n^{-1}(c)$ and $T_n^{-1}(\infty)$ are inverse points with respect to $\partial\mathbf{D}$. Let z be the point on $\partial\mathbf{D}$ that lies between $T_n^{-1}(c)$ and $T_n^{-1}(\infty)$; thus, we can write

$$T_n^{-1}(\infty) = \mu_n, \qquad z = \mu_n/|\mu_n|, \qquad T_n^{-1}(c) = \mu_n/|\mu_n|^2.$$

The invariance of cross-ratios under Möbius maps now gives

$$\begin{aligned}[\mu_n, \mu_n/|\mu_n|^2, \infty, \mu_n/|\mu_n|] &= [T_n^{-1}(\infty), T_n^{-1}(c), \infty, z] \\ &= [\infty, c, T_n(\infty), T_n(z)],\end{aligned}$$

and after taking the modulus of each side, we obtain the first equation in (3.4). The second equation in (3.4) follows because $T_n(0)$ and $T_n(\infty)$ are inverse points with respect to $T_n(\partial\mathbf{D})$. Following Thron [**7**] and using the fact that $T_n(\infty) = T_{n-1}(0)$, these equations give

$$\begin{aligned} r_n|T_n^{-1}(\infty)| - \frac{r_{n-1}}{|T_{n-1}^{-1}(\infty)|} &= |T_n(\infty) - c_n| - |T_{n-1}(0) - c_{n-1}| \\ &\le |(T_n(\infty) - c_n) - (T_{n-1}(0) - c_{n-1})| \qquad (3.5) \\ &= |c_n - c_{n-1}| \\ &\le r_{n-1} - r_n. \qquad (3.6) \end{aligned}$$

This simplifies to

$$r_n(1+|T_n^{-1}(\infty)|) \leq \frac{r_{n-1}(1+|T_{n-1}^{-1}(\infty)|}{|T_{n-1}^{-1}(\infty)|},$$

which, in turn, yields

$$r_n(1+|T_n^{-1}(\infty)|) \leq \left(\prod_{k=1}^{n} |T_k^{-1}(\infty)\right)^{-1} r_1(1+|T_1^{-1}(\infty)|.$$

Because $T_1(\infty) = t_1(\infty) = 0$, (3.4) gives

$$r_1(1+|T_1^{-1}(\infty)| = r_1 + |c_1| \leq 1,$$

and (3.1) follows.

Proof of Theorem 3.4. The conclusion (3.3) in Theorem 3.4 follows once we have justified the assertion that, in the tangency case, each inequality (3.5) and (3.6) is an equality. It is a trivial geometric fact that, in the context of this discussion, equality holds in (3.6) if and only if the circles $T_n(\partial\mathbf{D})$ and $T_{n-1}(\partial\mathbf{D})$ are tangent to each other. Further, if we assume that all $T_n(\partial\mathbf{D})$ are tangent to each other at a point ζ on $\partial\mathbf{D}$, then assuming that $\zeta = -1$ (as we may), we see that the points -1, c_n, c_{n-1} and $T_{n-1}(0)$ lie, in this order, along the diameter $(-1,1)$ of $\mathbf{D}$ (see, for example, [**2**]) so that equality also holds in (3.5). This completes the proof of Theorem 3.4.

We end this paper with a remark concerning numerical estimates. It is well known that we can write

$$T_n(z) = \frac{A_{n-1}z + A_n}{B_{n-1}z + B_n},$$

where

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = \begin{pmatrix} A_{n-2} & A_{n-1} \\ B_{n-2} & B_{n-1} \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \qquad \begin{pmatrix} A_{-1} & A_0 \\ B_{-1} & B_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In this case

$$T_1^{-1}(\infty)\cdots T_m^{-1}(\infty) = \left(-\frac{B_1}{B_0}\right)\left(-\frac{B_2}{B_1}\right)\cdots\left(-\frac{B_m}{B_{m-1}}\right) = (-1)^m B_m.$$

It follows that in the tangency case we have

$$\tilde{r} = \lim_{k\to\infty} \frac{1}{2|B_k|}.$$

As the B_n can in principle easily be computed (from the difference equation given above) we see that we can now obtain numerical estimates of the size of the limit circle (or point) in the tangency case.

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