

DECOMPOSING THE DISENTANGLING ALGEBRA ON WIENER SPACE

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ABSTRACT. For any functional H in the disentangling algebra $\mathcal{A}_{a,t}^c$ and any partition $a = r_0 < r_1 < \cdots < r_h = t$ of $[a, t]$, we show that H can be decomposed into a sum of terms where each term is the $*$ product of h functions, one from each of the disentangling algebras $\mathcal{A}_{r_{j-1}, r_j}^c$, $j = 1, \dots, h$. Correspondingly, we show that the operator-valued path integral of H , $K_\lambda^{a,t}(H)$, can be written as the sum of time-ordered products of the h operators associated with the h subintervals of $[a, t]$. We also consider various special cases of these results. The $*$ product is a noncommutative multiplication (or concatenation) of functions on Wiener space.

1. Introduction. A family $\{\mathcal{A}_t, t > 0\}$ of commutative Banach algebras of functionals on Wiener space was introduced in [4] and it was shown that for every $F \in \mathcal{A}_t$, the functional integral $K_\lambda^t(F)$ exists and is given by a time-ordered perturbation expansion which serves to disentangle, in the sense of Feynman's operational calculus for noncommuting operators, the operator $K_\lambda^t(F)$. The first author and Lapidus introduced in [6] the noncommutative operations $*$ and \dagger on Wiener functionals, and they showed as one of their main results that if $F \in \mathcal{A}_{t_1}$ and $G \in \mathcal{A}_{t_2}$, then $F * G \in \mathcal{A}_{t_1+t_2}$ and $K_\lambda^{t_1+t_2}(F * G) = K_\lambda^{t_1}(F)K_\lambda^{t_2}(G)$. It follows then that the product of operators which can be disentangled (in their framework) can itself be disentangled.

Our aim in this paper is related but quite different. It is to show that for any $H \in \mathcal{A}_{a,t}^c$ (see Definition 2.4 below) and any partition $a = r_0 < r_1 < \cdots < r_h = t$ of $[a, t]$, H can be decomposed into a sum of terms where each term is the $*$ product of F_j 's one from each of the algebras $\mathcal{A}_{r_{j-1}, r_j}^c$, $j = 1, \dots, h$. This decomposition of the function H along with the relationship between the functional integrals $K_\lambda^{r_{j-1}, r_j}(\cdot)$,

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$j = 1, \dots, h$ and the $*$ operation yields a corresponding decomposition of the operator $K_\lambda^{a,t}(H)$ into a sum of products in the Banach algebra $\mathcal{L}(L^2(\mathbf{R}^d))$ of the h operators associated with the h subintervals of the partition. The general formula for the decomposition of an arbitrary function $H \in \mathcal{A}_{a,t}^c$ (see Theorem 2.10 below) is quite complicated. But there are interesting special cases that are relatively simple. A particularly interesting case involves the exponential function (other analytic functions can be treated as well). There is also a type of multinomial formula. This formula and others from the present paper have been found useful for doing certain explicit calculations in the thesis of L. Johnson [9] (in preparation). The initial motivation for the present work is discussed briefly in the remark at the end of this paper.

2. Decomposing the Banach algebra. We begin with some definitions and notations from [4].

Let $\mathbf{C}, \mathbf{C}_+, \tilde{\mathbf{C}}_+$ denote, respectively, the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part. Let $L^2(\mathbf{R}^d)$ denote the space of Borel measurable \mathbf{C} -valued functions ψ on \mathbf{R}^d such that $|\psi|^2$ is integrable with respect to Lebesgue measure on \mathbf{R}^d .

Given $t > 0$, let $C[0, t] = C^t = C^{0,t}$ denote the space of continuous functions x on $[0, t]$ with values in \mathbf{R}^d . Let $C_0[0, t] = C_0^t = C_0^{0,t}$ denote Wiener space, that is, the set of all x in C^t which vanish at 0. $m_t = m_{0,t}$ will denote Wiener measure on C_0^t . Let $\mathcal{L}(L^2(\mathbf{R}^d))$ be the space of bounded linear operators from $L^2(\mathbf{R}^d)$ into itself. The space of Borel measurable \mathbf{C} -valued functions on \mathbf{R}^d which are essentially bounded will be denoted $L^\infty(\mathbf{R}^d)$. $U(0, t)$ will denote the space of complex continuous Borel measures μ on the open interval $(0, t)$. A \mathbf{C} -valued Borel measurable function θ on $(0, t) \times \mathbf{R}^d$ is said to belong to $L_{\infty 1; \mu} = L_{\infty 1; \mu}(0, t)$ if

$$\|\theta\|_{\infty 1, \mu} := \int_0^t \|\theta(s, \cdot)\|_\infty d|\mu|(s) < \infty.$$

Note that if $\theta \in L_{\infty 1; \mu}$, then $\theta(s, \cdot) \in L^\infty(\mathbf{R}^d)$ for μ almost everywhere s in $(0, t)$.

Definition 2.1. Fix $t > 0$. Let F be a function from C^t to \mathbf{C} . Given

$\lambda > 0$, $\psi \in L^2(\mathbf{R}^d)$ and $\xi \in \mathbf{R}^d$, we consider the expression

$$(2.1) \quad (K_\lambda^t(F)\psi)(\xi) = \int_{C_0^t} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}x(t) + \xi) dm(x).$$

The operator-valued function space integral $K_\lambda^t(F)$ exists for $\lambda > 0$ if (2.1) defines $K_\lambda^t(F)$ as an element of $\mathcal{L}(L^2(\mathbf{R}^d))$. If, in addition, $K_\lambda^t(F)$, as a function of λ , has an extension to an analytic function on \mathbf{C}_+ and a strongly continuous function on $\tilde{\mathbf{C}}_+$, we say that $K_\lambda^t(F)$ exists for $\lambda \in \tilde{\mathbf{C}}_+$. When λ is purely imaginary $K_\lambda^t(F)$ is called the analytic (in mass) operator-valued Feynman integral of F .

Definition 2.2. Given functions F and G defined on C^t , we say that F is equivalent to G (write $F \sim G$) if, for every $\lambda > 0$, $F(\lambda^{-1/2}x + \xi) = G(\lambda^{-1/2}x + \xi)$ for $m_t \times Leb$ almost everywhere $(x, \xi) \in C_0^t \times \mathbf{R}^d$.

We restrict attention to continuous measures whereas that restriction is not made in [4]. Let \mathcal{A}_0 be the collection of functions of the form $H(x) = \prod_{u=1}^m \int_0^t \theta_u(s, x(s)) d\mu_u(s)$, where $\theta_u \in L_{\infty 1; \mu_u}$ and $\mu_u \in U(0, t)$, $u = 1, \dots, m$.

