ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 33, Number 2, Summer 2003

POSITIVITY OF CONTINUED FRACTIONS ASSOCIATED WITH RATIONAL STIELTJES MOMENT PROBLEMS

A. BULTHEEL, P. GONZÁLEZ-VERA, E. HENDRIKSEN AND OLAV NJÅSTAD

Dedicated to W.B. Jones on the occasion of his 70th birthday

ABSTRACT. The constructive solution of the strong Stieltjes moment problem can be linked with positive Perron-Carathéodory continued fractions (PC-fractions) which are contradictions of positive Thron continued fractions (T-fractions). Their approximants are two-point Padé approximants for a related function. The multi-point moment problem similarly leads to positive multi-point Padé continued fractions (MP-fraction) and positive extended multi-point Padé continued fractions (EMP-fraction) whose approximants are multi-point Padé approximants. These relationships are explored also in the situation where the positivity condition is dropped.

1. Introduction. MP-fractions, multi-point Padé continued fractions, have similar relationship to multi-point Padé approximants as (general) T-fractions (Thron continued fractions) have to two-point Padé approximants. An MP-fraction is normally the even contraction of an EMP-fraction (extended MP-fraction) just as a (modified) T-fraction is the even contraction of a (modified) PC-fraction (Perron-Carathéodory continued fraction). The odd contraction of an EMP-fraction is normally equivalent to an MP-fraction, just as the odd contraction of a PC-fraction is basically a T-fraction (more precisely, an M-fraction).

Positive T-fractions and positive PC-fractions are related to two-point Padé approximants arising from strong (or two-point) Stieltjes moment problems. In this note we discuss EMP-fractions associated with multi-point Padé approximants arising from rational (or multi-point) Stieltjes moment problems.

1991 *AMS Mathematics Subject Classification.* 30B70, 41A21.
Key words and phrases. Multi-point Padé approximation, continued fraction, MP-fraction, multi-point Stieltjes moment problem.
Received by the editors on September 30, 2001, and in revised form on February 26, 2002.

For information on the above relationship in the two-point situation we refer to [**2, 19, 20, 23**] and references found therein. For earlier work on the corresponding relationship in the multi-point situation, see [**5, 8, 10, 11, 13, 16, 17**].

General information on (multi-point) Padé approximation can be found, e.g., in [**1, 4, 9, 10, 13**–**15**].

2. Orthogonal rational functions. Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of not necessarily distinct points on the real axis. We define the functions ω_n by

$$\omega_0 = 1, \quad \omega_n(z) = (z-\zeta_1)\cdots(z-\alpha_n), \quad n = 1, 2, \dots ,$$

and the spaces $\mathcal{L}_n$ and $\mathcal{L}$ by

$$\mathcal{L}_n = \operatorname{span}\left\{\frac{1}{\omega_0}, \frac{1}{\omega_1}, \dots, \frac{1}{\omega_n}\right\}, \quad \mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n.$$

The elements of $\mathcal{L}_n$ are exactly the functions $f(z) = p(z)/\omega_n(z)$, $p \in \Pi_n$, where Π_n denotes the space of polynomials of degree at most n.

Let μ be a probability measure on $(-\infty, \infty)$ such that all the functions in the product space $\mathcal{L}\cdot\mathcal{L}$ are absolutely integrable. (A more general situation can be considered where the measure is replaced by a linear functional. See, for example, [**3, 6, 7, 10**].) The measure μ induces an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{L}$ defined by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}\, d\mu(t), \quad f, g \in \mathcal{L}.$$

Let $\{\varphi_n\}_{n=0}^{\infty}$ be the essentially unique orthonormal sequence corresponding to the basis $\{1/\omega_n\}_{n=0}^{\infty}$. Each φ_n can be expressed in the form

$$\varphi_n(z) = \frac{p_n(z)}{\omega_n(z)}, \quad p_n \in \Pi_n.$$

By the defining property of φ_n, we have

$$p_n(\alpha_n) \neq 0, \quad n = 1, 2, \dots .$$

The sequence $\{\varphi_n\}$ is called *regular* if

$$\deg p_1 = 1, \quad p_n(\alpha_{n-1}) \neq 0, \quad n = 2, 3, \dots .$$

It will be assumed throughout this paper that the sequence $\{\varphi_n\}$ is regular.

