ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 3, Fall 2001

IDEALS GENERATED BY PROJECTIONS AND INDUCTIVE LIMIT C^*-ALGEBRAS

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ABSTRACT. We introduce two classes of inductive limit C^*-algebras which generalize the AH algebras: the GAH algebras (GAH stands for "generalized AH") and a subclass of it, the *strong* GAH algebras. We give necessary and sufficient conditions for an ideal of a GAH algebra to be generated by projections which, in particular, gives necessary and sufficient conditions for a GAH algebra to have the ideal property, i.e., any ideal is generated by projections. We prove that if $0 \to I \to A \to B \to 0$ is an exact sequence of C^*-algebras such that A is a GAH algebra, then A has the ideal property if and only if I and B have the ideal property. We describe the lattice of ideals generated by projections of a strong GAH algebra and also the partially ordered set of the stably cofinite ideals generated by projections of a strong GAH algebra A under the additional assumption that the projections in $M_\infty(A)$ satisfy the Riesz decomposition property. These results generalize some of our previous theorems involving AH algebras.

1. Introduction. A C^*-algebra has the ideal property if any ideal is generated, as an ideal, by projections ([**13**]). In this paper, by "ideal" we shall mean "closed, two-sided ideal". An AH algebra is the inductive limit of a sequence of C^*-algebras which are finite direct sums of C^*-algebras of the form $PC(X, M_n)P$, where X is a connected, finite CW complex and P is a projection in $C(X, M_n)$ ([**1**]). The AH algebras with the ideal property present interest since they include two important classes of C^*-algebras: the *simple* AH algebras and the *real rank zero* AH algebras ([**4**]), about which a lot of interesting results have been proved in the last years. The study of the AH algebras with the ideal property is related to a problem of Effros ([**6**]) (namely, find suitable topological invariants for AH algebras), and also to Elliott's project on the classification of the separable, amenable C^*-algebras by invariants including K-theory ([**7**]). The AH algebras with the ideal property have been studied in [**13**], [**9**], [**10**], [**11**] and [**12**].

1991 AMS *Mathematics Subject Classification.* 46L05, 46L99.
This research was partially supported by NSF grant DMS-9622250.
Received by the editors on September 13, 1998, and in revised form on August 10, 2000.

In this paper we generalize some of our results concerning AH algebras with the ideal property or ideals generated by projections in a given AH algebra. For this purpose, we introduce two new classes of inductive limit C^*-algebras: the GAH algebras (see Definition 2.1) (GAH stands for "generalized AH") and a subclass of it, the strong GAH algebras (see Definition 2.11). Note that any AH algebra or any inductive limit of a sequence of finite direct sums of unital, simple C^*-algebras is a strong GAH algebra. Also, any inductive limit of a sequence of C^*-algebras which are finite direct sums of C^*-algebras each of which is either unital and simple, or unital and projectionless (i.e., there are no nontrivial projections in the algebra), is a GAH algebra.

We give several necessary and sufficient conditions for an ideal of a GAH algebra to be generated by projections (see Theorem 2.2 and Remark 2.3). In particular, this result gives necessary and sufficient conditions for a GAH algebra A to have the ideal property, and one of these conditions is that any ideal of A has a countable approximate unit consisting of projections (see Corollary 2.4). Corollary 2.4 generalizes in part [**9**, Theorem 3.1 and Remark 3.2 b)], where A is an AH algebra.

Answering a question of Pedersen, we showed in [**11**], jointly with Dadarlat, that an extension of two C^*-algebras with the ideal property does not necessarily have the ideal property. However, we prove that if $0 \to I \to A \to B \to 0$ is an exact sequence of C^*-algebras such that A is a GAH algebra, then A has the ideal property if and only if I and B have the ideal property (see Theorem 2.6). This theorem generalizes [**10**, Theorem 3.1] where I, A and B are AH algebras and [**12**, Theorem 7.2] where A is an AH algebra. Theorem 2.6 required a completely new idea of proof, since the methods used in proving [**10**, Theorem 3.1] and [**12**, Theorem 7.2] don't work here.

