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THE EQUIVARIANT CATEGORY OF PROPER G-SPACES

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Introduction. If G is a Lie group, then by a G-space we mean a completely regular space X together with a fixed action of G on X. We follow the standard notation of the theory of transformation groups used in [**5**] and [**9**]. In this paper we deal with "minimax invariants" of a G-space. If one restricts consideration to compact Lie groups, then a substantial general theory of G-genus, G-index, as well as G-category in the sense of Lusternik-Schnirelmann has already been developed. See [**1**], [**11**] and [**17**]. In contrast, if G is not compact, results on such invariants become scarce in the literature. The aim of this paper is to give a general overview of the invariants of type Lusternik-Schnirelmann for an interesting class of G-spaces without the assumption of compactness for the group G; that is, G-spaces with proper actions. The crucial result that allows such a generalization is due to Palais [**19**]. Namely, Palais shows that "slices" still exist for proper G-spaces. This fact leads us to extend a remarkable amount of the theory of G-spaces with G compact to G-spaces with proper actions.

With this paper we intend to point out that there is no real difficulty in extending to proper actions the notion of equivariant Lusternik-Schnirelmann category defined for compact transformation groups in [**11**] and [**17**]. In fact, we show that the basic properties of equivariant "minimax" invariants still hold for proper G-spaces and more generally for Cartan G-spaces.[1] See Section 1 for definitions. Moreover, in many cases the computation of the equivariant category of a proper G-space can be reduced to the compact case, see (2.6) below. The main result of this paper states a Kranosielski type theorem for proper actions of discrete groups on acyclic manifolds (Theorem 4.2). Incidentally, we give in addition an alternative proof of a Kranosielski type theorem for (co)homology spheres due to Marzantowicz (Theorem 4.4). Finally we

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give a Lusternik-Schnirelmann theorem for critical points of G-invariant real functions on proper G-manifolds (Theorem 5.3). Most of the proofs of the basic results are formally similar to the corresponding ones for G compact already in the literature. See [**7**], [**8**], [**11**] and [**17**]. However, we include here several proofs to emphasize the crucial role played by Palais's results as well as to avoid an excessively dull exposition.

1. Basic results on proper actions. In this section we collect the basic notations and results on proper actions contained in [**3**], [**9**], [**16**] and [**19**]. Let G be a Lie group (with trivial element $e \in G$) not necessarily compact. If X is a space, an *action* of G on X is a homomorphism T of G into the group of homeomorphisms of X such that the map $G \times X \to X$ given by $(g, x) \mapsto T(g)x$ is continuous. We will write gx for $T(g)x$ if the action T is clear from the context. A *G-space* is a completely regular space together with a fixed action of G on X. A map between two G-spaces $f : X \to Y$ is said to be *G-invariant* (or a *G-map*) if $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. A real function $h : X \to \mathbf{R}$ is called *invariant* if $h(gx) = hx$. A *G-homotopy* $H : X \times I \to Y$ is a homotopy such that $H(gx, t) = gH(x, t)$.

The *isotropy group* of $x \in X$ is the subgroup $G_x \subseteq G$, $G_x = \{g \in G; gx = x\}$. The action is said to be *free* if $G_x = \{e\}$ for all $x \in X$. The *set of fixed points* is the set $X^G = \{x \in X; G_x = G\}$. The *orbit* of $x \in X$ is the subset $Gx = \{gx; g \in G\}$. Moreover the *saturation* of the subset $S \subseteq X$ is the subset $GS = \{gs; s \in S\}$. The subset $S \subseteq X$ is said to be *G-invariant* (or a *G-set*) if $GS = S$. Finally, X/G stands for the set of orbits of X topologized with the usual quotient topology.

Following Bourbaki [**3**, Chapter 3], the action of G on X is said to be *proper* if the map $\psi : G \times X \to X \times X$ defined by $\psi(g, x) = (gx, x)$ is proper. We recall that a continuous map $f : Y \to Z$ is proper if f is closed and $f^{-1}(z)$ is compact for any $z \in Z$. If the action of G on X is proper, then X is called a *proper G-space*. Equivalently, a G-space X is proper if for each pair of points $x, y \in X$ there exist neighborhoods V_x and V_y of x and y respectively, such that the set $\{g \in G; gV_x \cap V_y \neq \varnothing\}$ has compact closure in G. See [**9**] and [**19**] for details. Obviously, any G-space with G compact is a proper G-space. Moreover, if G acts properly on X, then the restriction of the action to any closed subgroup $H \subseteq G$ is also proper. Clearly if X is a proper G-space, the isotropy group G_x is compact for all $x \in X$, and hence $X^G = \varnothing$ if G is not compact. Moreover, each orbit Gx is closed in X and the natural map $G/G_x \to Gx$ defined by $[g] \mapsto gx$ is a homeomorphism; see [**9**].

Remark 1.1. Palais [**19**] considers a more restrictive definition of a proper G-space X by requiring that for each $x \in X$ there exists a neighborhood U_x such that for all $y \in X$ there exists a neighborhood U_y such that the set $\{g \in G; gU_x \cap U_y \neq \varnothing\}$ has compact closure in G. Palais also introduces the weaker notion of *Cartan G-space* by requiring that for each $x \in X$ there exists a neighborhood U_x such that the set $\{g \in G; gU_x \cap U_x \neq \varnothing\}$ has compact closure in G. Clearly one has *Palais-proper* $\Rightarrow$ *Bourbaki-proper* $\Rightarrow$ *Cartan G-space*. In general, the converses are not true. See [**13**] and [**16**], respectively. However, the three notions coincide for locally compact spaces and more generally if X/G is regular. See [**19**, p. 303].

Examples 1.2. Examples of proper actions are the following:

(1) If G is a Lie group, any closed subgroup $H \subseteq G$ acts properly on G by left translations. Moreover, if $K \subseteq G$ is a compact subgroup, then the left translation action of H on G/K is proper. See [**9**, Example 24].

(2) It follows from the previous observations that every closed (e.g., discrete) subgroup $H \subseteq G$ of a connected Lie group acts properly on some Euclidean space. The proof of this fact follows from a well-known result which states that G contains a maximal compact subgroup $K \subseteq G$ (unique up to conjugation) such that G/K is diffeomorphic to a Euclidean space. See [**14**].