Lemma 2.3. Let H_n be a sequence from \mathcal{A}_0 such that $\sum_{n=0}^\infty \prod_{u=1}^{m_n} \|\theta_{n,u}\|_{\infty 1; \mu_{n,u}} < \infty$. Then for every $\lambda \in \tilde{\mathbf{C}}_+$, the individual terms of the series $\sum_{n=0}^\infty H_n(\lambda^{-1/2}x + \xi)$ are defined and the series converges absolutely for $m_t \times Leb$ almost everywhere $(x, \xi) \in C_0^t \times \mathbf{R}^d$.

Proof. This is a special case of Corollary 2.1 in [4] and so follows immediately. \square

Definition 2.4. Let (H_n) be a sequence from \mathcal{A}_0 satisfying

$$(2.2) \quad \sum_{n=0}^\infty \prod_{u=1}^{m_n} \|\theta_{n,u}\|_{\infty 1; \mu_{n,u}} < \infty.$$

Let H be defined by

$$(2.3) \quad H(\lambda^{-1/2}x + \xi) = \sum_{n=0}^\infty H_n(\lambda^{-1/2}x + \xi).$$

We define \mathcal{A}_t^c to be the collection of equivalence class of functionals, each of which contains a function H which arises as just described from a sequence (H_n) in \mathcal{A}_0 . For H in \mathcal{A}_t^c , we define $\|H\|_t^c$ as the infimum of the left side of (2.2) for all choices of sequences (H_n) from \mathcal{A}_0 satisfying (2.3).

The proof of the next result is almost exactly the same as the proof of Theorem 6.1 from [4] and so will not be given.

Theorem 2.5. *($\mathcal{A}_t^c, \|\cdot\|_t^c$) is a commutative Banach algebra with identity. Moreover, given H in \mathcal{A}_t^c , $K_\lambda^t(H)$ exists for all $\lambda \in \tilde{\mathbf{C}}_+$ and satisfies the norm estimate $\|K_\lambda^t(H)\| \leq \|H\|_t^c$.*

Remark. (1) \mathcal{A}_t^c is a subalgebra of the algebra \mathcal{A}_t .

(2) Since t varies and $\theta \in L_{\infty 1; \mu}(a, t)$ and μ is a complex continuous Borel measure on (a, t) , we are using the symbols $K_\lambda^{a,t}$, $\mathcal{A}_{a,t}^c$ and $\|\cdot\|_{a,t}^c$ instead of K_λ^t , \mathcal{A}_t and $\|\cdot\|_t$, respectively, which were used in [4].

We will need the following definitions. Fix r_1, \dots, r_h such that

$$(2.4) \quad a = r_0 < r_1 < \dots < r_h = t.$$

Let R_j be the map which restricts $x \in C^{a,t}$ to the subinterval $[r_{j-1}, r_j]$, $j = 1, \dots, h$. Given $x \in C^{a,t}$ and $j \in \{1, \dots, h\}$, we define

$$(2.5) \quad x_j(s) = (R_j x)(s) = x(s), \quad s \in [r_{j-1}, r_j].$$

Now, given functions F_j on C^{r_{j-1}, r_j} , $j = 1, \dots, h$, we define $F_1 \dot{+} \dots \dot{+} F_h$ and $F_1 * \dots * F_h$ on $C^{a,t}$ by

$$(2.6) \quad (F_1 \dot{+} \dots \dot{+} F_h)(x) := F_1(x_1) + \dots + F_h(x_h),$$

$$(2.7) \quad (F_1 * \dots * F_h)(x) := F_1(x_1) \dots F_h(x_h).$$

These operations $\dot{+}, *$ are different from, but very similar to, the operations $\dot{+}, *$ in [6].

We are seeking a decomposition of any $H \in \mathcal{A}_{a,t}^c$ with respect to any partition as in (2.4) of $[a, t]$. The general case is rather complicated but we start here with a simple example for the case $h = 2$ which allows us to illustrate notation which will be useful as we continue.

Example 2.6. Let $a < r < t$. Let $H(x) = \int_a^t \theta(s, x(s)) d\mu(s)$ where $\theta \in L_{\infty 1; \mu}(a, t)$, μ is a complex continuous Borel measure on (a, t) , and $x \in C^{a, t}$. Then H is a function on $C^{a, t}$ which clearly belongs to $\mathcal{A}_{a, t}^c$. Now we define

$$(2.8) \quad F(x_1) := \int_a^r \theta(s, x_1(s)) d\mu(s) = \int_a^r \theta(s, x(s)) d\mu(s),$$

$$(2.9) \quad G(x_2) := \int_r^t \theta(s, x_2(s)) d\mu(s) = \int_r^t \theta(s, x(s)) d\mu(s).$$

Then F and G are functions on $C^{a, r}$ and $C^{r, t}$, respectively, and $F \in \mathcal{A}_{a, r}^c$ and $G \in \mathcal{A}_{r, t}^c$. So we can write

$$(2.10) \quad \begin{aligned} H(x) &= \int_a^t \theta(s, x(s)) d\mu(s) \\ &= \int_a^r \theta(s, x_1(s)) d\mu(s) + \int_r^t \theta(s, x_2(s)) d\mu(s) \\ &= F(x_1) + G(x_2) = (F \dot{+} G)(x). \end{aligned}$$

Hence we can write

$$H(x) = F(x_1) + G(x_2) = (F * G^\circ)(x) + (F^\circ * G)(x),$$

where we interpret F° to be $1_{[a, r]}$, where $1_{[a, r]}$ is the function which is identically one on $C^{a, r}$. Similarly, G° is interpreted as $1_{[r, t]}$ where $1_{[r, t]}$ is identically one on $C^{r, t}$.

The simple decomposition of the special function H in (2.11) leads to simple decompositions of H^n and $\exp(H)$ in the proposition to follow. The case of a general function H in the disentangling algebra $\mathcal{A}_{a, t}^c$ is much more complicated (see Theorem 2.10 below).

