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A FOURTH ORDER q-DIFFERENCE EQUATION FOR ASSOCIATED DISCRETE q-ORTHOGONAL POLYNOMIALS

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ABSTRACT. In this work we prove that the associated polynomials of general q-orthogonal polynomials satisfy a fourth order q-difference equation. We provide two algorithms for constructing this equation and we identify its solution basis.

1. Introduction. Let $w(x)$ be a positive weight defined on a q-linear lattice $\{aq^n, bq^n : n \in \mathbf{N}_0\}$ with $|q| < 1$. The corresponding discrete q-orthonormal polynomials satisfy the orthogonality relation

$$\int_a^b p_m(x)p_n(x)w(x)d_qx = \delta_{m,n}, \tag{1.1}$$

where the q-integral, see [**6, 8**], is defined by

(1.2)
$$\int_a^b f(x)\,d_qx = \sum_{n=0}^{\infty}(bq^n - bq^{n+1})f(bq^n) - \sum_{n=0}^{\infty}(aq^n - aq^{n+1})f(aq^n),$$

and

(1.3)
$$\int_0^{\infty} f(x)\,d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n).$$

If we normalize the weight so that

$$\int_a^b w(x)\,d_qx = 1$$

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then the polynomial sequence $\{p_n(x)\}$ will satisfy initial conditions of the form

$$p_0(x) = 1, \quad p_1(x) = (x - b_0)/a_1, \tag{1.4}$$

and can be generated by a three-term recurrence relation of the form

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \in \mathbf{N}. \tag{1.5}$$

Note that (1.5) also holds for $n = 0$ if we set $p_{-1}(x) = 0$.

In [**3, 2, 4**] it was proved in the case $q = 1$ that, if $d_q x$ in (1.1) is replaced by dx, then $p_n(x)$ satisfies a linear second order differential equation. This was extended to the discrete q-case in [**7**]. A q-analogue of d/dx is the q-difference operator D_q defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}. \tag{1.6}$$

In [**7**] it was established that q-orthonormal polynomials satisfy a second order linear q-difference equation of the form

$$D_q^2 p_n(x) + R_n(x) D_q p_n(x) + S_n(x) p_n(x) = 0. \tag{1.7}$$

The coefficients $R_n(x)$ and $S_n(x)$ are defined by

(1.8)
$$R_n(x) = B_n(qx) - \frac{D_q A_n(x)}{A_n(x)} + \frac{A_n(qx)}{A_n(x)}\left(B_{n-1}(x) - \frac{(x - b_{n-1})A_{n-1}(x)}{a_{n-1}}\right),$$

(1.9)
$$S_n(x) = \frac{a_n}{a_{n-1}} A_n(qx) A_{n-1}(x) + D_q B_n(x) - \frac{B_n(x)}{A_n(x)} D_q A_n(x) + B_n(x)\frac{A_n(qx)}{A_n(x)}\left(B_{n-1}(x) - \frac{(x - b_{n-1})A_{n-1}(x)}{a_{n-1}}\right),$$

where the functions $A_n(x)$ and $B_n(x)$ are defined by

(1.10)
$$A_n(x) = a_n \frac{w(y/q)p_n(y)p_n(y/q)}{x-y/q}\bigg|_a^b + a_n \int_a^b \frac{u(qx)-u(y)}{qx-y} p_n(y)p_n(y/q)w(y)\, d_q y,$$

(1.11)
$$B_n = a_n \frac{w(y/q)p_n(y)p_{n-1}(y/q)}{x-y/q}\bigg|_a^b + a_n \int_a^b \frac{u(qx)-u(y)}{qx-y} p_n(y)p_{n-1}(y/q)w(y)\, d_q y.$$