(3) A G-space M is called a *G-manifold* if M is a manifold and the map $x \mapsto gx$ is differentiable for each $g \in G$. Note that G acts on the tangent space TM via the differential; that is, $gv = (dg)_x v$ if $v \in T_x M$. In addition, M is called a *Reimannian G-manifold* if M is endowed of a Riemannian metric R for which the maps $x \mapsto gx$ are isometries. That is, R is a section of the dual bundle $(TM \oplus TM)^*$ such that for each $x \in M$, R_x is an inner product and for each $g \in G$, $R_{gx}((dg)_x v, (dg)_x w) = R_x(v, w)$ for all $v, w \in T_x M$. If M is a Riemannian G-manifold and G is a closed subgroup of the isometry group $\mathrm{Iso}\,(M)$ of M, then the action of G is proper. Moreover, if M is a proper G-manifold, then there exists a G-invariant Riemannian metric on M (i.e., the action of G is an isometric with respect to that

metric). Therefore, a G-manifold is proper if and only if there exists a Riemannian metric on M such that G is a closed subgroup of Iso(M). See [**20**] for details.

(4) The group of deck transformations of a covering $p : \tilde{X} \to X$ acts properly on $\tilde{X}$. More generally, if the completely regular space X is the total space of a principal G-bundle, then X is a Cartan G-space (1.1). In particular, X is a proper G-space if X is locally compact or, more generally, if X/G is regular. Conversely, if X is a proper G-space or, more generally, a Cartan G-space, and G acts freely on X, the orbit map $X \to X/G$ is a locally trivial principal G-bundle. See (1.1) and [**19**, 4.1].

(5) In the language of dynamical systems (i.e., $\mathbf{R}$-spaces) proper flows are called dispersive flows, and they coincide with parallelizable flows for metrizable locally compact spaces. We recall that a flow $\phi : X \times \mathbf{R} \to X$ on the space X is said to be parallelizable if the orbit map $X \to X/\mathbf{R}$ is a trivial bundle. See [**13**].

Important objects in a transformation group theory are tubes and slices. We recall here some basic properties of them which will be used below. A *tube* about x in a G-space X is an open G-invariant neighborhood $U \subseteq X$ of x in X for which there exists a G-invariant map $\pi : U \to Gx$ called a projection of U. A *slice* at x is a subset $S \subseteq X$ which is G_x-invariant and such that the map $G \times_{G_x} S \to X$ taking $[g, s]$ to gs yields a homeomorphism onto a tube U about x. We recall that, for a subgroup $H \subseteq G$ and an H-space X, the *twisted product* $G \times_H X$ is the orbit space of the H-action $h(g, x) = (gh^{-1}, hx)$ on $G \times X$. Tubes and slices come in pairs. Actually, if $\pi : U \to Gx$ is a projection of the tube U about x, then $S = \pi^{-1}(x)$ is a slice at x. Conversely, if S is a slice at x, then GS is a tube about x with projection $\pi(gs) = gx$. Also, if S is a slice at x in the G-space X, then the natural map $S/G_x \to X/G$ is a homeomorphism onto the open subspace GS/G. See [**5**, II.4.7]. The following existence theorem is crucial in dealing with proper G-spaces. See [**5**] for compact groups and [**19**] for the general case.

Theorem 1.3 (Palais). *Let X be a proper G-space. Then there exists a slice at each point $x \in X$. More precisely, Cartan G-spaces are exactly the G-spaces for which there is a slice at each point.*

Theorem 1.3 allows one to generalize many results on compact transformation groups to proper actions. We next state some of them which will be used below. The first one is the *path lifting theorem* for proper G-spaces. We give a proof to illustrate the way of generalizing results already known for G-spaces with G compact to proper G-spaces.

Theorem 1.4. *Let X be a proper G-space and let $\alpha : I \to X/G$ be any path. Then there exists a lifting $\alpha' : I \to X$ of α with respect to the orbit map $p : X \to X/G$.*

Proof. If S_x is a slice at $x \in X$, the family $\{GS_x/G\}$ is an open cover of the orbit space X/G. Then by Lebesgue's lemma one finds a positive integer n such that for each $0 \leq i \leq n$, the restriction $\alpha_i = \alpha|[i/n, (i+1)/n]$ is contained in some $GS_{x_i}/G \cong S_{x_i}/G_{x_i}$ for some $x_i \in X$. We now apply the path lifting theorem for compact groups [**5**, II.6.2] to the G_{x_i}-space S_{x_i}, and we get a lifting of α_i, $\tilde{\alpha}_i : [i/n, (i+1)/n] \to S_{x_i} \subseteq GS_{x_i}$ for all $0 \leq i \leq n$. Moreover, $p(\tilde{\alpha}_i((i+1)/n)) = p(\tilde{\alpha}_{i+1}((i+1)/n))$ for $0 \leq i \leq n-1$. Hence there exists $g_{i+1} \in G$ with $\tilde{\alpha}_i((i+1)/n) = g_{i+1}\tilde{\alpha}_{i+1}((i+1)/n)$, $0 \leq i \leq n-1$. Then $\tilde{\alpha}(t) = g_1 \ldots g_i \tilde{\alpha}_i(t)$ if $i/n \leq t \leq (i+1)/n$ is a lifting of α. $\square$

As it was pointed out by Palais [**19**, 4.5], the *covering homotopy theorem* [**5**, II.7.3] can also be extended to proper G-spaces. Namely, the following theorem holds

Theorem 1.5. *Let X and Y be separable metrizable proper G-spaces, and let $\tilde{f}_0 : X \to Y$ be a G-map. If $F : X/G \times I \to Y/G$ is any homotopy with F_0 the map induced by $\tilde{f}_0$ and $F(p(X_{(H)}) \times I) \subseteq p(Y_{(H)}))$ for all orbit type (H), then there exists a G-homotopy $\tilde{F} : X \times I \to Y$ with $\tilde{F}_0 = \tilde{f}_0$ which covers F.*

We recall that for a subgroup $H \subseteq G$, the *orbit type* of $x \in X$ is said to be (H) if $G_x = gHg^{-1}$ for some $g \in G$. The subset $X_{(H)} \subseteq X$ is the union of all orbits in X of type (H). A proof of Theorem 1.5 can be given along the lines of the proof in [**5**] for compact groups. Moreover, since for a proper G-space the set of fixed points X^G is empty, one actually skips Part D in that proof. In addition, that proof also shows that the previous theorem also holds for hereditarily paracompact spaces X and Y.

2. The equivariant category of proper G-spaces. We now proceed to extend the definition and basic properties of the equivariant category as introduced in [**11**] and [**17**] for compact Lie groups to proper actions of arbitrary Lie groups. This extension is completely natural, and it will be carried out more or less by way of showing what can be done.

Definition 2.1. Let X be a proper G-space or, more generally, a Cartan G-space in the sense of Remark 1.1. A G-set $A \subseteq X$ is said to be of *G-category* 1 if there exists an orbit G_x such that the inclusion $i_A : A \subseteq X$ is G-homotopic in X to a G-map $A \to Gx \subseteq X$.