On the other hand, we describe the lattice of ideals generated by projections of a strong GAH algebra (see Theorem 2.13). We also describe the partially ordered set of stably cofinite ideals generated by projections of a strong GAH algebra A such that the projections of $M_\infty(A)$ (= the algebraic direct limit of matrix algebras $M_n(A)$ under the embeddings $a \mapsto a \oplus 0$) satisfy the Riesz decomposition property (i.e., if p, q_1, q_2 are projections in $M_\infty(A)$ such that p is Murray-von Neumann equivalent to a subprojection of $q_1 \oplus q_2$, then $p = p_1 \oplus p_2$ for some projections $p_i \in M_\infty(A)$ with p_i Murray-von Neumann equivalent to a subprojection of q_i, for $i = 1, 2$) (see

Theorem 2.16). These two theorems taken together (see also [**11**, Lemma 4.6]) generalize [**11**, Theorem 4.1], where the lattice of the ideals generated by projections of an AH algebra has been considered. The ideals generated by projections of an AH algebra played an important role in the proof given by Dadarlat and Eilers in [**5**], leading to the surprising fact that there are inductive limits of AH algebras which are not AH algebras.

2. Results.

Definition 2.1. A C^*-algebra A is called a GAH algebra (GAH stands for "generalized AH") if $A = \varinjlim (A_n, \Phi_{n,m})$ (the $*$-homomorphisms $\Phi_{n,m}$ need not be either injective or unital), $A_n = \oplus_{i=1}^{k_n} A_n^i$ for $n \in \mathbf{N}$, and for each $n \in \mathbf{N}$ and $1 \leq i \leq k_n$, we have

(1) A_n^i is a unital C^*-algebra.

(2) Any proper ideal of A_n^i has no nonzero projections.

It is useful to point out that even though a GAH algebra may be nonunital, it must have a countable approximate unit of projections.

The proof of the next theorem uses techniques from [**9**] and [**10**].

Theorem 2.2. *Let $A = \varinjlim (A_n, \Phi_{n,m})$ be a GAH algebra, where $A_n = \oplus_{i=1}^{k_n} A_n^i$ and each A_n^i is as in Definition* 2.1. *Let I be an ideal of A. Then the following are equivalent*:

(1) *I is generated as an ideal by projections.*

(2) *$I = \varinjlim (I_n, \Phi_{n,m}|_{I_n})$ where each I_n is a direct sum of full blocks of A_n.*

(3) *I has a countable approximate unit consisting of projections.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). By [**2**] it follows that $I = \varinjlim (I_n, \Phi_{n,m}|_{I_n})$, where each I_n is an ideal of A_n. Let $I_n = \oplus_{i=1}^{k_n} I_n^i \subseteq \oplus_{i=1}^{k_n} A_n^i = A_n$, $n \in \mathbf{N}$. For any $n \in \mathbf{N}$ define a projection $p_n = \oplus_{i=1}^{k_n} p_n^i$

by

$$p_n^i = \begin{cases} 1_{A_n^i} & \text{if } I_n^i = A_n^i, \\ 0 & \text{if } I_n^i \neq A_n^i. \end{cases}$$

Observe that for each $n \in \mathbf{N}$, we have $p_n \in I_n$ and by hypothesis, $\Phi_{n,n+1}(p_n) \leq p_{n+1}$. Define for any $n \in \mathbf{N}$ that $P_n = \Phi_{n,\infty}(p_n)$ where $\Phi_{n,\infty} : A_n \to A = \varinjlim (A_m, \Phi_{m,k})$ is the canonical homomorphism. Obviously, $(P_n)_{n\geq 1}$ is an increasing sequence of projections in I. We want to prove first that $(P_n)_{n\geq 1}$ is an approximate unit of I. To prove this, since I is generated by projections, it will be enough to prove that for an arbitrary fixed $n \in \mathbf{N}$ and for any $a, b \in A_n$ and any $e = e^* = e^2 \in I_n$ we have