The function $u(x)$ is related to the weight through the generalized Pearson equation

$$D_q w(x) = -u(qx)w(qx). \tag{1.12}$$

The second order q-difference equation follows from the lowering operator relationship [**7**]

$$D_q p_n(x) = A_n(x)p_{n-1}(x) - B_n(x)p_n(x), \quad n \in \mathbf{N} \tag{1.13}$$

the three-term recurrence relation (1.5), and the property

$$\begin{aligned} D_q(f(x)g(x)) &= f(x)D_q g(x) + g(qx)D_q f(x) \\ &= f(x)D_q g(x) + g(x)D_q f(x) \\ &\quad + (q-1)xD_q f(x)D_q g(x). \end{aligned} \tag{1.14}$$

When the definition of a_n and b_n in (1.4) and (1.5) can be extended from integers to nonnegative real numbers, as for example when a_n and b_n are rational functions of n or of q^n, then the associated orthogonal polynomials of order c, $\{p_n^{(c)}(x)\}$ are generated by the initial conditions

$$p_0^{(c)}(x) = 1, \quad p_1^{(c)}(x) = (x-b_c)/a_{c+1}, \tag{1.15}$$

and the recurrence relation

(1.16)
$$xp_n^{(c)}(x) = a_{n+c+1}p_{n+1}^{(c)}(x) + b_{n+c}p_n^{(c)}(x) + a_{n+c}p_{n-1}^{(c)}(x), \quad n \in \mathbf{N}.$$

2. The fourth order q-difference equation for the associated polynomials. A basis of solutions for (1.5) is formed by $p_n(x)$ and the function of the second kind $Q_n(x)$ [**7**],

$$Q_n(x) = \frac{1}{w(x)} \int_a^b \frac{p_n(t)}{x-t} w(t)\, d_q t, \tag{2.1}$$

defined for $x \notin \{aq^n, bq^n, n \in \mathbf{N}_0\}$. Moreover, it was shown in [**7**] that $Q_n(x)$ satisfies the q-difference equation (1.7) provided that

$$w(a/q) = w(b/q) = 0. \tag{2.2}$$

From now on we shall assume that the weight w satisfies (2.2).

Let $c \in \mathbf{N}_0$. Since $p_{n+c}(x)$ and $Q_{n+c}(x)$ form a solution basis for (1.16), using (1.5) and the initial conditions (1.15) we obtain

$$p_n^{(c)}(x) = \frac{Q_{c-1}(x)p_{n+c}(x) - p_{c-1}(x)Q_{n+c}(x)}{Q_{c-1}(x)p_c(x) - p_{c-1}(x)Q_c(x)}. \tag{2.3}$$

Let $\Delta_c(x)$ denote the denominator in (2.3). Using (1.5) we get

$$\begin{aligned}\Delta_c(x) &= p_c(x)[(x-b_c)Q_c(x) - a_{c+1}Q_{c+1}(x)]/a_c \\ &\quad - Q_c(x)[(x-b_c)p_c(x) - a_{c+1}p_{c+1}(x)]/a_c \\ &= a_{c+1}\Delta_{c+1}(x)/a_c.\end{aligned}$$

Thus, $a_{c+1}\Delta_{c+1}(x) = a_c\Delta_c(x)$ for every $c \geq 0$. Then

$$\begin{aligned}\Delta_c(x) &= \frac{a_1}{a_c}(Q_0(x)p_1(x) - p_0(x)Q_1(x)) \\ &= \frac{1}{a_c w(x)} \int_a^b \frac{a_1(p_1(x) - p_1(t))}{x-t} w(t)\, d_q t \\ &= \frac{1}{a_c w(x)}.\end{aligned}$$

Hence,

$$p_n^{(c)}(x)/w(x) = a_c[Q_{c-1}(x)p_{n+c}(x) - p_{c-1}(x)Q_{n+c}(x)]. \tag{2.4}$$

Note that $p_{c-1}(x)$ and $Q_{c-1}(x)$ satisfy the q-difference equation

$$D_q^2 y(x) + R_{c-1}(x)D_q y(x) + S_{c-1}(x)y(x) = 0, \tag{2.5}$$

$p_{n+c}(x)$ and $Q_{n+c}(x)$ satisfy the q-difference equation

$$D_q^2 y(x) + R_{n+c}(x) D_q y(x) + S_{n+c}(x) y(x) = 0, \tag{2.6}$$

and $p_n^{(c)}/w$ is a linear combination of $Q_{c-1}p_{n+c}$ and $p_{c-1}Q_{n+c}$.