Proposition 2.7. *Let the function $H : C^{a, t} \rightarrow \mathbf{C}$ be defined by*

$$(2.11) \quad H(x) = \int_a^t \theta(s, x(s)) d\mu(s)$$

where $\theta \in L_{\infty 1;\mu}(a, t)$, and let R_j be the restriction map from $C^{a,t}$ to C^{r_{j-1}, r_j} , where $j = 1, 2, \dots, h$, $r_0 = a$, $r_h = t$. For the function F_j on C^{r_{j-1}, r_j} given by $F_j(x_j) = \int_{r_{j-1}}^{r_j} \theta(s, x_j(s)) d\mu(s)$, $j = 1, \dots, h$, we have $H = F_1 \dot{+} \dots \dot{+} F_h$. Further,

$$(1) \exp(F_1 \dot{+} \dots \dot{+} F_h) = \exp(F_1) * \dots * \exp(F_h).$$

$$(2) (F_1 \dot{+} \dots \dot{+} F_h)^n = \sum_{q_1 + \dots + q_h = n} \frac{n!}{q_1! \dots q_h!} F_1^{q_1} * \dots * F_h^{q_h}.$$

Proof. Closely related formulas are proved in Theorem 3.4 of [6] in the case $h = 2$. The simple proofs of (1) and (2) above follow the same lines. \square

Lemma 2.8 and Theorem 2.9 will establish our main result in the case $h = 2$. The proof of this case already includes the main ideas used in proving Theorem 2.10, but the combinatorics involved are somewhat simpler.

Lemma 2.8. *Let $H_u(x) = \int_a^t \theta_u(s, x(s)) d\mu_u(s)$ where $\theta_u \in L_{\infty 1;\mu}(a, t)$, $u = 1, 2, \dots, m$, $x \in C^{a,t}$, and let $a < r < t$. Then, for the function $F_{u1}(x) = \int_a^r \theta_u(s, x(s)) d\mu_u(s)$ and $F_{u2}(x) = \int_r^t \theta_u(s, x(s)) d\mu_u(s)$, we have*

$$\prod_{u=1}^m H_u = \prod_{u=1}^m (F_{u1} \dot{+} F_{u2}) = \sum_{i_1=1}^2 \dots \sum_{i_m=1}^2 \prod_{u=1}^m F_{u1}^{\delta(i_u,1)} * \prod_{u=1}^m F_{u2}^{\delta(i_u,2)},$$

where δ is the Kronecker delta and $F_{u1}^\circ, F_{u2}^\circ$ are identically one on $C^{a,r}$ and $C^{r,t}$, respectively.

Proof. For any u , we define $F_{u1}(x) := \int_a^r \theta_u(s, x(s)) d\mu_u(s)$ and $F_{u2}(x) := \int_r^t \theta_u(s, x(s)) d\mu_u(s)$. Then $H_u(x) = F_{u1}(x_1) + F_{u2}(x_2)$ by (2.8) and (2.9). We can also write $H_u(x) = (F_{u1}^1 * F_{u2}^\circ)(x) + (F_{u1}^\circ * F_{u2}^1)(x)$. Note that this last equality is the desired formula when $m = 1$. We will proceed by induction. We now examine the case $m = 2$. Using

the case $m = 1$, we obtain

$$\begin{aligned}
 (H_1 H_2)(x) &= (F_{11}^1 * F_{12}^\circ + F_{11}^\circ * F_{12}^1)(F_{21}^1 * F_{22}^\circ + F_{21}^\circ * F_{22}^1)(x) \\
 &= (F_{11}^1 * F_{12}^\circ)(F_{21}^1 * F_{22}^\circ)(x) + (F_{11}^1 * F_{12}^\circ)(F_{21}^\circ * F_{22}^1)(x) \\
 &\quad + (F_{11}^\circ * F_{12}^1)(F_{21}^1 * F_{22}^\circ)(x) + (F_{11}^\circ * F_{12}^1)(F_{21}^\circ * F_{22}^1)(x) \\
 (2.12) \quad &= (F_{11}^1 F_{21}^1)(x_1)(F_{12}^\circ F_{22}^\circ)(x_2) + (F_{11}^1 F_{21}^\circ)(x_1)(F_{12}^\circ F_{22}^1)(x_2) \\
 &\quad + (F_{11}^\circ F_{21}^1)(x_1)(F_{12}^1 F_{22}^\circ)(x_2) + (F_{11}^\circ F_{21}^\circ)(x_1)(F_{12}^1 F_{22}^1)(x_2) \\
 &= (F_{11}^1 F_{21}^1 * F_{12}^\circ F_{22}^\circ)(x) + (F_{11}^1 F_{21}^\circ * F_{12}^\circ F_{22}^1)(x) \\
 &\quad + (F_{11}^\circ F_{21}^1 * F_{12}^1 F_{22}^\circ)(x) + (F_{11}^\circ F_{21}^\circ * F_{12}^1 F_{22}^1)(x).
 \end{aligned}$$

So

$$H_1 H_2 = \sum_{i_1=1}^2 \sum_{i_2=1}^2 F_{11}^{\delta(i_1,1)} F_{21}^{\delta(i_2,1)} * F_{12}^{\delta(i_1,2)} F_{22}^{\delta(i_2,2)},$$

and this is the desired formula for $m = 2$. Now we assume that the formula holds for m and examine the case $m + 1$.

$$\begin{aligned}
 &H_1 \cdots H_m H_{m+1} \\
 &= (H_1 \cdots H_m) H_{m+1} \\
 &= \left\{ \sum_{i_1=1}^2 \cdots \sum_{i_m=1}^2 \prod_{u=1}^m F_{u1}^{\delta(i_u,1)} * \prod_{u=1}^m F_{u2}^{\delta(i_u,2)} \right\} \\
 &\quad \cdot \{ F_{m+1,1}^1 * F_{m+1,2}^\circ + F_{m+1,1}^\circ * F_{m+1,2}^1 \} \\
 &= \left\{ \sum_{i_1=1}^2 \cdots \sum_{i_m=1}^2 \left[\left(\prod_{u=1}^m F_{u1}^{\delta(i_u,1)} \right) F_{m+1,1}^1 \right] * \left[\left(\prod_{u=1}^m F_{u2}^{\delta(i_u,2)} \right) F_{m+1,2}^\circ \right] \right\} \\
 &\quad + \left\{ \sum_{i_1=1}^2 \cdots \sum_{i_m=1}^2 \left[\left(\prod_{u=1}^m F_{u1}^{\delta(i_u,1)} \right) F_{m+1,1}^\circ \right] * \left[\left(\prod_{u=1}^m F_{u2}^{\delta(i_u,2)} \right) F_{m+1,2}^1 \right] \right\} \\
 &= \sum_{i_1=1}^2 \cdots \sum_{i_m=1}^2 \sum_{i_{m+1}=1}^2 \prod_{u=1}^{m+1} F_{u1}^{\delta(i_u,1)} * \prod_{u=1}^{m+1} F_{u2}^{\delta(i_u,2)},
 \end{aligned}$$

where the last equality will now be explained. Let $u = m + 1$. When $i_{m+1} = 1$, the last factor in the lefthand product is $F_{m+1,1}^{\delta(1,1)} = F_{m+1,1}^1$. The last factor in the righthand product is $F_{m+1,2}^{\delta(1,2)} = F_{m+1,2}^\circ$. Now we

consider the case when $i_{m+1} = 2$. Then the last factor in the lefthand product is $F_{m+1,1}^{\delta(2,1)} = F_{m+1,1}^\circ$. On the other hand, the last factor in the righthand product is $F_{m+1,2}^{\delta(2,2)} = F_{m+1,2}^1$. Thus we see that the last equality in the string of four equalities above is justified, and so our induction proof is complete. \square