$$(*) \qquad aeb \cdot p_n = p_n \cdot aeb = aeb.$$

Let $a = \oplus_{i=1}^{k_n} a^i$, $b = \oplus_{i=1}^{k_n} b^i$, $e = \oplus_{i=1}^{k_n} e^i$ be the canonical decompositions of these elements in $\oplus_{i=1}^{k_n} A_n^i$. Fix an arbitrary $1 \leq i \leq k_n$. To prove $(*)$ we have to show that

$$(**) \qquad a^i e^i b^i \cdot p_n^i = p_n^i \cdot a^i e^i b^i = a^i e^i b^i.$$

We have two cases:

(a) If $p_n^i = 1_{A_n^i}$, then $(**)$ is obvious.

(b) If $p_n^i = 0$, then $I_n^i \neq A_n^i$ and then, by hypothesis, I_n^i has no nonzero projections. Since $e^i = (e^i)^* = (e^i)^2 \in I_n^i$, it follows that $e^i = 0$ and hence $(**)$ is satisfied.

Hence $(P_n)_{n\geq 1}$ is an approximate unit of projections of I, and this proves (1) implies (3). But then, obviously,

$$I = \varinjlim (p_n A_n p_n, \Phi_{n,m}|_{p_n A_n p_n}).$$

Observe that since $p_n A_n p_n = \oplus_{i=1}^{k_n} p_n^i A_n^i p_n^i$, the definition of p_n^i implies that $p_n^i A_n^i p_n^i$ is 0 or A_n^i for any n and i. This ends the proof of (1) implies (2).

The implication (3) $\Rightarrow$ (1) is trivial, and the implication (2) $\Rightarrow$ (1) follows from the fact that a C^*-algebra which is an inductive limit of unital C^*-algebras is generated by projections as an ideal.

Remark 2.3. (i) It is easily seen that conditions (1), (2) and (3) in Theorem 2.2 are also equivalent to the following two:

(4) For any integer n, any $\varepsilon > 0$ and any $x \in A_n \cap I$, there is an $m > n$ and a projection $p \in A_m \cap I$ such that

$$\|\Phi_{n,m}(x) - p\,\Phi_{n,m}(x)\| \leq \varepsilon.$$

(5) For any integer n, any $\varepsilon > 0$ and any $x \in A_n \cap I$, there is an $m > n$ and a projection $p \in A_m \cap I$ such that

$$\|\Phi_{n,m}(x) - p\,\Phi_{n,m}(x)\,p\| \leq \varepsilon.$$

(Above we used the notation $A_k \cap I = \{y \in A_k : \Phi_{k,\infty}(y) \in I\}$).

(ii) Let I be an ideal of a GAH algebra A. Suppose that I is generated by projections. Then I and A/I are also GAH algebras. (This follows immediately from the implication (1) $\Rightarrow$ (2) in the above theorem).

(iii) The proof of the implication (1) $\Rightarrow$ (2) of Theorem 2.2 in the case when A is an AH algebra is contained in the proof of [**10**, Theorem 3.1]. The above equivalence (1) $\Leftrightarrow$ (2) for AH algebras A was obtained independently in [**5**].

Corollary 2.4. *Let* $A = \varinjlim (A_n, \Phi_{n,m})$ *be a* GAH *algebra, where* $A_n = \oplus_{i=1}^{k_n} A_n^i$ *and each* A_n^i *is as in Definition* 2.1. *Then the following are equivalent*:

(1) *A has the ideal property.*

(2) *Any ideal I of A is given by $I = \varinjlim (I_n, \Phi_{n,m}|_{I_n})$ where each I_n is a direct sum of full blocks of A_n.*

(3) *Any ideal of A has a countable approximate unit of projections.*

(4) *For any ideal I of A we have*: *for any integer n, any $\varepsilon > 0$ and any $x \in A_n \cap I$, there is an $m > n$ and a projection $p \in A_m \cap I$ such that*

$$\|\Phi_{n,m}(x) - p\,\Phi_{n,m}(x)\| \leq \varepsilon.$$

(5) *For any ideal I of A, we have*: *for any integer n, any $\varepsilon > 0$ and any $x \in A_n \cap I$ there is an $m > n$ and a projection $p \in A_m \cap I$ such that*:

$$\|\Phi_{n,m}(x) - p\,\Phi_{n,m}(x)\,p\| \leq \varepsilon.$$

Remark 2.5. The above corollary is a generalization of part of [**9**, Theorem 3.1 and Remark 3.2 b)], where A is an AH algebra.