In several parts of this paper, we shall be especially interested in the following situation, which may be considered as an analog of the two-point Stieltjes situation. We assume the existence of real numbers α and β such that

$$\alpha_{2m} \leq \alpha < \beta \leq \alpha_{2m-1} \leq 0, \quad m = 1, 2, \dots , \tag{2.1}$$

and we furthermore assume that the support of the measure μ is contained in $[0, \infty)$, i.e.,

$$\operatorname{supp}(\mu) \subset [0, \infty). \tag{2.2}$$

In this case the polynomial p_n and hence the function φ_n have all their zeros in $(0, \infty)$ and the sequence $\{\varphi_n\}$ is regular. Whenever we deal with this situation, we shall normalize the sign of φ_n such that

$$p_{4m}(x) > 0, \quad p_{4m+1}(x) > 0, \quad p_{4m+2}(x) < 0, \quad p_{4m+3}(x) < 0, \tag{2.3}$$

for $x \in (-\infty, 0)$. It follows from (2.7)–(2.9) that $\varphi_n(x) > 0$ for $\alpha < x < \beta$.

Note that the two-point Stieltjes situation is obtained as the limiting situation when α tends to ∞ along the negative axis and $\beta = 0$, $\alpha_{2m} = \alpha$, $\alpha_{2m-1} = 0$ for all m.

For more information on orthogonal rational functions and associated moment theory, we refer to [**3, 6, 7, 10**–**12**], cf. also [**18**]. For the relationship of this theory to the theory of multi-point Padé approximants, see especially [**4, 8, 9, 13, 16, 17, 24**].

3. MP-fractions. When the sequence $\{\varphi_n\}$ is regular, it satisfies a recurrence relation of the form

(3.1)

$$\varphi_n(z) = \left(h_n + \frac{g_n}{z - \alpha_n}\right)\varphi_{n-1}(z) + F_n \frac{z - \alpha_{n-2}}{z - \alpha_n}\varphi_{n-2}(z), \quad n = 1, 2, \dots ,$$

with initial conditions

$$\varphi_0 = 1, \quad \varphi_{-1} = 0.$$

(For simplicity, we have used the following convention: $z - \alpha_0$ means 1, $z - \alpha_{-1}$ means z.) The coefficients satisfy the inequalities

$$F_n \neq 0, \quad g_n + h_n(\alpha_{n-1} - \alpha_n) \neq 0, \quad n = 1, 2, \dots . \tag{3.2}$$

Proof of this result can be found in [**10**] and in [**3**] for the analogous situation when all the points α_k lie on the unit circle. See also [**5**–**8**]. In the special situation when the points α_k arise by cyclic repetition of a finite number of points, the recurrence formula was obtained in [**16**].

Let the functions ρ_n be defined by the same recurrence (3.1), i.e.,

$$\rho_n(z) = \left(h_n + \frac{g_n}{z - \alpha_n}\right)\rho_{n-1}(z) + F_n \frac{z - \alpha_{n-2}}{z - \alpha_n}\rho_{n-2}(z), \quad n = 1, 2, \dots , \tag{3.3}$$

with initial conditions

$$\rho_0 = 1, \quad \rho_{-1} = 1. \tag{3.4}$$

Then ρ_n and φ_n are canonical numerators and denominators of a continued fraction

$$1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{\theta_n}{\kappa_n} \tag{3.5}$$

where

$$\begin{aligned} \theta_n &= F_n \frac{z - \alpha_{n-2}}{z - \alpha_n}, \quad n = 1, 2, \dots \\ \kappa_n &= h_n + \frac{g_n}{z - \alpha_n}, \quad n = 1, 2, \dots . \end{aligned} \tag{3.6}$$

See [**21, 22**]. Continued fractions of this kind are called *Multi-point Padé continued fractions*, or *MP-fractions*. The reason for this terminology may be explained as follows. The sequence of approximants $\{\rho_n/\varphi_n\}$ determines a sequence of interpolation values at the table

$\{\infty, 0, \alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \dots\}$ such that $\{\rho_n/\varphi_n\}$ is an $[n/n]$ multi-point Padé approximant of the corresponding Newton series. See [**8**] and also [**4, 9, 13, 17**].

The sum $\kappa_n = h_n + \dfrac{g_n}{z-\alpha_n}$ may be written in various ways:

$$h_n + \frac{g_n}{z-\alpha_n} = \frac{x_n \zeta_n(z) + y_n \tau_n(z)}{z-\alpha_n},$$

where $\{\zeta_n, \tau_n\}$ is an arbitrary basis for the space Π_1 and x_n and y_n are constants. In particular, we find that if

$$\alpha_{2m} \neq \alpha_{2m-1}, \quad m = 1, 2, \dots, \tag{3.7}$$

then we may write

$$\kappa_{2m} = H_{2m} + G_{2m} \frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}, \quad m = 1, 2, \dots, \tag{3.8}$$

$$\kappa_{2m+1} = H_{2m+1} \frac{z-\alpha_{2m}}{z-\alpha_{2m+1}} + G_{2m+1} \frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}, \quad m = 0, 1, 2, \dots. \tag{3.9}$$