Theorem 2.9. *Let H be in $\mathcal{A}_{a,t}^c$, and let $a < r < t$. Then there exist $F_{nu1} \in \mathcal{A}_{a,r}^c$ and $F_{nu2} \in \mathcal{A}_{r,t}^c$, $u = 1, \dots, m_n$, $n = 1, 2, \dots$, defined in (2.14) below such that*

$$H = \sum_{n=0}^{\infty} \sum_{i_1=1}^2 \cdots \sum_{i_{m_n}=1}^2 \prod_{u=1}^{m_n} F_{nu1}^{\delta(i_u,1)} * \prod_{u=1}^{m_n} F_{nu2}^{\delta(i_u,2)},$$

where δ is the Kronecker delta and $F_{nu1}^\circ, F_{nu2}^\circ$ are identically one on $C^{a,r}$ and $C^{r,t}$, respectively.

Proof. Let H be in $\mathcal{A}_{a,t}^c$, and let $a < r < t$. We take an arbitrary representation of H in terms of measures $\mu_{n,u} \in U(a, t)$ and functions $\theta_{n,u} \in L_{\infty 1; \mu_{n,u}}(a, t)$, $u = 1, \dots, m_n$, $n = 1, 2, \dots$, such that

$$H(x) = \sum_{n=0}^{\infty} \prod_{u=1}^{m_n} \int_a^t \theta_{n,u}(s, x(s)) d\mu_{n,u}(s), \quad x \in C^{a,t},$$

where

$$(2.13) \quad \sum_{n=0}^{\infty} \prod_{u=1}^{m_n} \|\theta_{n,u}\|_{\infty 1; \mu_{n,u}} < \infty.$$

Also we can write $H(x)$ as follows. $H(x) = \sum_{n=0}^{\infty} \prod_{u=1}^{m_n} H_{nu}(x)$ where $H_{nu}(x) = \int_a^t \theta_{n,u}(s, x(s)) d\mu_{n,u}(s)$. Now we define

$$(2.14) \quad \begin{aligned} F_{nu1}(x) &= \int_a^r \theta_{n,u}(s, x(s)) d\mu_{n,u}(s), \\ F_{nu2}(x) &= \int_r^t \theta_{n,u}(s, x(s)) d\mu_{n,u}(s). \end{aligned}$$

Then $F_{nu1} \in \mathcal{A}_{a,r}^c$ and $F_{nu2} \in \mathcal{A}_{r,t}^c$ because of (2.13). Hence, by Lemma 2.8,

$$\begin{aligned} H(x) &= \sum_{n=0}^{\infty} \prod_{u=1}^{m_n} H_{nu}(x) \\ &= \sum_{n=0}^{\infty} \sum_{i_1=1}^2 \cdots \sum_{i_{m_n}=1}^2 \left(\prod_{u=1}^{m_n} F_{nu1}^{\delta(i_u,1)} * \prod_{u=1}^{m_n} F_{nu2}^{\delta(i_u,2)} \right)(x). \quad \square \end{aligned}$$

More generally, if we use the same method from Lemma 2.8 and Theorem 2.9, we obtain that for any $H \in \mathcal{A}_{a,t}^c$ and any partition $a = r_0 < r_1 < \cdots < r_h = t$ of $[a, t]$, H can be written as the sum of terms where each term is the $*$ product of F_j 's, one from each of the algebras $\mathcal{A}_{r_{j-1},r_j}^c$, $j = 1, \dots, h$. We state this result precisely as the following theorem. This theorem and the corresponding result for operators, Theorem 3.8, are our main results.

Theorem 2.10. *Let H be in $\mathcal{A}_{a,t}^c$, and let σ be any partition of $[a, t]$ such that $\sigma : a = r_0 < r_1 < \cdots < r_h = t$. Let R_j be the restriction map from $C^{a,t}$ to C^{r_{j-1},r_j} where $j = 1, \dots, h$, $r_0 = a$, $r_h = t$. Then there exist $F_{nu1} \in \mathcal{A}_{a,r_1}^c, F_{nu2} \in \mathcal{A}_{r_1,r_2}^c, \dots, F_{nuh} \in \mathcal{A}_{r_{h-1},r_h}^c$, $u = 1, 2, \dots, m_n$, $n = 1, 2, \dots$, defined much as in (2.14) above such that*

$$H = \sum_{n=0}^{\infty} \sum_{i_1=1}^h \cdots \sum_{i_{m_n}=1}^h \prod_{u=1}^{m_n} F_{nu1}^{\delta(i_u,1)} * \cdots * \prod_{u=1}^{m_n} F_{nuh}^{\delta(i_u,h)},$$

where F_{nuj}° is identically one on C^{r_{j-1},r_j} , F_{nuj} is a function on C^{r_{j-1},r_j} , $j = 1, 2, \dots, h$, respectively, and δ is the Kronecker delta.

Proof. Let $a = r_0 < r_1 < \cdots < r_h = t$. The key is to extend Lemma 2.8. We define

$$F_{u1}(x) = \int_a^{r_1} \theta_u(s, x(s)) d\mu_u(s), \dots, F_{uh}(x) = \int_{r_{h-1}}^{r_h} \theta_u(s, x(s)) d\mu_u(s).$$

Then $H_u(x) = F_{u1}(x_1) + F_{u2}(x_2) + \cdots + F_{uh}(x_h)$, where $x_j \in C^{r_{j-1},r_j}$, $j = 1, \dots, h$, and we can write

$$\prod_{u=1}^m H_u = \prod_{u=1}^m (F_{u1}^1 * F_{u2}^\circ * \cdots * F_{uh}^\circ + \cdots + F_{u1}^\circ * F_{u2}^\circ * \cdots * F_{uh}^1).$$