In [**11**] we constructed, jointly with Dadarlat, extensions of C^*-algebras with the ideal property which don't have the ideal property, answering thus a question of Pedersen. However, the next theorem shows that an extension of two C^*-algebras with the ideal property which is a GAH algebra *has* the ideal property. Theorem 2.6 below generalizes [**10**, Theorem 3.1] where it is proved that if $0 \to I \to A \to B \to 0$ is an exact sequence of AH algebras, then A has the ideal property if and only if I and B have the ideal property; its proof, which has to use a completely different idea, involves quasidiagonal extensions. Theorem 2.6 also generalizes [**12**, Theorem 7.2].

Theorem 2.6. *Let* $0 \to I \xrightarrow{i} A \xrightarrow{\pi} B \to 0$ *be an exact sequence of* C^**-algebras. Suppose that* A *is a* GAH *algebra. Then the following are equivalent*:

(1) A *has the ideal property.*

(2) I *and* B *have the ideal property.*

The proof of this theorem needs some preparatory results. The first one is a generalization of a joint result of Brown and Dadarlat (namely, of [**3**, Proposition 11]). To state it, we need to recall the following definition which goes back to Murphy and Salinas:

Definition 2.7. An extension of C^*-algebras

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

is called *quasidiagonal* if there is an approximate unit $(p_n)_{n=1}^{\infty}$ of I consisting of projections such that

$$\lim_{n\to\infty} \|ap_n - p_n a\| = 0$$

for all $a \in A$.

Lemma 2.8. *Let* I *be an ideal of a* GAH *algebra* A*. Suppose that* J *is an ideal of* I *and that* J *has a countable approximate unit of projections.*

Then the extension

$$0 \longrightarrow J \longrightarrow I \longrightarrow I/J \longrightarrow 0$$

is quasidiagonal (*here the map $J \to I$ is the canonical inclusion and the map $I \to I/J$ is the canonical surjection*).

Proof. The proof is a modification of the argument used in the proof of [**3**, Proposition 11]. Suppose that $A = \varinjlim (A_n, \Phi_{n,m})$ where $A_n = \oplus_{i=1}^{k_n} A_n^i$, $n \in \mathbf{N}$, and each A_n^i is a unital C^*-algebra such that any proper ideal has no nonzero projections. By [**2**], $I = \varinjlim (I_n, \Phi_{n,m}|_{I_n})$ where $I_n = \oplus_{i=1}^{k_n} I_n^i$ is an ideal of $A_n = \oplus_{i=1}^{k_n} A_n^i$ and $J = \varinjlim (J_n, \Phi_{n,m}|_{J_n})$ where $J_n = \oplus_{i=1}^{k_n} J_n^i$ is an ideal of $I_n = \oplus_{i=1}^{k_n} I_n^i$ for any $n \in \mathbf{N}$. Since J has a countable approximate unit of projections, by a standard approximation argument we may suppose, after replacing $(A_n)_{n=1}^{\infty}$ with a subsequence if needed, that there are projections $p_n \in J_n$, $n \in \mathbf{N}$, such that if $\Phi_{n,\infty} : A_n \to \varinjlim (A_k, \Phi_{k,m})$ is the canonical homomorphism, $n \in \mathbf{N}$, then we have

$$\lim_{n\to\infty} \|\Phi_{n,\infty}(p_n) j - j\| = 0$$

for any $j \in J$.