Given a G-set $A \subseteq X$, the *G-category* (or *equivariant category*), of A in X, $\mathrm{cat}_G(A; X)$, is the smallest number k such that there exists a cover $A \subseteq \cup_{j=1}^{k} A_j$ consisting of closed G-sets in X of G-category 1. If no such cover exists for all k, then $\mathrm{cat}_G(A, X) = \infty$. As usual, $\mathrm{cat}_G(X)$ will replace $\mathrm{cat}_G(X, X)$. The classical Lusternik-Schnirelmann category, $\mathrm{cat}\,(-)$, defined with closed covers [**15**] is obtained as $\mathrm{cat}\,(A, X) = \mathrm{cat}_G(A, X)$ for the trivial group $G = \{e\}$.

Remark 2.2. As usual, the invariant $\mathrm{cat}_G(-)$ can be extended to G-maps $f : X \to Y$ in such a way that $\mathrm{cat}_G(\mathrm{id}_X) = \mathrm{cat}_G(X)$. Namely, $\mathrm{cat}_G(f)$ is the smallest k such that there exists a cover $\{X_1, \dots, X_k\}$ of X consisting of closed G-sets such that for each i one has an orbit $Gy_i \subseteq Y$ such that the restriction $f|X_i$ is G-homotopic in Y to a G-map $X_i \to Gy_i \subseteq Y$.

At this point we should mention the connection of the above definition with the general definition of equivariant category introduced by Clapp and Puppe in [**7**]. That definition starts by fixing a class $\mathcal{A}$ of G-spaces for some compact Lie group G. Then they define $\mathcal{A} - \mathrm{cat}\,(f) \leq k$ if there exist spaces $A_i \in \mathcal{A}$ such that the restriction $f|X_i$ is G-homotopic to a composition of G-maps $X_i \to A_i \to Y$. If G is now an arbitrary Lie group, we can consider the corresponding definition by choosing a family $\mathcal{A}_{\text{proper}}$ of proper G-spaces. We recover the definition of $\mathrm{cat}_G(-)$ above if $\mathcal{A}_{\text{proper}} = \{G/H; H \subseteq G \text{ is a compact subgroup}\}$.

Notions of relative equivariant category as well as *G-genus* and *$\mathcal{A}$-genus* can also be studied for proper G-spaces. See [**21**], [**8**], [**1**] and [**7**].

As one expects, the fundamental properties of "minimax" invariants extend to proper G-spaces. Namely,

Proposition 2.3. *Let A, B be closed G-sets of the proper G-space X. Then*

(1) (*Subadditivity*). $\mathrm{cat}_G(A \cup B, X) \leq \mathrm{cat}_G(A, X) + \mathrm{cat}_G(B, X)$.

(2) (*Deformation monotonicity*). *Assume that there exists a G-homotopy $H : A \times I \to X$ such that $H_0 = \mathrm{id}_A$ and $H_1(A) \subseteq B$. Then* $\mathrm{cat}_G(A, X) \leq \mathit{cat}_G(B, X)$. *In particular* $\mathrm{cat}_G(X)$ *is a G-homotopy invariant of X.*

(3) (*Continuity*). *If X is a G-*ANR*, then there exists a G-neighborhood U of A in X with* $\mathrm{cat}_G(A, X) = \mathrm{cat}_G(U, X)$.

Properties (1) and (2) are easily checked following the classical arguments, and they are left to the reader. Compare [**15**] and [**7**]. Similarly for property (3), but some further remarks are needed in this case since G-ANR spaces are usually studied in the literature only for compact groups (see [**18**]). In Section 3 we include the results on proper G-ANR needed for the proof of (3). We now continue with some results concerning proper free actions.

Proposition 2.4. *If X is a proper G-space and A is a closed G-set, then* $\mathrm{cat}\,(A/G, X/G) \leq \mathrm{cat}_G(A, X)$. *Moreover, if X is separable metrizable and G acts with only one orbit type, in particular freely, then* $\mathrm{cat}_G(A, X) = \mathrm{cat}\,(A/G, X/G)$.

Proof. Let $p : X \to X/G$ be the orbit map. The first part is obvious since given a cover $A \subseteq \cup_{i=1}^{m} A_i$ of closed G-sets with $\mathrm{cat}_G(A_i, X) = 1$ one has that $\{p(A_i)\}$ is a closed categorical cover of X/G and hence the first inequality is proved. Assume now the extra hypothesis, then we will show $\mathrm{cat}_G(A, X) \leq \mathrm{cat}\,(A/G, X/G)$. Let $\{U_j\}_{1 \leq j \leq n}$ be a closed cover of X/G with $\mathrm{cat}\,(U_j, X/G) = 1$. Let $H^j : U_j \times I \to X/G$ be a deformation of the inclusion $U_j \subseteq X/G$

to a point $p(x_j) \in X/G$. Clearly, the inclusion $k_j : p^{-1}(U_j) \subseteq X$ is an equivariant lifting of the inclusion $H_0^j : U_j \subseteq S/G$. Since G acts with only one orbit type, we can apply the covering homotopy theorem (1.5), and one finds an equivariant homotopy $\tilde{H}^j$ with covers H^j and with $\tilde{H}_0^j = k_j$. In particular, $\tilde{H}_1^j(x) \in Gx_j$ for all $x \in p^{-1}(U_j)$. Hence $\mathrm{cat}_G(p^{-1}(U_j), X) = 1$, and the proof is finished. $\square$

We now consider the relationship between cat_G and cat_H when $H \subseteq G$ is a closed subgroup. The following general result is available.

Proposition 2.5. *Let X be a proper G-space and $H \subseteq G$ be a closed subgroup. Then* $\mathrm{cat}_H(X) \leq \mathrm{cat}_G(X) \max\{\mathrm{cat}_H(Gx) : x \in X\}$.

The proof is similar to that of [**1**, 2.14]. In particular, if G acts freely and X is separable metrizable, then (2.4) yields $\mathrm{cat}_H(X) \leq \mathrm{cat}_G(X)\mathrm{cat}(G/H)$. Moreover, if H is a maximal compact subgroup of the connected Lie group G, then $G/H \cong \mathbf{R}^m$ for some m (see (1.2)) and one gets $\mathrm{cat}_H(X) \leq \mathrm{cat}_G(X)$. The next result shows that the latter inequality actually holds for any G-space, and furthermore it turns out to be an equality in many cases.

Proposition 2.6. *Let G be a connected Lie group, and let X be a proper G-space. If $H \subseteq G$ is a maximal compact subgroup in G, then* $\mathrm{cat}_H(X) \leq \mathrm{cat}_G(X)$. *Moreover, if the action is free, the space X is metrizable and separable, and the orbit spaces X/G and X/H have the homotopy type of CW-complexes, then* $\mathrm{cat}_H(X) = \mathrm{cat}_G(X)$.