Now we examine the case $m = 2$. We have

$$\begin{aligned} H_1 H_2 &= (F_{11}^1 * F_{12}^\circ * \cdots * F_{1h}^\circ + F_{11}^\circ * F_{12}^1 * \cdots * F_{1h}^\circ + \cdots \\ &\quad + F_{11}^\circ * F_{12}^\circ * \cdots * F_{1h}^1) \\ &\quad \cdot (F_{21}^1 * F_{22}^\circ * \cdots * F_{2h}^\circ + F_{21}^\circ * F_{22}^1 * \cdots * F_{2h}^\circ + \cdots \\ &\quad + F_{21}^\circ * F_{22}^\circ * \cdots * F_{2h}^1). \end{aligned}$$

In order to obtain H_1, H_2 , we use the formula $(a_1 + \cdots + a_h)(b_1 + \cdots + b_h) = \sum_{i=1}^h \sum_{j=1}^h a_i b_j$. If, for example, we multiply the second term of H_1 with the first term of H_2 , we obtain

$$\begin{aligned} (F_{11}^\circ * F_{12}^1 * \cdots * F_{1h}^\circ)(F_{21}^1 * F_{22}^\circ * \cdots * F_{2h}^\circ)(x) \\ = (F_{11}^\circ F_{21}^1 * F_{12}^1 F_{22}^\circ * \cdots * F_{1h}^\circ F_{2h}^\circ)(x). \end{aligned}$$

This equality is the $i_1 = 2, i_2 = 1$ term of the equation

$$\sum_{i_1=1}^h \sum_{i_2=1}^h F_{11}^{\delta(i_1,1)} F_{21}^{\delta(i_2,1)} * F_{12}^{\delta(i_1,2)} F_{22}^{\delta(i_2,2)} * \cdots * F_{1h}^{\delta(i_1,h)} F_{2h}^{\delta(i_2,h)}.$$

Therefore,

$$H_1 H_2 = \sum_{i_1=1}^h \sum_{i_2=1}^h F_{11}^{\delta(i_1,1)} F_{21}^{\delta(i_2,1)} * F_{12}^{\delta(i_1,2)} F_{22}^{\delta(i_2,2)} * \cdots * F_{1h}^{\delta(i_1,h)} F_{2h}^{\delta(i_2,h)}.$$

The proof of the desired formula can now be completed by induction. We omit this step since it is essentially the same as in the proof of Lemma 2.8. The final formula in this theorem now follows in essentially the same way as in the proof of Theorem 2.9. \square

3. Consequences of the decomposition for the functional integral $K_\lambda^{a,t}$. Our main result in this section, Theorem 3.8, gives a time-ordered decomposition of the operator $K_\lambda^{a,t}(H)$ where $H \in \mathcal{A}_{a,t}^c$ has the decomposition given by Theorem 2.10. The operator $K_\lambda^{a,t}(H)$ is written as the sum of products of the h operators associated with the h subintervals of a given partition of $[a, t]$. The other results or examples in this section are either (i) of a general nature and closely follow results from [6], [7] or (ii) start with the special function

$H(x) := \int_a^t \theta(s, x(s)) d\mu(s)$ and consider the operator decomposition associated with a partition of the interval $[a, t]$ and the decomposition of the functions H^n and $\exp(H)$.

Theorem 3.1. *Let σ be any partition of $[a, t]$ such that $\sigma : a = r_0 < r_1 < \dots < r_h = t$. Let F_j be in $\mathcal{A}_{r_{j-1}, r_j}^c$, $j = 1, \dots, h$. Then $F_1 * \dots * F_h$ is in $\mathcal{A}_{a, t}^c$ and, for all $\lambda \in \tilde{\mathbf{C}}_+$, $K_\lambda^{r_{j-1}, r_j}(F_j)$, $j = 1, \dots, h$, and $K_\lambda^{a, t}(F_1 * \dots * F_h)$ exist and*

$$K_\lambda^{a, t}(F_1 * \dots * F_h) = K_\lambda^{a, r_1}(F_1) \dots K_\lambda^{r_{h-1}, t}(F_h).$$

Proof. The proof of this result proceeds much like the proofs of Theorems 5.2 and 5.3 in [6]. \square

Theorem 3.2. *Let $H \in \mathcal{A}_{a, t}^c$, and let f be a \mathbf{C} -valued function which is analytic in a disk about the origin with radius strictly greater than $\|H\|_{a, t}^c$. Then the function G defined by $G(x) := f(H(x))$ is in $\mathcal{A}_{a, t}^c$ and so $K_\lambda^{a, t}(G)$ exists for all $\lambda \in \tilde{\mathbf{C}}_+$.*

Proof. This follows using some standard facts about Banach algebras [10, pp. 202–205]. \square

Theorem 3.3. *Let $H(x) = \int_a^t \theta(s, x(s)) d\mu(s)$ where $\theta \in L_{\infty 1; \mu}(a, t)$. Then for F and G as in (2.8) and (2.9), $\exp(F \dot{+} G)$, $\exp(F) * \exp(G)$ are in $\mathcal{A}_{a, t}^c$ and, for all $\lambda \in \tilde{\mathbf{C}}_+$,*

$$K_\lambda^{a, t}(\exp(F \dot{+} G)) = K_\lambda^{a, r}(\exp(F))K_\lambda^{r, t}(\exp(G)).$$

Proof. Since $H \in \mathcal{A}_{a, t}^c$, by Theorem 3.2, $K_\lambda^{a, t}(\exp(H))$ exists for all $\lambda \in \tilde{\mathbf{C}}_+$. So by (1) of Proposition 2.7 and Theorem 3.1,

$$\begin{aligned} K_\lambda^{a, t}(\exp(H)) &= K_\lambda^{a, t}(\exp(F \dot{+} G)) \\ &= K_\lambda^{a, t}(\exp(F) * \exp(G)) \\ &= K_\lambda^{a, r}(\exp(F))K_\lambda^{r, t}(\exp(G)). \quad \square \end{aligned}$$

The next result follows from the $h = 2$ case of formula (2) in Proposition 2.7 and Theorem 3.4 of [6] and the equalities $K_\lambda^{a,r}(F^\circ) = \exp[-(r - a)(H_\circ/\lambda)]$, $K_\lambda^{r,t}(G^\circ) = \exp[-(t - r)(H_\circ/\lambda)]$ where $H_\circ = -1/2\Delta$ is the free Hamiltonian acting in $L^2(\mathbf{R}^d)$.