Observe that by hypothesis, if $J_n^i \neq A_n^i$, then $p_n^i = 0$ (since p_n^i is a projection in J_n^i).

We define now for any $n \in \mathbf{N}$ a projection q_n in A_n such that $q_n = \oplus_{i=1}^{k_n} q_n^i \in A_n = \oplus_{i=1}^{k_n} A_n^i$, where

$$q_n^i = \begin{cases} 1_{A_n^i} & \text{if } J_n^i = A_n^i \\ 0 & \text{if } J_n^i \neq A_n^i \end{cases}$$

for any $1 \leq i \leq k_n$. Observe that in fact $q_n \in J_n$ and $p_n \leq q_n$, $\Phi_{n,n+1}(q_n) \leq q_{n+1}$ for any $n \in \mathbf{N}$.

For every $n \in \mathbf{N}$ and any $x \in J_n$ we have

$$\|x(1-q_n)\|^2 = \|(x(1-q_n)x^*\| \leq \|x(1-p_n)x^*\| = \|x(1-p_n)\|^2.$$

Since also each q_n is a central projection of I_n, it follows that

$$\lim_{n\to\infty} \|\Phi_{n,\infty}(q_n) j - j\| = 0, \quad \forall\, j \in J$$

and

$$\lim_{n\to\infty} \|\Phi_{n,\infty}(q_n)i - i\Phi_{n,\infty}(q_n)\| = 0, \quad \forall\, i \in I.$$

These equalities imply that the extension

$$0 \longrightarrow J \longrightarrow I \longrightarrow I/J \longrightarrow 0$$

is quasidiagonal, since $\Phi_{n,\infty}(q_n) \le \Phi_{n+1,\infty}(q_{n+1})$ for all $n \in \mathbf{N}$.

Lemma 2.9 ([**10**, Lemma 3.9 (1)]). *Let*

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

be a quasidiagonal extension of C^*-algebras. Then any projection in B lifts to a projection in A.

The following lemma is a remark of Dadarlat:

Lemma 2.10. *Let*

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi} B \longrightarrow 0$$

be an exact sequence of C^*-algebras such that any projection in B lifts to a projection in A.

If I is generated as an ideal (of I) by projections and if B is generated as an ideal (of B) by projections, then A is also generated as an ideal (of A) by projections.

Proof. Let $a \in A$ and $\varepsilon > 0$. By our hypothesis on B, it follows that there are $n \in \mathbf{N}$ and $\tilde{x}_i, \tilde{y}_i, \tilde{e}_i \in B$ with $\tilde{e}_i$ a projection for any $1 \le i \le n$ such that

$$\left\|\pi(a) - \sum_{i=1}^{n} \tilde{x}_i\tilde{e}_i\tilde{y}_i\right\| < \varepsilon/2.$$

Since π is surjective, there are $x_i, y_i \in A$ such that $\pi(x_i) = \tilde{x}_i$, $\pi(y_i) = \tilde{y}_i$, $1 \le i \le n$. Using the fact that any projection in B lifts to a projection of A, it follows that there is $e_i = e_i^* = e_i^2 \in A$ such that $\pi(e_i) = \tilde{e}_i$ for each $1 \leq i \leq n$. By the definition of the norm in $A/i(I)(\cong B)$ it turns out that there is a $c \in I$ such that

$$\left\| a - i(c) - \sum_{i=1}^{n} x_i e_i y_i \right\| < \varepsilon/2. \tag{1}$$

Since I is generated as an ideal (of I) by projections, it follows that there are $m \in \mathbf{N}$ and $u_j, v_j, f_j = f_j^* = f_j^2 \in I$, $1 \leq j \leq m$, such that

$$\left\| i(c) - \sum_{j=1}^{m} i(u_j) i(f_j) i(v_j) \right\| < \varepsilon/2 \tag{2}$$

But (1) and (2) imply that

$$\left\| a - \sum_{j=1}^{m} i(u_j) i(f_j) i(v_j) - \sum_{i=1}^{n} x_i e_i y_i \right\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This ends the proof.