It follows from (3.2) that

$$H_{2m} \neq 0, \quad m = 1, 2, \dots, \quad G_{2m} \neq 0, \quad m = 0, 1, 2, \dots. \tag{3.10}$$

The representation (3.8)–(3.9) is used in [**8**] and we shall make use of it in later sections. We note that, by the formal substitution $(z-\alpha_{2m+1}) \to z$, $(z-\alpha_{2m}) \to 1$ for $m = 1, 2, \dots$, we obtain recurrence formulas for orthogonal Laurent polynomials. See [**2**], (where the Laurent polynomials are not orthonormal but normalized such that the left coefficient, i.e., H_n, is 1).

Now assume that (3.7) is replaced by the stronger condition

$$\alpha_{2p} \neq \alpha_{2q-1}, \quad p, q = 1, 2, \dots. \tag{3.11}$$

(Note that this is the case in the Stieltjes situation (2.1)–(2.2)). We may then use the basis $\{z-\alpha_{n-2}, z-\alpha_{n-1}\}$ for Π_1 for any $n > 2$ to express the elements κ_n. Thus we may write

$$\begin{aligned}\theta_n &= W_n \frac{z-\alpha_{n-2}}{z-\alpha_n}, \quad n=3,4,\dots\\ \theta_2 &= W_2 \frac{1}{z-\alpha_2},\\ \theta_1 &= W_1 \frac{z}{z-\alpha_1},\\ \kappa_n &= \frac{U_n(z-\alpha_{n-2})+V_n(z-\alpha_{n-1})}{z-\alpha_n}, \quad n=3,4,\dots,\\ \kappa_2 &= \frac{U_2(z-\alpha_2)+V_2(z-\alpha_1)}{z-\alpha_2}\\ \kappa_1 &= \frac{U_1(z-\alpha_1)+V_1(z-\alpha_2)}{z-\alpha_1}.\end{aligned}$$

This representation is used in [**11**] except that θ_1 is replaced by $W_1/(z-\alpha_1)$. This has no influence on the recursion for $\{\varphi_n\}$, since $\varphi_{-1}=0$.

It should be pointed out that the functions ρ_k that were introduced by (3.3)–(3.4) are *not* the same as the associated functions as they were defined in [**11**]. They are however closely related as we shall presently show. In [**11**] the associated functions σ_n are defined by

$$\sigma_n(z) = \int_{-\infty}^{\infty} \frac{\varphi_n(t)-\varphi_n(z)}{t-z}\, d\mu(t), \quad n=0,1,2,\dots.$$

The sequence $\{\sigma_n\}_{n=0}^{\infty}$ satisfies the recursion

$$\begin{aligned}\sigma_n(z) &= \kappa_n\sigma_{n-1}+\theta_n\sigma_{n-2}, \quad n=2,3,\dots,\\ \sigma_1(z) &= \kappa_1\sigma_0+\frac{W_1}{z-\alpha_1}\sigma_{-1},\end{aligned}$$

with initial conditions

$$\sigma_0=0, \quad \sigma_{-1}=-1.$$

Thus the sequence $\{\pi_n\}_{n=0}^{\infty}$ with $\pi_n(z)=z\sigma_n(z)$ satisfies the recursion

$$\pi_n = \kappa_n\pi_{n-1}+\theta_n\pi_{n-2}, \quad n=1,2,\dots$$

with initial conditions

$$\pi_0 = 0, \quad \pi_{-1} = -1.$$

It follows from this that

$$\rho_n = \varphi_n - \pi_n, \quad n = 0, 1, 2, \ldots .$$

4. Positive MP-fractions. We shall in this section assume that (3.11) holds. Note that in the Stieltjes situation described by (2.1)–(2.2), the sequence $\{\varphi_n\}$ is regular and (3.11) is satisfied.

The coefficients F_n, G_n, H_n can be expressed in terms of U_n, V_n, W_n as follows:

$$F_n = W_n, \quad n = 1, 2, \ldots \tag{4.1}$$

$$G_1 = U_1 + V_1, \quad G_{2m+1} = U_{2m+1}, \quad m = 1, 2, \ldots \tag{4.2}$$

$$G_2 = V_2, \quad G_{2m} = \frac{U_{2m}(\alpha_{2m} - \alpha_{2m-2}) + V_{2m}(\alpha_{2m} - \alpha_{2m-1})}{\alpha_{2m} - \alpha_{2m-1}}, \quad m = 2, 3, \ldots \tag{4.3}$$

$$H_1 = -(\alpha_1 U_1 + \alpha_2 V_1), \quad H_{2m+1} = V_{2m+1}, \quad m = 1, 2, \ldots \tag{4.4}$$

$$H_2 = U_2, \quad H_{2m} = U_{2m} \frac{\alpha_{2m-1} - \alpha_{2m-2}}{\alpha_{2m-1} - \alpha_{2m}}, \quad m = 2, 3, \ldots . \tag{4.5}$$

(This is found by direct comparison of the expressions for θ_n and κ_n.)