Proof. Assume that H contains the isotropy group G_x for some $x \in X$. Then the bundle $G/G_x \to G/H$ of fiber H/G_x and group H is trivial since $G/H \cong \mathbf{R}^m$ for some m. Hence, the composition of the trivialization $G/G_x \cong G/H \times H/G_x$ with the second projection $\pi_2 : G/H \times H/G_x \to H/G_x$ is a homotopy H-equivalence $Gx \simeq Hx$.

Now let A be a G-set such that the inclusion $A \subseteq X$ is G-homotopic (and hence H-homotopic) to a G-map $A \to Gz \subseteq X$. As G_z is compact and the maximal compact subgroups are uniquely determined up to conjugation, there exists $g \in G$ such that $gG_zg^{-1} = G_{gz} \subseteq H$. Therefore, Hgz is homotopy H-equivalent to $Gz = Ggz$. Hence,

$\mathrm{cat}_H(A, X) = 1$, and it readily follows that $\mathrm{cat}_H(X) \leq \mathrm{cat}_G(X)$.

Assume now that G acts freely. According to (1.2)(4), $X \to X/G$ and $X \to X/H$ are principal bundles of fibers G and H, respectively. Then the exact sequences of homotopy groups of these bundles as well as the bundle $G \to G/H$ yield isomorphisms $\pi_n(X/H) \cong \pi_n(X/G)$ for all n. Hence X/H and X/G have the same homotopy type by Whitehead's theorem [**24**, V.3.5] and so $\mathrm{cat}_G(X) = \mathrm{cat}(X/G) = \mathrm{cat}(X/H) = \mathrm{cat}_H(X)$. □

Corollary 2.7. *Let G be a contractible Lie group acting properly on a separable metrizable space X. Then* $\mathrm{cat}_G(X) = \mathrm{cat}(X)$.

Proof. Since G is contractible, then G contains no compact subgroup other than the trivial subgroup $\{e\}$. Therefore, any proper action of G is necessarily free and hence the result follows from (2.6). □

Let G now be an arbitrary Lie group acting properly on X. Assume that $H \subseteq G$ is a closed and open subgroup. Then the H-orbit Hx is closed and open in the G-orbit Gx for all $x \in X$. This follows from the fact that the canonical maps $H \to Hx$ and $G \to Gx$ are both closed and open. For this we use the homeomorphisms $Hx \cong H/H_x$ and $Gx \cong G/G_x$ since G acts properly. After these observations, one can prove

Proposition 2.8. *Assume that X is a path connected separable and metrizable G-space. Assume that H is a closed and open subgroup such that the action restricted to H has only one orbit type. Then* $\mathrm{cat}_H(X) \leq \mathrm{cat}_G(X)$.

Proof. Let $\{U_1, \dots, U_k\}$ be a cover of X by closed G-sets such that for all i there is a G-deformation (and so an H-deformation) ϕ_i in X to G-orbits Gx_i. Let $Gx_i = \cup Hy_j^i$ be the decomposition of G_x into H-orbits. According to the previous observations, each Hy_j^i is open and closed in Gx_i, and hence Gx_i/H is a discrete space. Since X/G is path connected, we consider paths $\gamma_j^i : I \to X/H$ running from $[y_j^i]$ to some fixed point $[x_0]$. Since the H-action has only one orbit type, we can apply the covering homotopy theorem (1.5) to lift each path γ_j^i to an H-homotopy $F_j^i : Hy_j^i \times I \to X$ such that $F_j^i(x,1) \in Hx_0$ for all $x \in Hy_j^i$. Hence $F^i = \cup F_j^i : Gx_i \times I \to X$ is an H-deformation of Gx_i to Hx_0. By using F^i and ϕ^i, one can easily construct an H-deformation of U_i to Hx_0 and so $\mathrm{cat}_H(X) \leq \mathrm{cat}_G(X)$. □

Corollary 2.9. *Let H be a subgroup of a discrete group G. If X is a separable metrizable proper G-space and the action of G is free, then* $\mathrm{cat}_H(X) \leq \mathrm{cat}_G(X)$. *In particular, if H is the trivial subgroup, one gets* $\mathrm{cat}(X) \leq \mathrm{cat}_G(X)$.

3. The equivariant category of proper G-ANR spaces. We recall that a separable metrizable G-space Y is called a G-equivariant absolute neighborhood retract (G-ANR *space*) if, given a closed G-subspace A of a metrizable separable G-space X and a G-map $f : A \to Y$, there exists an extension $\tilde{f} : U \to Y$ to some G-neighborhood U of A in X. Usually G-ANR spaces are studied in equivariant topology for compact Lie groups. See [**18**]. We give here some basic properties of proper G-ANR. Here by a *proper G-ANR* we mean a G-ANR for which the action of G is proper. Again, Palais's results in [**19**] allow us to extend the theory of G-ANR spaces with G compact to proper G-ANR spaces in a straightforward way. Moreover, proper G-manifolds remain examples of proper G-ANR spaces.[2] We prove it in (3.3) below. For this we need the following result.

Lemma 3.1. *Let X be a metrizable separable proper G-space. Then*

(a) *For any cover $\mathcal{U} = \{U_\alpha\}$ of X consisting of open G-sets, there exists a partition of unity of invariant maps $\{\tilde{f}_\alpha : X \to I\}$ subordinate to $\mathcal{U}$.*

(b) *If $A_1, A_2 \subseteq X$ are disjoint closed G-sets, there exists an invariant map $f : X \to I$ with $f|A_1 = 0$ and $f|A_2 = 1$.*

(c) *If $A \subseteq X$ is a closed G-set and U is an open G-neighborhood of A, there is a closed G-neighborhood V of A with $A \subseteq V \subseteq U$.*

Proof. (a) In fact the orbit space X/G is metrizable by (4.3.4) in [**19**]. Hence X/G is paracompact and the open cover $\{p(U_\alpha)\}$ admits an ordinary partition of unity $\{f_\alpha\}$. Then we set $\tilde{f}_\alpha = f_\alpha p$.

(b) The open G-sets $U_i = X - A_i$, $i = 1, 2$, cover X and by (a) one finds two G-maps $f_i : X \to I$ with $f_1 + f_2 = 1$ and $\{x; f_i(x) \neq 0\} \subseteq U_i$. Then f_1 is the required map.