Lemma 2.1. *Let y_1 and y_2 be solutions of the second order q-difference equations*

$$D_q^2 y(x) = f_j(x) D_q y(x) + g_j(x) y(x), \quad j = 1, 2, \tag{2.7}$$

respectively. Then $y_1 y_2$ satisfies a q-difference equation of order less than five.

Proof. We set $u_1 = y_1 y_2$, $u_2 = y_1 D_q y_2$, $u_3 = y_2 D_q y_1$, $u_4 = D_q y_1 D_q y_2$ and $\tau(x) = (q-1)x$. From (1.14) and (2.7) we obtain

$$\begin{aligned}
D_q u_1 &= u_2 + u_3 + \tau u_4, \\
D_q u_2 &= y_1 D_q^2 y_2 + D_q y_2 D_q y_1 + \tau D_q y_1 D_q^2 y_2 \\
&= u_4 + (y_1 + \tau D_q y_1)(f_2 D_q y_2 + g_2 y_2) \\
&= g_2 u_1 + f_2 u_2 + \tau g_2 u_3 + (1 + \tau f_2) u_4, \\
D_q u_3 &= g_1 u_1 + \tau g_1 u_2 + f_1 u_3 + (1 + \tau f_1) u_4, \\
D_q u_4 &= D_q y_1 D_q^2 y_2 + D_q y_2 D_q^2 y_1 + \tau D_q^2 y_1 D_q^2 y_2 \\
&= f_2 u_4 + g_2 u_3 + f_1 u_4 + g_1 u_2 \\
&\quad + \tau f_1 f_2 u_4 + \tau f_1 g_2 u_3 + \tau f_2 g_1 u_2 + \tau g_1 g_2 u_1.
\end{aligned}$$

These equations can be written in a matrix form. Set $\bar{u} = (u_1, u_2, u_3, u_r)^t$ and define

$$A = \begin{bmatrix} 0 & 1 & 1 & \tau \\ g_2 & f_2 & \tau g_2 & 1 + \tau f_2 \\ g_1 & \tau g_1 & f_1 & 1 + \tau f_1 \\ \tau g_1 g_2 & g_1 + \tau f_2 g_1 & g_2 + \tau f_1 g_2 & f_1 + f_2 + \tau f_1 f_2 \end{bmatrix}.$$

Then we have

$$D_q \bar{u} = A \bar{u}. \tag{2.8}$$

Let M be a 4×4 function matrix. From the matrix version of (1.14) and (2.8) we get

$$\begin{aligned} D_q(M\bar{u}) &= MD_q\bar{u} + (D_qM)\bar{u} + \tau(D_qM)D_q\bar{u} \\ &= (MA + D_qM + \tau(D_qM)A)\bar{u} \\ &=: (L_AM)\bar{u}. \end{aligned} \tag{2.9}$$

From (2.8)–(2.9), it follows that

$$D_q^n\bar{u} = (L_A^{n-1}A)\bar{u}, \quad n \in \mathbf{N}, \tag{2.10}$$

where L_A^0 is the identity operator, and the operator $L_A^{n+1} = L_A \circ L_A^n$ is defined inductively by composition. Let M_1, M_2, M_3, M_4 be the 5×4 matrices formed in the following way: the first, second, third, fourth and fifth rows of M_j are the jth rows of the matrices E (the 4×4 identity matrix), A, L_AA, L_A^2A and L_A^3A, respectively. Then

$$(u_j, D_qu_j, D_q^2u_j, D_q^3u_j, D_q^4u_j)^t = M_j\bar{u}, \quad j = 1, \dots, 4. \tag{2.11}$$

Since $\operatorname{rank}(M_j) \leq 4$, a nonzero vector $\bar{\lambda}_j = (\lambda_{j,1}, \lambda_{j,2}, \lambda_{j,3}, \lambda_{j,4}, \lambda_{j,5})^t$ exists such that $\bar{\lambda}_j^tM_j = 0$, that is,

$$\sum_{k=0}^{4} \lambda_{j,k+1}D_q^ku_j = \bar{\lambda}_j^tM_j\bar{u} = 0. \tag{2.12}$$