Furthermore, it follows from [**11**, Section 3] that, for $n = 3, 4, \ldots,$

$$\begin{aligned} U_n &= \frac{p_n(\alpha_{n-1})}{(\alpha_{n-1} - \alpha_{n-2}) p_{n-1}(\alpha_{n-1})}, \\ V_n &= \frac{p_n(\alpha_{n-2})}{(\alpha_{n-2} - \alpha_{n-1}) p_{n-1}(\alpha_{n-2})}, \end{aligned} \tag{4.6}$$

while

$$U_2 = \frac{p_2(\alpha_1)}{(\alpha_1 - \alpha_2) p_1(\alpha_1)}, \quad V_2 = \frac{c_{0,1} p_2(\alpha_2) p_1(\alpha_1)^2 + p_2(\alpha_1)(\alpha_2 - \alpha_1)}{(\alpha_2 - \alpha_1) c_{0,1} p_1(\alpha_1)^2 p_1(\alpha_2)}, \tag{4.7}$$

$$U_1 = \frac{p_1(\alpha_2)}{(\alpha_2 - \alpha_1)}, \quad V_1 = \frac{p_1(\alpha_1)}{(\alpha_1 - \alpha_2)}, \tag{4.8}$$

and

$$W_n = -\frac{p_n(\alpha_{n-1})p_{n-2}(\alpha_{n-2})}{p_{n-1}(\alpha_{n-1})p_{n-1}(\alpha_{n-2})}, \quad n = 3, 4, \dots, \tag{4.9}$$

with

$$W_2 = -\frac{p_2(\alpha_1)}{p_1(\alpha_2)^2 c_{0,1}} \tag{4.10}$$

$$W_1 = c_{0,1}p_1(\alpha_1) \tag{4.11}$$

$\big($where $c_{0,1} = \displaystyle\int_{-\infty}^{\infty} \frac{d\mu(t)}{t-\alpha_1}$ $\big)$.

Now we assume that we are in the Stieltjes situation with the normalization (2.3), so that (2.1)–(2.3) holds and consequently (3.11) is satisfied. We then find from (4.6)–(4.11) together with (2.1)–(2.3) that

$$U_n < 0, \quad n = 1, 2, 3, \dots \tag{4.12}$$

$$V_n > 0, \quad n = 1, 2, 3, \dots \tag{4.13}$$

$$W_n > 0, \quad n = 1, 2, 3, \dots, \tag{4.14}$$

(See [**11**, Section 3]).

Theorem 4.1. *In the Stieltjes situation with the normalization* (2.3), (*i.e., when* (2.1)–(2.3) *are satisfied*), *the following inequalities hold*:

$$F_n > 0, \quad n = 1, 2, 3, \dots \tag{4.15}$$

$$G_{2m} > 0, \quad m = 1, 2, 3, \dots, \tag{4.16}$$

$$G_{2m+1} < 0, \quad m = 0, 1, 2, 3, \dots, \tag{4.17}$$

$$H_{2m} < 0, \quad m = 1, 2, 3, \dots \tag{4.18}$$

$$H_{2m+1} > 0, \quad m = 1, 2, 3, \dots. \tag{4.19}$$

Proof. First note that $G_1 = U_1 + V_1 = [p(\alpha_1) - p_1(\alpha_2)]/(\alpha_1 - \alpha_2)$. Because p_1 has its zero in $(0, \infty)$, it follows from (2.3) that p_1 is decreasing in $(-\infty, 0)$. Hence $G_1 < 0$. Furthermore, because $p_1(x) =$

$H_1 + G_1 x$, we have for $x_0 \in (0, \infty)$ the zero of p_1 that $0 = p_1(x_0) = H_1 + G_1 x_0$, so that $H_1 = -G_1 x_0 > 0$. The rest of the inequalities (4.15) and (4.17)–(4.19) follow immediately from (4.1)–(4.5) together with (4.12)–(4.14).