(c) Let $f : X \to I$ be the invariant map obtained by applying (b) to $A_1 = A$ and $A_2 = X - U$. Then $V = f^{-1}([0, 1/2])$ satisfies the required properties. □

Lemma 3.2. *Any orbit Gx in a proper G-*ANR *X is a proper G-*ANR*.*

Proof. One easily checks that any open G-set $U \subseteq X$ is a G-ANR. Then one applies that Gx is a G-retract of an open G-neighborhood U of Gx in X. See [**19**, 2.3.3.1]. □

Proposition 3.3. *Any proper G-manifold M is a proper G-*ANR *space.*

Proof. When G is compact, the result is proved in [**18**, 8.8]. Let $x \in M$ and S_x be a slice at x. Then S_x is a G_x-manifold [**20**, 5.1 and 5.2]. Since G_x is compact, we apply the compact case in [**18**, 8.7] to get that S_x is a G_x-ANR space. Now $G \times_{G_x} S_x$ is G-homeomorphic to a tube $U_x \subseteq M$ about x which is G-ANR. The proof of this fact mimics the proof of [**18**, 8.5] by using (3.2). Since M is separable metrizable, only countably many tubes U_n are needed to cover M. Next one checks that the countable union of open G-ANR subspaces $\{U_n\}$ is again a G-ANR. For this, the proof for countable unions of ordinary ANR [**2**, IV.10.2] can be mimicked by taking into account that any separable metrizable proper G-space admits an invariant metric [**19**, 4.3.4]. □

We state below some properties of proper G-ANR spaces in connection with the equivariant category. The next result yields that all proper G-ANR spaces admit open covers consisting of G-sets of equivariant category 1.

Proposition 3.4. *Let X be a proper G-*ANR*. Then each orbit Gx*

has a G-neighborhood W which can be deformed equivariantly to Gx inside X.

Proof. Let $r : V \to Gx$ be the retraction of a G-neighborhood of Gx given by [**19**, 2.3.3.1]. By (3.1)(c) we can assume that V is actually closed. Now the proof fits the usual pattern. One considers the G-map $H : C = V \times \{0,1\} \cup Gx \times I \to X$ given by $H(v,0) = v$, $H(v,1) = r(v)$ and $H(y,t) = y$ for $v \in V$, $y \in Gx$ and $t \in I$. Since X is G-ANR, we find a G-neighborhood Ω of the closed G-set C in $X \times I$ and a G-extension $\tilde{H} : \Omega \to X$ of H. Let U_y be a neighborhood of $y \in Gx$ with $U_y \times I \subseteq \Omega$. Then $W = \cup_{y \in Gx} GU_y \subseteq \Omega$ is a G-neighborhood of Gx with $W \times I \subseteq \Omega$. Hence $\tilde{H}|V \times I$ is a G-deformation of the inclusion $V \subseteq X$ onto Gx in X. □

The equivariant version of Borsuk's theorem also holds for proper G-ANR spaces. Namely, by using (3.1)(c), one can repeat the classical proof [**2**, IV.8.1] to get

Proposition 3.5. *Let Y be a proper G*-ANR. *If A is a closed G-subset of a metrizable separable G-space X, then the pair (X, A) has the G-homotopy extension property with respect to Y.*

At this point we can give a proof of (2.3)(3). Compare [**7**, Appendix B].

Proof of (2.3)(3). We can assume $\mathrm{cat}_G(A, X) = n < \infty$ (otherwise take $U = X$). Let $A \subseteq A_1 \cup \cdots \cup A_n$ with A_i a closed G-set with $\mathrm{cat}_G(A_i, X) = 1$. It will suffice to show that each A_i has a closed G-neighborhood V_i with $\mathrm{cat}_G(V_i, X) = 1$. Let H^i be a G-homotopy with H^i_0 the inclusion $k_i : A_i \subseteq X$ and $H^i_1(A_i) \subseteq Gx_i$ for some $x_i \in X$. By (3.5) we can extend H^i to a G-homotopy $\tilde{H}^i : X \times I \to X$ with $\tilde{H}^i_0 = \mathrm{id}_X$. By (3.4), we choose a G-neighborhood W_i of Gx_i which is deformed equivariantly inside X. Then $\tilde{H}^i_1(A_i) \subseteq W_i$. Now (3.1)(c) gives a closed G-neighborhood V_i of A_i with $\tilde{H}^i_1(V_i) \subseteq W_i$ and so by (2.3)(2) one gets $\mathrm{cat}_G(V_i, X) \le \mathrm{cat}_G(\tilde{H}^i_1(V_i), X) \le \mathrm{cat}_G(W_i, X) = 1$. Hence $\mathrm{cat}_G(V_i, X) = 1$. □

Corollary 3.6. *If X is a G-*ANR *space, one can indistinctly use open or closed covers of invariant sets to define* $\mathrm{cat}_G(X)$.

Proof. Let $X = U_1 \cup \cdots \cup U_k$ be a cover with $\mathrm{cat}_G(U_i, X) = 1$ and each U_i an open G-set. We use (3.1)(a) to get an equivariant partition of unity $\{f_i\}$ subordinated to $\{U_i\}$. Then the family of supports $A_i = \overline{\{x; f_i(x) \neq 0\}}$ is a cover of X by closed G-sets and $\mathrm{cat}_G(A_i, X) = 1$ for all i. The converse follows from (2.3)(3). $\square$

For a proper G-ANR space X, there is also available an upper bound for the G-category in terms of $\dim X/G$. For this we recall that an orbit type (H) of X is said to be *minimal* if there is no $x \in X$ with $H \not\subseteq G_x$. See [**5**, p. 42]. Then one can prove

Proposition 3.7. *If X is a path connected proper G-*ANR *space and $A \subseteq X$ is a closed G-set, then*

$$\mathrm{cat}_G(A, X) \leq (1 + \mathit{dim}\, A/G) \sum c(H).$$

Here $c(H)$ denotes the (cardinal) number of components of the orbit space GX^H/G and (H) ranges over the set of minimal orbit types of X.

Of course, the above formula is only of interest if $\sum c(H) < \infty$. The proof of (3.7) uses the same arguments as the proof of the compact case in [**17**, 1.10]. The crucial point in the proof is the path lifting theorem which still holds for proper actions, see (1.4). We leave the details to the reader. As a corollary, we have

Corollary 3.8. *If X is a path connected proper G-*ANR *space with only one orbit type (H), then* $\mathrm{cat}_G(X) \leq (1 + \dim X/G)c(H)$. *Moreover, if the fixed point set X^H is connected, then* $\mathrm{cat}_G(X) \leq 1 + \dim X/G$.

4. Kranosielski type theorems for proper free actions of discrete groups. The classical Kranosielski theorem states $\mathrm{cat}(S^n/G) = n+1$ for any free action of a finite group G on the n-sphere S^n. Several generlizations can be found in the literature. In particular, Marzantowicz has shown that the same result holds for any (co)homology sphere [**17**, 2.6]. We prove here similar results for proper free actions of infinite discrete groups on acyclic manifolds. Incidentally, we use the same arguments to give an alternative proof of Marzantowicz's result. The crucial point in the results below is the following theorem in [**12**, Proposition 3].