This is a q-difference equation for u_j of order at most 4. Such vector $\bar{\lambda}_j$ can be found using that if $m_{s,l}^{(j)}$ are the entries of M_j and $\Delta_k(M_j)$ is the determinant of the 4×4 matrix obtained from M_j by removing its kth row, then $\sum_{k=1}^{5} m_{k,l}^{(j)}(-1)^k\Delta_k(M_j) = 0$, $l = 1, \dots, 4$. Thus we can take

$$\bar{\lambda}_j = (\Delta_1(M_j), -\Delta_2(M_j), \Delta_3(M_j), -\Delta_4(M_j), \Delta_5(M_j))^t,$$

provided that it is a nonzero vector. $\square$

In most cases it is more convenient to write the q-difference equation (1.7) as a functional equation involving $p_n(x)$, $p_n(qx)$ and $p_n(q^2x)$. In fact, any q-difference equation of degree n can be written as a functional equation of the form

$$\sum_{j=0}^{n} a_j(x) f(q^j x) = 0, \tag{2.13}$$

and, conversely, any equation of the form (2.13) with $a_n(x) \neq 0$ is equivalent to a q-difference equation of degree n. This follows from the transformation formulas

(2.14)
$$D_q^n f(x) = \frac{1}{((1-q)x)^n} \sum_{j=0}^{n} (-1)^j q^{\binom{j+1}{2} - jn} \begin{bmatrix} n \\ j \end{bmatrix}_q f(q^j x), \quad n \in \mathbf{N}_0,$$

(2.15)
$$f(q^n x) = \sum_{j=0}^{n} (-1)^j ((1-q)x)^j q^{\binom{j}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q D_q^j f(x), \quad n \in \mathbf{N}_0,$$

where

$$\begin{bmatrix} n \\ j \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_j (q;q)_{n-j}}, \quad j = 0, \dots, n,$$

are the so-called q-binomial coefficients, and

$$(a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad n \in \mathbf{N}, \quad (a;q)_0 := 1.$$

The next lemma follows immediately from Lemma 2.1 and (2.14)–(2.15). We will give a different proof that provides a simpler algorithm for constructing a q-difference equation for the associated polynomials.

Lemma 2.2. *Let y_1 and y_2 be solutions of the functional equations*

$$y(q^2 x) = \tilde{f}_j(x) y(qx) + \tilde{g}_j(x) y(x), \quad j = 1, 2, \tag{2.16}$$

respectively. Then $v(x) = y_1(x) y_2(x)$ satisfies a functional equation of the form

$$\sum_{k=0}^{4} c_k(x) v(q^k x) = 0. \tag{2.17}$$

Proof. We set $v_1(x) = v(x)$, $v_2(x) = y_1(x)y_2(qx)$, $v_3(x) = y_1(qx)y_2(x)$ and $v_4(x) = y_1(qx)y_2(qx)$. Then $v_1(qx) = v_4(x)$ and, from equations (2.16) we obtain

$$\begin{aligned} v_2(qx) &= y_1(qx)y_2(q^2x) = \tilde{f}_2(x)v_4(x) + \tilde{g}_2(x)v_3(x), \\ v_3(qx) &= y_1(q^2x)y_2(qx) = \tilde{f}_1(x)v_4(x) + \tilde{g}_1(x)v_2(x), \\ v_4(qx) &= y_1(q^2x)y_2(q^2x) \\ &= \tilde{f}_1(x)\tilde{f}_2(x)v_4(x) + \tilde{f}_1(x)\tilde{g}_2(x)v_3(x) \\ &\quad + \tilde{f}_2(x)\tilde{g}_1(x)v_2(x) + \tilde{g}_1(x)\tilde{g}_2(x)v_1(x). \end{aligned}$$