Theorem 3.4. *Let $H(x) = \int_a^t \theta(s, x(s)) d\mu(s)$ where $\theta \in L_{\infty 1;\mu}(a, t)$, and let n be a positive integer. Let $a < r < t$. Then for F and G as in (2.8) and (2.9), $(F \dot{+} G)^n$ is in $\mathcal{A}_{a,t}^c$ and, for all $\lambda \in \tilde{\mathbf{C}}_+$,*

$$K_\lambda^{a,t}[(F \dot{+} G)^n] = \sum_{p+q=n} \frac{n!}{p!q!} K_\lambda^{a,r}(F^p) K_\lambda^{r,t}(G^q).$$

The proof of Theorem 3.4 as given above is simple. One can instead give the proof by computing appropriate Wiener integrals. However, here and certainly in more involved situations (for example, Corollary 3.6 of the present paper), the resulting computations are much lengthier and it is not easy to recognize general patterns. The results in this paper have helped considerably in formulating and proving some of the results and examples in the thesis of Johnson [9] and in a related paper of hers with the first author of this paper.

In Example 3.5 to follow we write explicit formulas for the operators involved in Theorem 3.4 in the case $h = n = 2$ where $a = 0$ and $r = t/2$.

Example 3.5. Let μ be a continuous measure and let $\theta \in L_{\infty 1;\mu}$. We consider the case $h = n = 2$, $a = 0$ and $r = t/2$ of Theorem 3.4. Let $F(x) := \{\int_0^t \theta(s, x(s)) d\mu(s)\}^2/2$, $F_1(x) := \int_0^{t/2} \theta(s, x(s)) d\mu(s)$ and $F_2(x) := \int_{t/2}^t \theta(s, x(s)) d\mu(s)$. Then

$$\begin{aligned} F(x) &= (F_1 \dot{+} F_2)^2(x)/2 = F_1^2(x_1)/2 + F_1(x_1)F_2(x_2) + F_2^2(x_2)/2 \\ &= (F_1^2 * F_2^\circ)(x)/2 + (F_1 * F_2)(x) + (F_1^\circ * F_2^2)(x)/2. \end{aligned}$$

By Corollary 1.2 in [4], we have

$$\begin{aligned} K_\lambda^t(F) &= \int_0^t \int_0^{s_2} e^{-s_1(H_\circ/\lambda)} \theta(s_1) e^{-(s_2-s_1)(H_\circ/\lambda)} \theta(s_2) \\ &\quad \cdot e^{-(t-s_2)(H_\circ/\lambda)} d(\mu \times \mu)(s_1, s_2), \end{aligned}$$

where $0 < s_1 < s_2 < t$. Similarly, we obtain $K_\lambda^{(0,t/2)}(F_1)$, $K_\lambda^{(t/2,t)}(F_2)$, $K_\lambda^{(0,t/2)}(F_1^\circ)$ and $K_\lambda^{(t/2,t)}(F_2^\circ)$. So by the semigroup property of $e^{-s(H_\circ/\lambda)}$, we have

$$\begin{aligned} & K_\lambda^{(0,t/2)}(F_1^2)K_\lambda^{(t/2,t)}(F_2^\circ)/2 \\ &= \int_0^{t/2} \int_0^{s_2} e^{-s_1(H_\circ/\lambda)} \theta(s_1) e^{-(s_2-s_1)(H_\circ/\lambda)} \theta(s_2) e^{-(t/2-s_2)(H_\circ/\lambda)} \\ & \quad \cdot e^{-(t-t/2)(H_\circ/\lambda)} d(\mu \times \mu)(s_1, s_2) \\ &= \int_0^{t/2} \int_0^{s_2} e^{-s_1(H_\circ/\lambda)} \theta(s_1) e^{-(s_2-s_1)(H_\circ/\lambda)} \theta(s_2) \\ & \quad \cdot e^{-(t-s_2)(H_\circ/\lambda)} d(\mu \times \mu)(s_1, s_2), \end{aligned}$$

where $0 < s_1 < s_2 < t/2$. Similarly, we have

$$\begin{aligned} & K_\lambda^{(0,t/2)}(F_1^\circ)K_\lambda^{(t/2,t)}(F_2^2)/2 \\ &= \int_{t/2}^t \int_0^{s_2} e^{-s_1(H_\circ/\lambda)} \theta(s_1) e^{-(s_2-s_1)(H_\circ/\lambda)} \theta(s_2) \\ & \quad \cdot e^{-(t-s_2)(H_\circ/\lambda)} d(\mu \times \mu)(s_1, s_2), \end{aligned}$$

where $t/2 < s_1 < s_2 < t$. Also,

$$\begin{aligned} & K_\lambda^{(0,t/2)}(F_1)K_\lambda^{(t/2,t)}(F_2) \\ &= \int_0^{t/2} e^{-s_1(H_\circ/\lambda)} \theta(s_1) e^{-(t/2-s_1)(H_\circ/\lambda)} d\mu(s_1) \\ & \quad \cdot \int_{t/2}^t e^{-(s_2-t/2)(H_\circ/\lambda)} \theta(s_2) e^{-(t-s_2)(H_\circ/\lambda)} d\mu(s_2) \\ &= \int_0^{t/2} \int_{t/2}^t e^{-s_1(H_\circ/\lambda)} \theta(s_1) e^{-(s_2-s_1)(H_\circ/\lambda)} \theta(s_2) \\ & \quad \cdot e^{-(t-s_2)(H_\circ/\lambda)} d(\mu \times \mu)(s_1, s_2), \end{aligned}$$

where $0 < s_1 < t/2 < s_2 < t$. Hence we have obtained the explicit formulas that we sought.

Our final corollary gives the full consequences for operators of Proposition 2.7 and the multivariate version of Theorems 3.3 and 3.4.

Corollary 3.6. *Let $H(x) = \int_a^t \theta(s, x(s)) d\mu(s)$ where $\theta \in L_{\infty 1; \mu}(a, t)$, and let R_j be the restriction map from $C^{a,t}$ to C^{r_{j-1}, r_j} , where $j = 1, 2, \dots, h$, $r_0 = a$, $r_h = t$. Then for the function F_j on C^{r_{j-1}, r_j} given by $F_j(x_j) = \int_{r_{j-1}}^{r_j} \theta(s, x_j(s)) d\mu(s)$, $j = 1, \dots, h$, we have that $\exp(F_1 \dot{+} \dots \dot{+} F_h)$ and $\exp(F_1) * \dots * \exp(F_h)$ are in $\mathcal{A}_{a,t}^c$, and for all $\lambda \in \tilde{\mathbf{C}}_+$, we obtain*

$$\begin{aligned} (1) \quad & K_\lambda^{a,t}[\exp(F_1 \dot{+} \dots \dot{+} F_h)] = K_\lambda^{a,r_1}(\exp(F_1)) \cdots K_\lambda^{r_{h-1},t}(\exp(F_h)). \\ (2) \quad & K_\lambda^{a,t}[(F_1 \dot{+} \dots \dot{+} F_h)^n] \\ & = \sum_{q_1 + \dots + q_h = n} \frac{n!}{q_1! \cdots q_h!} K_\lambda^{a,r_1}(F_1^{q_1}) \cdots K_\lambda^{r_{h-1},t}(F_h^{q_h}). \end{aligned}$$