Theorem 4.1 (Eilenberg-Ganea). *If X is a paracompact space and* $\mathrm{cat}\,(X) \leq n$, *there exist an $(n{-}1)$-dimensional polyhedron L and a map $g : X \to L$ which induce an isomorphism $g_* : \pi_1(X) \cong \pi_1(L)$ between fundamental groups.*

We are now ready to show

Theorem 4.2. *Let X be an acyclic n-manifold, $n \geq 3$. Assume that G is a discrete group acting properly and freely on X with X/G compact. Then* $\mathrm{cat}_G(X) = \mathrm{cat}\,(X/G) = n + 1$.

Proof. For the Eilenberg-Maclane space $K(G, 1)$ we consider the associated universal G-bundle $q_G : E(G, 1) \to K(G, 1)$. Moreover, we have the diagram

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & E(G,1) \arrow[d, "q_G"] \\ X/G \arrow[r, "f"'] & K(G,1) \end{tikzcd}$$

where f is the classifying map of the G-bundle $\pi : X \to X/G$. See (1.2)(4).

Assume now that $\mathrm{cat}_G(X) = \mathrm{cat}(X/G) \leq n$. Then by (4.1) there exists a map $g : X/G \to L$ where L is an $(n-1)$-polyhedron and $g_* : \pi_1(X/G) \to \pi_1(L)$ is an isomorphism. It is easily checked that the inverse g_*^{-1} can be realized by a map $h : L^2 \to X/G$ from the 2-skeleton

$L^2 \subseteq L$. That is, the diagram

$$\begin{tikzcd} & \pi_1(L) \\ \pi_1(L^2) \arrow[ur, "i_*"] \arrow[r, "h_*"] & \pi_1(X/G) \arrow[u, "g_*"'] \end{tikzcd}$$

is commutative. As $\pi_n(K(G,1))$ is trivial for all $n \geq 2$, one readily extends the map $fh : L^2 \to K(G,1)$ to a map $f' : L \to K(G,1)$ for which the following diagram commutes up to homotopy

$$\begin{tikzcd} X/G \arrow[r, "f"] \arrow[d, "g"] & K(G,1) \\ L \arrow[ur, "f'"'] & \end{tikzcd}$$

Since X/G is compact, we can assume without loss of generality that in the above diagram L is actually a finite complex. Moreover, as f is the classifying map of $\pi : X \to X/G$, we have a commutative diagram of total spaces of the corresponding pullback constructions

$$\begin{tikzcd} g^* f'^* E \cong X \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"] & E = E(G,1) \\ f'^* E \arrow[ur, "\tilde{f}'"'] & \end{tikzcd}$$

We next observe that both the augmented cellular chain complex $\mathcal{C}(E(G,1))$ and the augmented cellular chain complex $\mathcal{C}(X)$ are free $\mathbf{Z}G$-resolutions of $\mathbf{Z}$, and by [**6**, I.7.5] it follows that the induced homomorphism $\tilde{f}_* : \mathcal{C}(X) \to \mathcal{C}(E(G,1))$ is a homotopy equivalence of chain complexes, and hence $\tilde{f}^* : H^*(G, \mathbf{Z}G) \cong H^*(G, \mathbf{Z}G)$ is an isomorphism. Therefore, the above diagram yields the following commutative diagram of augmented cellular chain complexes

$$\begin{tikzcd} \mathcal{C}(X) \arrow[r, "\tilde{f}_*", "\simeq"'] \arrow[d, "\tilde{g}_*"] & \mathcal{C}(E) \\ \mathcal{C}(f'^* E) \arrow[ur, "\tilde{f}'_*"'] & \end{tikzcd}$$

From this we get a commutative diagram

$$\begin{tikzcd} H^n(G;\mathbf{Z}G) \arrow[r, "\tilde{f}^*", "\cong"'] \arrow[d, "\tilde{f}'^*"] & H^n(G;\mathbf{Z}G) \cong H^n_c(X;\mathbf{Z}) \cong \mathbf{Z} \\ H^n(\mathcal{C}(f'^*E);\mathbf{Z}G) = 0 \arrow[ur, "\tilde{g}^*"'] & \end{tikzcd}$$

where the isomorphism $H^n_c(X,\mathbf{Z}) \cong H^n(G;\mathbf{Z}G)$ is given by the proof of [**6**, VIII.7.5] since X/G is compact and $\mathcal{C}_m(X) = 0$ for $m \geq n+1$. Finally $H^n(\mathcal{C}(f'^*E);\mathbf{Z}G) = 0$ since $\dim f'^*E = \dim L \leq n-1$. This finishes the proof. $\square$

Remark 4.3. a) In case X is contractible, the orbit space X/G is an Eilenberg-MacLane space $K(G,1)$ (i.e., X/G is an aspherical n-manifold with $\pi_1(X/G) = G$) and Theorem 4.2 follows, in fact, from a well-known theorem by Eilenberg and Ganea [**12**] which states $\mathrm{cat}\,(K(G,1)) = \mathrm{cd}\,(G)+1$ except possibly for the case $\mathrm{cat}\,(K(G,1)) = 3$ and $\mathrm{cd}\,(G) = 2$. Here "cd" stands for cohomological dimension; that is, $\mathrm{cd}\,(G)$ is the smallest n such that $H^q(G;M) = 0$ for all $q > n$ and all $\mathbf{Z}G$-modules M. See [**6**]. Moreover, for G as above, one gets $\mathrm{cd}\,(G) = n$ by [**6**, VIII.8.1], and so $\mathrm{cat}_G(X) = n+1$ when $n \neq 2$. For $n = 2$, X/G is an orientable surface ($\neq S^2$) and it is well known that $\mathrm{cat}\,(X/G) = 3$.

b) The group G in Theorem 4.2 is necessarily torsion free. More explicitly, let G be a discrete group acting properly on an acyclic n-manifold X with X/G compact. Then the action is free if and only if G is torsion free. Indeed, if the action is free, then we have $\mathrm{cd}\,(G) = n$ as in a) above, and so G is torsion free [**6**, VIII.2.5]. Conversely, since the action is proper all isotropy groups G_x are finite and so trivial since G is torsion free.

Theorem 4.2 above can be regarded as the noncompact counterpart of the result by Marzantowicz [**17**, 2.6] on free actions of finite groups on (co)homology spheres. Actually, the arguments used in the proof of (4.2) allow us to give an alternative proof of Marzantowicz's result. Namely, we now prove

Theorem 4.4. *Let X be a (co)homology n-sphere, and let G be a finite group acting freely on X. Then* $\mathrm{cat}_G(X) = \mathrm{cat}\,(X/G) = n+1$.