Proof of Theorem 2.6. As noticed in the proof of [**10**, Theorem 3.1], the implication (1) $\Rightarrow$ (2) is trivially true for any C^*-algebras, I, A and B.

Now let us prove the implication (2) $\Rightarrow$ (1). Assume that I and B have the ideal property. Let J be an ideal of A. We shall prove that J, as an ideal of A, is generated by projections. We may suppose that $i : I \to A$ is the canonical inclusion and that $\pi : A \to B = A/I$ is the canonical surjection. The extension

$$0 \longrightarrow I \xrightarrow{i} A \xrightarrow{\pi} B \longrightarrow 0$$

induces in a canonical way an extension

$$0 \longrightarrow J \cap I \longrightarrow J \longrightarrow J/J \cap I \longrightarrow 0. \tag{$*$}$$

(Observe that by Remark 2.3 (ii) I and B are both GAH algebras). Since $J \cap I$ is an ideal of I and I has the ideal property, it follows that

$J \cap I$ is an ideal of the GAH algebra A generated by projections. By Theorem 2.2 it follows that $J \cap I$ has a countable approximate unit of projections. Hence, Lemma 2.8 implies that the extension $(*)$ is quasidiagonal. Applying now Lemma 2.9, we find that any projection in $J/J \cap I$ lifts to a projection in J. But since $J/J \cap I \cong \pi(J)$ (= an ideal of B) and B has the ideal property, it follows that $J/J \cap I$ is generated as an ideal of B by projections. This easily implies that $J/J \cap I$ is generated as an ideal of $J/J \cap I$ by projections. Finally, since $J \cap I$ is generated as an ideal of $J \cap I$ by projections, by Lemma 2.10, J is generated as an ideal of J, and hence also as an ideal of A by projections. This ends the proof.

Definition 2.11. A C^*-algebra is called a *strong* GAH algebra if $A = \varinjlim (A_n, \Phi_{n,m})$, $A_n = \oplus_{i=1}^{k_n} A_n^i$ for $n \in \mathbf{N}$ and for each $n \in \mathbf{N}$ and $1 \le i \le k_n$, we have

(1) A_n^i is a unital C^*-algebra.

(2) For any $m \in \mathbf{N}$, any proper ideal of $M_m(A_n^i)$ has no nonzero projections.

Remark 2.12. Let A be a strong GAH algebra. If B is an AF algebra, then $A \otimes B$ is a strong GAH algebra.

In what follows we shall describe the partially ordered set of all the ideals generated by projections of a strong GAH algebra.

Theorem 2.13. *Let A be a strong GAH algebra. Then there is a canonical lattice isomorphism*:

$$\{I \mid I \textit{ is an ideal of } A \textit{ generated by projections}\} \xrightarrow{\sim} \{J \mid J \textit{ is an ideal of } D(A \otimes \mathcal{K})\}.$$

In the above theorem, we used the standard notation $D(B)$, where B is a C^*-algebra, to denote the abelian local semi-group of Murray-von Neumann equivalence classes of projections in B, the addition of two classes being defined when they have orthogonal representatives. Also, an ideal in $D(B)$ is a nonempty hereditary subset which is closed under addition, where defined. Above, $\mathcal{K}$ is the C^*-algebra of compact operators on $l^2(\mathbf{N})$.

The proof of Theorem 2.13 will need the following result, which generalizes [**11**, Lemma 4.5].

Lemma 2.14. *Let A be a strong GAH algebra. Then there is a canonical lattice isomorphism*:

$$\{I \mid I \text{ is an ideal of } A \text{ generated by projections}\} \\ \xrightarrow{\sim} \{J \mid J \text{ is an ideal of } A \otimes \mathcal{K} \text{ generated by projections}\}.$$

Proof. It is similar to the proof of [**11**, Lemma 4.5] and uses the equivalence (1) $\Leftrightarrow$ (2) in Theorem 2.2 and Remark 2.12.