These equations can be written in a matrix form. With $\bar{v}(x) = (v_1(x), v_2(x), v_3(x), v_4(x))^t$ and

$$T(x) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \tilde{g}_2(x) & \tilde{f}_2(x) \\ 0 & \tilde{g}_1(x) & 0 & \tilde{f}_1(x) \\ \tilde{g}_1(x)\tilde{g}_2(x) & \tilde{f}_2(x)\tilde{g}_1(x) & \tilde{f}_1(x)\tilde{g}_2(x) & \tilde{f}_1(x)\tilde{f}_2(x) \end{bmatrix}$$

we get the matrix equation

$$\bar{v}(qx) = T(x)\bar{v}(x). \tag{2.18}$$

We set $T_1(x) = T(x)$ and $T_{n+1}(x) = T(q^n x)T_n(x)$, $n \in \mathbf{N}$. Then equation (2.18) implies $\bar{v}(q^n x) = T_n(x)\bar{v}(x)$, $n \in \mathbf{N}$. As in the proof of Lemma 2.1, for $j = 1, \dots, 4$, we define $\tilde{M}_j$ to be the 5×4 matrix, the ith row of which is the jth row of $T_i(x)$, $i = 0, \dots, 4$, where $T_0(x)$ is the 4×4 identity matrix E. Let $\bar{\lambda}_j = (\lambda_{j,1}, \lambda_{j,2}, \lambda_{j,3}, \lambda_{j,4}, \lambda_{j,5})^t$ be a nonzero vector such that $\bar{\lambda}_j^t \tilde{M}_j = 0$. Then

$$\sum_{k=0}^{4} \lambda_{j,k+1} v_j(q^k x) = \bar{\lambda}_j^t \tilde{M}_j \bar{v} = 0, \tag{2.19}$$

is a functional equation for $v_j(x)$ of the form (2.17). In particular, for $j = 1$, we get such an equation for $v_1(x) = y_1(x)y_2(x)$ that can be written as a q-difference equation using (2.15). □

The main result concerning associated q-orthogonal polynomials is the following

Theorem 2.3. *The associated q-orthogonal polynomials $p_n^{(c)}$ with $c \in \mathbf{N}$ satisfy a fourth order q-difference equation.*

Proof. Applying Lemma 2.1 to (2.5)–(2.6) and using (2.4), we obtain a q-difference equation for $p_n^{(c)}/w$ of order at most four. Using formula (2.14) we can write this equation as an equation of the form (2.13) for $p_n^{(c)}/w$ and then for $p_n^{(c)}$ itself. Then applying (2.15) we get a q-difference equation for $p_n^{(c)}$ of order at most four. From Lemma 2.1 it follows that each one of the functions $wp_{c-1}p_{n+c}, wp_{c-1}Q_{n+c}, wQ_{c-1}p_{n+c}$ and $wQ_{c-1}Q_{n+c}$ satisfy this q-difference equation. We will show that these four functions form a solution basis for the q-difference equation for $p_n^{(c)}$, in particular the order of this equation is exactly four.

Indeed, assume that for some constants A, B, C and D,

$$\begin{aligned}(2.20)\quad & Aw(x)p_{c-1}(x)p_{n+c}(x) + Bw(x)p_{c-1}(x)Q_{n+c}(x) \\ & + Cw(x)Q_{c-1}(x)p_{n+c}(x) + Dw(x)Q_{c-1}(x)Q_{n+c}(x) = 0, \quad x \in S_w,\end{aligned}$$

where $S_w = \{x : w(x) > 0\}$. Since Q_{c-1} and Q_{n+c} have simple poles at infinitely many elements of the q-lattice, see (2.1), $D = 0$. Then $B = C = 0$ in which case $A = 0$ or $C = -B \neq 0$. For the latter case we use (2.4).