Here a *(co)homology n-sphere* is an n-manifold whose integral (co)-homology coincides with the (co)homology of the n-sphere S^n. The universal coefficient theorem [**23**, 5.5.3] shows that homology and cohomology spheres coincide. As in the original, Marzantowicz's proof of (4.4), we shall use the following crucial result on degrees of G-maps between (co)homology spheres also due to Marzantowicz [**17**, 2.9].

Theorem 4.5 (Marzantowicz). *Let $f : X \to Y$ be a $\mathbf{Z}_p$-map between (co)homology spheres. If the action of $\mathbf{Z}_p$ is free, then deg$(f) \neq 0$ (mod p).*

In the proof of (4.4) we shall also use the following elementary fact for (co)homology spheres of even dimension.

Lemma 4.6. *If $\dim X = n$ is even, then $\mathbf{Z}_2$ is the only finite group acting freely on X.*

This result is an immediate consequence of the formula $\mathcal{L}(f) = 1 + (-1)^n \mathrm{deg}\,(f)$ for any map $f : S^n \to S^n$ where $\mathcal{L}(f)$ stands for the Lefschetz number of f. See [**23**, 4.7.9].

Proof of 4.4. Let p be a prime divisor of the order of G. Then $\mathbf{Z}_p$ is a subgroup of G and according to (2.9) and (3.8) we have $\mathrm{cat}_{\mathbf{Z}_p}(X) \leq \mathrm{cat}_G(X) \leq n+1$. Therefore, it will suffice to show $\mathrm{cat}_{\mathbf{Z}_p}(X) = n+1$. Assume on the contrary that $\mathrm{cat}_{\mathbf{Z}_p}(X) = \mathrm{cat}(X/\mathbf{Z}_p) \leq n$. Then by (4.1) there exist an $(n-1)$-dimensional polyhedron L and a map $g : X/G \to L$ which induce an isomorphism $g_* : \pi_1(X/G) \to \pi_1(L)$. Next we consider the diagram

$$\begin{tikzcd} X \arrow[r, "\tilde{f}"] \arrow[d, "\pi"'] & E(\mathbf{Z}_p, 1) = E_p \arrow[d, "q_p"] \\ X/\mathbf{Z}_p \arrow[r, "f"'] & K(\mathbf{Z}_p, 1) = K_p \end{tikzcd}$$

defined by the classifying map f of the $\mathbf{Z}_p$-bundle $\pi : X \to X/\mathbf{Z}_p$. Now we construct a map $h : L^2 \to X/\mathbf{Z}_p$ realizing the inverse g_*^{-1} as in the proof of (4.2). Moreover, by using the asphericity of K_p, we can extend fh to a map $f' : L \to K_p$ such that $f'g \simeq f$.

At this point we consider two cases:

a) $p = 2$. Then $E_2 = S^\infty$ and $K_2 = \mathbf{R}P^\infty$. Moreover, by the cellular approximation theorem [**23**, 7.6.18], we can assume that $f(X/\mathbf{Z}_2) \subset \mathbf{R}P^n$ and $\tilde{f}(X) \subseteq S^n$. Now we apply (4.5) to the $\mathbf{Z}_2$-equivariant map $\tilde{f} : X \to S^n$ and we get $\deg(\tilde{f}) = 1 \pmod 2$. Then the differentiable (or simplicial) definition of degree yields that $\deg(f) = 1 \pmod 2$, and so

$$f_* : H_n(X/\mathbf{Z}_2; \mathbf{Z}_2) \cong \mathbf{Z}_2 \longrightarrow H_n(\mathbf{R}P^n; \mathbf{Z}_2) \cong H_n(\mathbf{R}P^\infty; \mathbf{Z}_2) \cong \mathbf{Z}_2$$

is an isomorphism. This leads to a contradiction since f_* factorizes through $H_n(L; \mathbf{Z}_2) = 0$.

b) $p \neq 2$. Then by (4.6) $\dim X = n = 2k+1$ is odd. Moreover, $E_p = S^\infty$ and $K_p = L^\infty(p)$ is the infinite lens space. It is known that $L^\infty(p)$ is a CW-complex such that the lens space $L^{2m+1}(p)$ is its $(2m+1)$-skeleton for each m. See [**24**, p. 91]. As in case a) we can assume $f(X/\mathbf{Z}_p) \subseteq L^{2k+1}(p)$ and $\tilde{f}(X) \subseteq S^{2k+1}$, and by (4.5), $\deg(\tilde{f}) \neq 0 \pmod p$. As $H_{2k+1}(X/\mathbf{Z}_p; \mathbf{Z}_p) \cong \mathbf{Z}_p$ [**5**, Example III.3], we argue as above to get that

$$\begin{aligned} f_* : H_{2k+1}(X/\mathbf{Z}_p; \mathbf{Z}_p) \cong \mathbf{Z}_p &\longrightarrow H_{2k+1}(L^{2k+1}(p); \mathbf{Z}_p) \\ &\cong H_{2k+1}(L^\infty(p); \mathbf{Z}_p) \cong \mathbf{Z}_p \end{aligned}$$

is an isomorphism, and we conclude as in case a). □

5. Critical points on proper G-manifolds. In this final section we illustrate how the equivariant category of proper G-manifolds provides an upper bound for the number of critical orbits of invariant smooth functions. In the action-free case, one requires the Palais-Smale condition (PS). Namely, if M is a complete Riemannian manifold, the smooth function $f : M \to \mathbf{R}$ satisfies condition (PS) if any sequence $\{x_n\} \subseteq M$ for which $|f(x_n)|$ is bounded and $\|\nabla f_{x_n}\|$ converges to zero admits a convergent subsequence. Then the Lusternik-Schnirelmann theorem states that a smooth function bounded below satisfying condition (PS) has at least $\operatorname{cat}(M)$ *critical points*. Recall that $\nabla f : M \to TM$ is the *gradient vector field* defined as the dual of the differential df. Explicitly, for any vector field Y and any $x \in M$, $df_x(Y_x) = R_x(Y_x, \nabla f_x)$. Here R is the Riemannian structure of M.

In the equivariant setting any invariant function $f : M \to \mathbf{R}$ gives rise to an equivariant gradient $\nabla f : M \to TM$; that is, $\nabla f_{gx} = dg_x \nabla f_x$. Moreover, if the Riemannian metric on M is G-invariant, then $\|\nabla f\| : M \to \mathbf{R}$ is an invariant function and the flow $\{\phi_t\}$ generated by ∇f commutes with the action (i.e., $\phi_t(gx) = g\phi_t(x)$). In addition, if x is a critical point of f, then the whole orbit Gx consists of critical points, and so if the group G is not compact, then the usual condition (PS) is not satisfied. In such a case we replace condition (PS) by the following *orbitwise Palais-Smale condition* (OPS).