Proof of Theorem 2.13. By the proof of [**11**, Lemma 4.2] and by [**11**, Remark 4.3], it follows that there is a canonical lattice isomorphism

$$\{I \mid I \text{ is an ideal of } A \otimes \mathcal{K} \text{ generated by projections}\} \\ \xrightarrow{\sim} \{J \mid J \text{ is an ideal of } D(A \otimes \mathcal{K})\}.$$

Now the proof follows, combining this fact with Lemma 2.14.

Let us recall the following:

Definition 2.15. An ideal I in a C^*-algebra A is said to be *stably cofinite* if the C^*-algebra A/I is stable finite, i.e., if there do not exist projections $p, q \in M_\infty(A/I)$ such that $p \oplus q \sim q$ and $p \neq 0$.

Theorem 2.16. *Let A be a strong GAH algebra such that the projections in $M_\infty(A)$ satisfy the Riesz decomposition property. Then there is an order isomorphism:*

$$\{I \mid I \textit{ is a stably confinite ideal of } A \textit{ generated by projections}\} \\ \xrightarrow{\sim} \{J \mid J \textit{ is an ideal of } K_0(A)\}.$$

More precisely, there are order-preserving inverse isomorphisms sending each stably cofinite ideal I of A, generated by projections, to the kernel of K_0 of the quotient map $A \to A/I$, and sending each ideal J of $K_0(A)$ to the ideal of A, generated by those projections $p \in A$ for which $[p] \in J$. (Here, by an ideal of $K_0(A)$ we mean a subgroup H of $K_0(A)$ such that $H^+ := H \cap K_0(A)^+$ is hereditary (i.e., if $0 \le g \le h$ for some $g \in K_0(A)$ and $h \in H^+$, then $g \in H$) and $H = H^+ - H^+$).

The proof of the above theorem will use the following:

Lemma 2.17. *Let A be a strong GAH algebra and let I be an ideal of A generated by projections. Then for any $n \in \mathbf{N}$, all projections in $M_n(A/I)$ lift to projections in $M_n(A)$.*

Proof. By Theorem 2.2, $M_n(I)$, which is an ideal of the strong GAH algebra $M_n(A)$ (see Remark 2.12), has a countable approximate unit consisting of projections for any arbitrary, fixed $n \in \mathbf{N}$. Then, by Lemma 2.8, the extension:

$$0 \longrightarrow M_n(I) \longrightarrow M_n(A) \longrightarrow M_n(A/I) \longrightarrow 0$$

is quasidiagonal. Now a simple application of Lemma 2.9 (i.e., [**10**, Lemma 3.9 (1)]) implies that all projections in $M_n(A/I)$ lift to projections in $M_n(A)$. This ends the proof.

Proof of Theorem 2.16. The proof uses the argument given in the proof of Lemma 4.10 in [**11**]. This is possible, essentially, because of the implication (1) $\Rightarrow$ (3) in Theorem 2.2. Note that if I is an ideal of A, and if $i : I \to A$ is the canonical inclusion and $\pi : A \to A/I$ is the canonical surjection, then by the six-term exact sequence in K-theory, $\ker K_0(\pi) = K_0(i)(K_0(I))$. Moreover, if I is generated by projections, then by the implication (1) $\Rightarrow$ (3) in Theorem 2.2, we have $K_0(I) = K_0(I)^+ - K_0(I)^+$ and hence, $\ker K_0(\pi) = \{[p]_{K_0(A)} - [q]_{K_0(A)} \mid p$ and q are projections in $M_\infty(I)\}$. Also, using Lemma 2.17 we can prove as in the proof of [**8**, Lemma 10.8 (a)] that, if J is an ideal of $K_0(A)$, then the ideal of A, generated by those projections $p \in A$ for which $[p] \in J$, is stably cofinite.

Remark 2.18. Theorem 2.13, together with Theorem 2.16, give a generalization of [**11**, Theorem 4.1] (see also [**11**, Lemma 4.6]) where the lattice of all the ideals generated by projections of an AH algebra is described.

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