If $C = -B \neq 0$ we get from (2.20) with $D = 0$ and using (2.4)

$$Aw(x)p_{c-1}(x)p_{n+c}(x) = (B/a_c)p_n^{(c)}(x), \quad x \in S_w,$$

hence

$$\int_a^b (p_n^{(c)}(x))^2 \, d_q x = (a_c A/B) \int_a^b p_n^{(c)}(x)p_{c-1}(x)p_{n+c}(x)w(x) \, d_q x = 0$$

since $p_{n+c}(x)$ is orthogonal to $p_n^{(c)}(x)p_{c-1}(x)$ which is a polynomial of degree $n + c - 1$. This is clearly impossible. $\square$

3. Some examples.

1. *Big q-Jacobi polynomials.* The big q-Jacobi polynomials [**10**] are defined by

$$(3.1)\qquad P_n(x;a,b,c;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix}\,\middle|\, q;q\right),$$

where

$${}_r\phi_s\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\,\middle|\, q;z\right) = \sum_{n=0}^{\infty} \frac{(a_q,\dots,a_r;q)_n}{(q,b_1,\dots,b_s;q)_n} z^n(-q^{(n-1)/2})^{n(s+1-r)}$$

denotes a basic hypergeometric series, and

$$(a_1,\dots,a_r;q)_n = (a_1;q)_n\cdots(a_r;q)_n$$

denotes a product of q-shifted factorials.

The normalized polynomials

$$(3.2)\qquad p_n(x) := (-acq^2)^{-n/2} q^{\binom{-n}{2}/2}\left(\frac{(1-abq^{2n+1})(abq,aq,cq;q)_n}{(1-abq)(q,bq,abc^{-1}q;q)_n}\right)^{1/2} \times P_n(x;a,b,c;q)$$

satisfy the orthogonality relation

$$(3.3)\qquad \int_{cq}^{aq} p_m(x)p_n(x)w(x)\,d_qx = \delta_{m,n},$$

with weight

$$(3.4)\qquad w(x) = \frac{(aq,bq,cq,abc^{-1}q;q)_\infty}{aq(1-q)(q,a^{-1}c,ac^{-1}q,abq^2;q)_\infty}\frac{(a^{-1}x,c^{-1}x;q)_\infty}{(x,bc^{-1}x;q)_\infty}.$$

Note that $p_0(x) = 1$ and, since $w(a) = w(c) = 0$, w satisfies (2.2). The q-difference equation in the form (2.16) is

$$(3.5)\qquad y(q^2x) = \left(1 + \frac{D(x) - (1-q^{-n})(1-abq^{n+1})q^2x^2}{B(x)}\right)y(qx) - \frac{D(x)}{B(x)}y(x),$$

where $B(x) = aq(qx-1)(bqx-c)$ and $D(x) = q^2(x-a)(x-c)$.

2. *Big q-Laguerre polynomials.* The big q-Laguerre polynomials [**10**] are defined by

$$P_n(x;a,b;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, 0, x \\ aq, bq \end{matrix}\middle| q;q\right). \tag{3.6}$$

The normalized polynomials

$$p_n(x) := (-abq^2)^{-n/2} q^{-\binom{n}{2}/2} \left(\frac{(aq,bq;q)_n}{(q;q)_n}\right)^{1/2} P_n(x;a,b;q) \tag{3.7}$$

satisfy the orthogonality relation

$$\int_{bq}^{aq} p_m(x)p_n(x)w(x)\,d_qx = \delta_{m,n}, \tag{3.8}$$

with weight

$$w(x) = \frac{(aq,bq;q)_\infty}{aq(1-q)(q,ba^{-1},ab^{-1}q;q)_\infty}\frac{(a^{-1}x,b^{-1}x;q)_\infty}{(x;q)_\infty}. \tag{3.9}$$

Note that $p_0(x) = 1$ and, since $w(a) = w(b) = 0$, w satisfies (2.2). The q-difference equation in the form (2.16) is

$$y(q^2x) = \left(1 + \frac{D(x) + (1-q^{-n})q^2x^2}{B(x)}\right)y(qx) - \frac{D(x)}{B(x)}y(x), \tag{3.10}$$

where $B(x) = abq(qx-1)$ and $D(x) = -q^2(x-a)(x-b)$.

Using the algorithm of Lemma 2.2, we can find fourth order q-difference equations for these two families of discrete q-orthogonal polynomials.

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