Definition 5.1. Let M be a complete Riemannian proper G-manifold and $f : M \to \mathbf{R}$ a smooth invariant function. The function f satisfies condition (OPS) if, given a sequence $\{x_n\} \subseteq M$ such that $|f(x_n)|$ is bounded and $\|\nabla f_{x_n}\|$ converges to zero, then the sequence of orbits $\{Gx_n\}$ contains a convergent subsequence in the orbit space M/G.

Remark 5.2. (1) Let $S_f \subseteq M$ denote the set of critical orbits of f. Since S_f is a G-set, the restriction $f|S_f$ yields a continuous map $\tilde{f} : S_f/G \to \mathbf{R}$. Then one readily checks that condition (OPS) implies that $\tilde{f}$ is proper. In particular, $\tilde{f}(S_f/G) = f(S_f) \subseteq \mathbf{R}$ is a closed set.

(2) Since the orbit space is not in general a smooth manifold, in order to avoid the possibly singular space M/G, condition (OPS) is often replaced by the *Palais-Smale condition modulo G* (G-PS): Given the sequence $\{x_n\} \subseteq M$ for which $|f(x_n)|$ is bounded and $\|\nabla f_{x_n}\| \to 0$, there exists a sequence $\{g_n\} \subseteq G$ such that the sequence $\{g_n x_n\}$ has a convergent subsequence in M. Clearly (G-PS) implies (OPS) and in case G is compact, (PS) implies (G-PS).

Theorem 5.3. *Let M be a complete Riemannian G-manifold and $f : M \to \mathbf{R}$ a smooth invariant function satisfying condition* (OPS). *Then f has at least* $\operatorname{cat}_G(M)$ *critical orbits.*

The proof of (5.3) just mimics the proof of the classical Lusternik-Schnirelmann theorem in [**20**]. Compare also with [**11**] for G compact. More precisely, one considers the (equivariant) flow $\{\phi_t\}$ generated by $-\nabla f$. Then one shows that for all $t > 0$, ϕ_t is defined on all of M, and for any $x \in M$, $\lim_{t\to\infty} \phi_t(x)$ is a critical point of M. The proof is the same as [**20**, 9.1.6]. Next one proceeds to use the flow $\{\phi_t\}$ to prove the following equivariant deformation theorem. For this, let S_c denote the union of all critical orbits in $f^{-1}(c)$ and $M_c = f^{-1}((-\infty, c])$, $c \in \mathbf{R}$.

Lemma 5.4 (Deformation theorem). *For any G-neighborhood U of S_c in M, there exists $\varepsilon > 0$ such that $\phi_1(M_{c+\varepsilon} - U) \subseteq M_{c-\varepsilon}$. In particular, if c is a regular value, (i.e., $S_c = \varnothing$), then $\phi_1(M_{c+\varepsilon}) \subseteq M_{c-\varepsilon}$.*

Proof. Let d denote the (invariant) distance on M induced by its Riemannian structure. Then the family $\mathcal{N}$ of G-sets $N_\delta(S_c) = \{x \in M; d(x, S_c) < \delta\}$, $\delta > 0$, is a basis of G-neighborhoods of S_c in M. Indeed, by [**19**, 4.3.4], if $p : M \to M/G$ is the orbit map $\tilde{d}(p(x), p(y)) = \inf\{d(z, w); z \in Gx, w \in Gy\}$ is a distance on M/G. Moreover, by (5.2)(1), the set $p(S_c)$ is compact in M/G and so the sets $\tilde{N}_\delta(p(S_c)) = \{\tilde{x} \in M/G; \tilde{d}(\tilde{x}, p(S_c)) < \delta\}$ form a basis of neighborhoods of $p(S_c)$ in M/G. Now the inclusion $p(N_\delta(S_c) \subseteq \tilde{N}_\delta(p(S_c)))$ easily implies that $\mathcal{N}$ is the required basis. At this point the proof is carried out formally in the same way as in [**20**, 9.2.3]. $\square$

Now we have the ingredients for a proof of (5.3) along the lines of the classical proof in [**20**, 9.2.9]. For this, one applies the properties of cat_G in (2.3). As usual, one defines for $m \leq \mathrm{cat}_G(M)$ the *minimax number* $c_m(f)$ of f with respect to the family $\mathcal{F}_m$ of G-sets $F \subseteq M$ with $\mathrm{cat}_G(F, M) \geq m$. Namely,

$$c_m(f) = \inf\{\sup f(F); F \in \mathcal{F}_m\} = \inf\{a \in \mathbf{R}; \mathrm{cat}_G(M_a, M) \geq m\}.$$

Then, clearly $c_1(f) = \inf f(M)$ and $c_m(f) \leq c_{m+1}(f)$. Moreover, the property that $\lim_{t\to\infty} \phi_t(x)$ is always a critical point yields that $c_1(f)$ is actually the minimum of f. Lemma (5.4) is used to check that the rest of minimax numbers $c_m(f)$ are critical values of f. Therefore, Theorem 5.3 will follow if one shows that $c = c_m(f) = c_{m+k}(f)$ implies that S_c contains at least $k+1$ critical orbits. Suppose not. Then S_c is the union $\cup\{Gx_j; 1 \leq j \leq r\}$ of $r \leq k$ critical orbits. Since M is a G-ANR space by (3.3), one finds by (2.3)(3) a G-neighborhood U of S_c with $\mathrm{cat}_G(U, M) = \mathrm{cat}_G(S_c, M) \leq r$.

Moreover, by (5.4) there exists $\varepsilon > 0$ such that $\mathrm{cat}_G(M_{c+\varepsilon} - U, M) \leq \mathrm{cat}_G(M_{c-\varepsilon}, M) \leq m - 1$. For this we use property (2.3)(2) of cat_G and $c = c_m(f)$. Now the equality $c = c_{m+1}(f)$ yields the contradiction $m+k \leq \mathrm{cat}_G(M_{c+\varepsilon}, M) \leq \mathrm{cat}_G(M_{c+\varepsilon}-U, M)+\mathrm{cat}_G(U, M) \leq m+k-1$. For this, we use property (2.3)(1) of cat_G. This finishes the proof of (5.3).

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ENDNOTES

1. After having finished the manuscript we learned about the thesis of E. Shaw [**22**] where the equivariant LS category of proper G-spaces is also considered. In fact, (2.3) and (2.4) also appear in Shaw's thesis.

2. This result also appears in the thesis of E. Elfving [**10**].

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