Proof. (1) We will show this by induction. First for $h = 2$, we are already done by Theorem 3.3. We assume that the property holds for $h = k - 1$. Then

$$\begin{aligned} & K_\lambda^{a,t}([\exp(F_1 \dot{+} \dots \dot{+} F_k)]) \\ & = K_\lambda^{a,t}(\exp[(F_1 \dot{+} \dots \dot{+} F_{k-1}) \dot{+} F_k]) \\ & = K_\lambda^{a,t}[\exp(F_1 \dot{+} \dots \dot{+} F_{k-1}) * \exp(F_k)] \\ & = K_\lambda^{a,r_{k-1}}(\exp(F_1 \dot{+} \dots \dot{+} F_{k-1})) K_\lambda^{r_{k-1},t}(\exp(F_k)) \\ & = K_\lambda^{a,r_1}(\exp(F_1)) \cdots K_\lambda^{r_{k-2},r_{k-1}}(\exp(F_{k-1})) K_\lambda^{r_{k-1},t}(\exp(F_k)). \end{aligned}$$

(2) By (2) of Proposition 2.7 and Theorem 3.1,

$$\begin{aligned} & K_\lambda^{a,t}(F_1 \dot{+} \dots \dot{+} F_h)^n \\ & = K_\lambda^{a,t} \left(\sum_{q_1 + \dots + q_h = n} \frac{n!}{q_1! \cdots q_h!} F_1^{q_1} * \dots * F_h^{q_h} \right) \\ & = \sum_{q_1 + \dots + q_h = n} \frac{n!}{q_1! \cdots q_h!} K_\lambda^{a,r_1}(F_1^{q_1}) \cdots K_\lambda^{r_{h-1},t}(F_h^{q_h}). \quad \square \end{aligned}$$

Theorem 3.7. *Let $H(x) = \int_a^t \theta(s, x(s)) d\mu(s)$ where $\theta \in L_{\infty 1; \mu}(a, t)$ is given. Let $g(z) = \sum_{n=0}^\infty a_n z^n$ be an analytic function with*

radius of convergence strictly greater than $\|\theta\|_{\infty 1; \mu}$. Let $I(x) := g(\int_a^t \theta(s, x(s)) d\mu(s))$ with g as above. Then $K_\lambda^{a,t}(I)$ exists for all $\lambda \in \tilde{\mathbf{C}}_+$. Further,

(1) For F and G as in (2.8) and (2.9), respectively, we have

$$\begin{aligned} K_\lambda^{a,t}(I) &= \sum_{n=0}^\infty a_n K_\lambda^{a,t}((F \dot{+} G)^n) \\ &= \sum_{n=0}^\infty a_n \sum_{p+q=n} \frac{n!}{p!q!} K_\lambda^{a,r}(F^p) K_\lambda^{r,t}(G^q). \end{aligned}$$

(2) For the function F_j on C^{r_{j-1}, r_j} given by

$$F_j(x_j) = \int_{r_{j-1}}^{r_j} \theta(s, x_j(s)) d\mu(s), \quad j = 1, \dots, h,$$

we have

$$K_\lambda^{a,t}(I) = \sum_{n=0}^\infty a_n \sum_{q_1 + \dots + q_h = n} \frac{n!}{q_1! \dots q_h!} K_\lambda^{a,r_1}(F_1^{q_1}) \dots K_\lambda^{r_{h-1},t}(F_h^{q_h}).$$

Proof. Clearly $I \in \mathcal{A}_{a,t}^c$ and so $K_\lambda^{a,t}(I)$ exists for all $\lambda \in \tilde{\mathbf{C}}_+$ and is given by $K_\lambda^{a,t}(I) = \sum_{n=0}^\infty a_n K_\lambda^{a,t}(H^n)$ (see [4]). (1) follows from Theorem 3.4 and (2) results from Corollary 3.6. \square

Next we give the time-ordered decomposition of operators $K_\lambda^{a,t}(H)$ that comes from our earlier decomposition result, Theorem 2.10, for arbitrary functionals H in the disentangling algebra $\mathcal{A}_{a,t}^c$. The proof of Theorem 3.8 below is easy, but this theorem and Theorem 2.10 are our main results.

Theorem 3.8. *Let H be in $\mathcal{A}_{a,t}^c$, and let σ be any partition of $[a, t]$ such that $\sigma : a = r_0 < r_1 < \dots < r_h = t$. Let R_j be the restriction map from $C^{a,t}$ to C^{r_{j-1}, r_j} where $j = 1, \dots, h$, $r_0 = a$, $r_h = t$. Then*

$K_\lambda^{a,t}(H)$ exists for all $\lambda \in \tilde{\mathbf{C}}_+$ and is given by

$$K_\lambda^{a,t}(H) = \sum_{n=0}^\infty \sum_{i_1=1}^h \dots \sum_{i_{m_n}=1}^h K_\lambda^{a,r_1} \left(\prod_{u=1}^{m_n} F_{nu1}^{\delta(i_u,1)} \right) \dots K_\lambda^{r_{h-1},r_h} \left(\prod_{u=1}^{m_n} F_{nuh}^{\delta(i_u,h)} \right)$$

where H is given by the formula in Theorem 2.10.

Proof. By Theorem 2.5, $K_\lambda^{a,t}(H)$ exists for all $\lambda \in \tilde{\mathbf{C}}_+$. So, by Theorem 2.10 and Theorem 3.1, we have the result. \square

We could now give a variety of examples illustrating the use of Theorems 2.10 and 3.8. For example, if θ_j and μ_j satisfy the usual conditions for $j = 1, \dots, n$, $a = r_0 < r_1 < \dots < r_h = t$ is a partition of $[a, t]$, and if g is an entire function of n complex variables, our results yield a decomposition of the operator $K_\lambda^{a,t}(H)$, for $\lambda \in \tilde{\mathbf{C}}_+$, where

$$H(x) := g \left(\int_a^t \theta_1(s, x(s)) d\mu_1(s), \dots, \int_a^t \theta_n(s, x(s)) d\mu_n(s) \right),$$

$$x \in C^{a,t}.$$

We now give an example which is different from our earlier explicit example but is a very simple special case of the situation just described above.

Example 3.9. Let

$$H(x) = \left(\int_0^t \theta_1(r, x(r)) d\mu_1(r) \right) \left(\int_0^t \theta_2(s, x(s)) d\mu_2(s) \right)$$

where μ_j is a continuous measure and $\theta_j \in L_{\infty 1; \mu_j}$, $j = 1, 2$. We define $F_{11}(x) := \int_0^{t/2} \theta_1(r, x(r)) d\mu_1(r)$, $F_{12}(x) := \int_{t/2}^t \theta_1(r, x(r)) d\mu_1(r)$, $F_{21}(x) := \int_0^{t/2} \theta_2(s, x(s)) d\mu_2(s)$ and $F_{22}(x) := \int_{t/2}^t \theta_2(s, x(s)) d\mu_2(s)$. Then, by Lemma 2.8,

$$H = F_{11}^1 F_{21}^1 * F_{12}^\circ F_{22}^\circ + F_{11} * F_{22} + F_{21} * F_{12} + F_{11}^\circ F_{21}^\circ * F_{12}^1 F_{22}^1.$$

Hence, by Theorem 3.8, we see that

$$K_\lambda^{0,t}(H) = K_\lambda^{0,t/2}(F_{11}^1 F_{21}^1) K_\lambda^{t/2,t}(F_{12}^\circ F_{22}^\circ) + K_\lambda^{0,t/2}(F_{11}) K_\lambda^{t/2,t}(F_{22}) \\ + K_\lambda^{0,t/2}(F_{21}) K_\lambda^{t/2,t}(F_{12}) + K_\lambda^{0,t/2}(F_{11}^\circ F_{21}^\circ) K_\lambda^{t/2,t}(F_{12}^1 F_{22}^1).$$

We finish this example by giving explicit expressions for all of the operators involved in the last equality. By Example 3.7 in [4], we obtain

$$K_\lambda^{0,t}(H) \\ = \int_0^t \int_0^r e^{-s(H_\circ/\lambda)} \theta_2(s) e^{-(r-s)H_\circ/\lambda} \theta_1(r) e^{-(t-r)H_\circ/\lambda} d\mu_2(s) d\mu_1(r) \\ + \int_0^t \int_0^s e^{-r(H_\circ/\lambda)} \theta_1(r) e^{-(s-r)H_\circ/\lambda} \theta_2(s) e^{-(t-s)H_\circ/\lambda} d\mu_1(r) d\mu_2(s).$$

Similarly, we can also calculate $K_\lambda^{0,t/2}(F_{11}^1 F_{21}^1)$, $K_\lambda^{t/2,t}(F_{12}^\circ F_{22}^\circ)$, $K_\lambda^{t/2,t}(F_{12}^1 F_{22}^1)$, $K_\lambda^{0,t/2}(F_{11}^\circ F_{21}^\circ)$. By the semigroup property of $e^{-s(H_\circ/\lambda)}$, we have

$$K_\lambda^{0,t/2}(F_{11}^1 F_{21}^1) K_\lambda^{t/2,t}(F_{12}^\circ F_{22}^\circ) \\ = \int_0^{t/2} \int_0^r e^{-s(H_\circ/\lambda)} \theta_2(s) e^{-(r-s)H_\circ/\lambda} \theta_1(r) e^{-(t-r)H_\circ/\lambda} d\mu_2(s) d\mu_1(r) \\ + \int_0^{t/2} \int_0^s e^{-r(H_\circ/\lambda)} \theta_1(r) e^{-(s-r)H_\circ/\lambda} \theta_2(s) e^{-(t-s)H_\circ/\lambda} d\mu_1(r) d\mu_2(s).$$

Similarly, we have

$$K_\lambda^{0,t/2}(F_{11}^\circ F_{21}^\circ) K_\lambda^{t/2,t}(F_{12}^1 F_{22}^1) \\ = \int_{t/2}^t \int_0^r e^{-s(H_\circ/\lambda)} \theta_2(s) e^{-(r-s)H_\circ/\lambda} \theta_1(r) e^{-(t-r)H_\circ/\lambda} d\mu_2(s) d\mu_1(r) \\ + \int_{t/2}^t \int_0^s e^{-r(H_\circ/\lambda)} \theta_1(r) e^{-(s-r)H_\circ/\lambda} \theta_2(s) e^{-(t-s)H_\circ/\lambda} d\mu_1(r) d\mu_2(s).$$

Also,

$$K_\lambda^{0,t/2}(F_{11}) K_\lambda^{t/2,t}(F_{22}) \\ = \int_0^{t/2} \int_{t/2}^t e^{-r(H_\circ/\lambda)} \theta_1(r) e^{-(s-r)H_\circ/\lambda} \theta_2(s) e^{-(t-s)H_\circ/\lambda} d\mu_2(s) d\mu_1(r).$$

where $0 < r < t/2 < s < t$. Similarly,

$$\begin{aligned} & K_\lambda^{0,t/2}(F_{21})K_\lambda^{t/2,t}(F_{12}) \\ &= \int_0^{t/2} \int_{t/2}^t e^{-s(H_\circ/\lambda)} \theta_2(s) e^{-(r-s)H_\circ/\lambda} \theta_1(r) e^{-(t-r)H_\circ/\lambda} d\mu_1(r) d\mu_2(s), \end{aligned}$$

where $0 < s < t/2 < r < t$.

Remark. The initial motivation for the main results considered in this paper, Theorems 2.10 and 3.8, came from a desire to begin to understand in a mathematically rigorous sense some of the recent influential heuristic work using the ‘Feynman integral.’ The hope is that understanding what is going on in related settings such as the present one, where rigorous path integrals are available, will yield some insight into that work. One of the recurring ideas in the heuristic work begins by expressing (at least locally) a manifold of interest in the form $T \times M$ where T is the unit circle or \mathbf{R} or some subinterval of \mathbf{R} . Then, by partitioning T , the manifold $T \times M$ can be cut into arbitrarily small pieces, and the ‘integral’ over ‘paths’ in $T \times M$ can be correspondingly decomposed. An operation of concatenation on the ‘paths’ and functions of the ‘paths’ plays a crucial role. This concatenation is akin to the operations $*$ and $\dot{+}$ which have been used throughout this paper. For us, the role of T is played by the interval $[a, t]$, and M becomes \mathbf{R}^d . The path integrals for us are Wiener integrals for $\lambda > 0$ and Feynman integrals for λ purely imaginary. We should mention that the ‘paths’ in the heuristic work are not truly paths. They can be, for example, equivalence classes of gauge fields on $T \times M$. A brief exposition of the recent heuristic work and its relationship to the Feynman integral and Feynman’s operational calculus for noncommuting operators along with many references can be found in Chapter 20 of the book [7]. Edward Witten has played a prominent role in the heuristic developments; the papers [11], [12] are among the many relevant references to his work.

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