To prove the inequality (4.16) we note that, for $m = 1, 2, \ldots,$

$$\varphi_{2m}(z) = \left[H_{2m} + G_{2m}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}\right]\varphi_{2m-1}(z) + F_{2m}\frac{z-\alpha_{2m-2}}{z-\alpha_{2m}}\varphi_{m-2}(z).$$

Taking the inner product with φ_{2m-2} gives

(4.20)

$$0 = G_{2m}\left\langle \frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}\varphi_{2m-1}, \varphi_{2m-2}\right\rangle + F_{2m}\left\langle \frac{z-\alpha_{2m-2}}{z-\alpha_{2m}}\varphi_{2m-2}, \varphi_{2m-2}\right\rangle$$

where the second term is positive. Again, using the recurrence for the φ_n, we get for $m = 0, 1, \ldots,$

$$\varphi_{2m+1} = \left[H_{2m+1}\frac{z-\alpha_{2m}}{z-\alpha_{2m+1}} + G_{2m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}\right]\varphi_{2m} + F_{2m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}\varphi_{2m-1}$$

which can be written as

$$\frac{z-\alpha_{2m+1}}{z-\alpha_{2m}}\varphi_{2m+1} = \left[H_{2m+1} + G_{2m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}\right]\varphi_{2m} + F_{2m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}\varphi_{2m-1}.$$

We take the inner product with φ_{2m-2} so that, because φ_{2m+1} is orthogonal to $\varphi_{2m-2}\dfrac{z-\alpha_{2m+1}}{z-\alpha_{2m}}$, we get

(4.21)

$$0 = G_{2m+1}\left\langle \tfrac{z-\alpha_{2m-1}}{z-\alpha_{2m}}\varphi_{2m}, \varphi_{2m-2}\right\rangle + F_{2m+1}\left\langle \tfrac{z-\alpha_{2m-1}}{z-\alpha_{2m}}\varphi_{2m-1}, \varphi_{2m-2}\right\rangle.$$

Next we note that

$$\frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}\varphi_{2m-2}(z) = \frac{(z-\alpha_{2m-1})^2 p_{2m-2}(z)}{\omega_{2m(z)}} = g(z) + \frac{(\alpha_{2m}-\alpha_{2m-1})^2 p_{2m-2}(\alpha_{2m})}{\omega_{2m}(z)}$$

with $g \in \mathcal{L}_{2m-1}$. Furthermore,

$$p_{2m}(z) = p_{2m}(\alpha_{2m}) + (z - \alpha_{2m})h(z), \quad h \in \Pi_{2m-1},$$

so that the orthonormal φ_{2m} has leading coefficient $p_{2m}(\alpha_{2m})$ with respect to the basis $\{1/\omega_k\}$, and hence $\langle 1/\omega_{2m}, \varphi_{2m}\rangle = 1/p_{2m}(\alpha_{2m})$. With these two observations we can conclude that

$$\left\langle \frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}\varphi_{2m}, \varphi_{2m-2}\right\rangle = \left\langle \frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}\varphi_{2m-2}, \varphi_{2m}\right\rangle = \frac{(\alpha_{2m}-\alpha_{2m-1})^2 p_{2m-2}(\alpha_{2m})}{p_{2m}(\alpha_{2m})}.$$

This is negative because of (2.3). On the other hand, because $F_n > 0$ and $G_{2m+1} < 0$, we can conclude from (4.20) and (4.21) that $G_{2m} > 0$. This proves (4.16). □

We shall call an MP-fraction with elements given by (3.6), (3.8)–(3.9) which satisfy (4.16)–(4.19) a *positive MP-fraction*. Thus, Theorem 4.1 states that in the Stieltjes situation (2.1)–(2.2) with the normalization (2.3), the corresponding MP-fraction is positive. The reason for calling this a positive MP-fraction is because, in the two-point case, i.e., when $\alpha_{2m} = \alpha \to -\infty$ and $\alpha_{2m+1} = \beta \to 0$, it becomes, with proper normalization, equivalent to a positive T-fraction. They play in the multi-point case the same role as the positive T-fractions do in the two-point case.

Remark 4.2. We remark that the two-point version of the formulas in this paper do not coincide directly with the formulas given in [**2**]. This is partly due to a different normalization, namely in [**2**] an equivalent continued fraction is considered in which all the coefficients H_n were chosen to be 1 (which is possible by (3.10)). More importantly, the moments were defined differently in [**2**]. The nth moment there is defined as the integral of $(-x)^n$ instead of x^n. This implies, for example, that the inequalities of [**2**] corresponding to our (4.15)–(4.19) indicate that these numbers are all positive instead of having the present alternating sign. It also explains why in [**2**] the denominator polynomials $Q_n(x)$ (which correspond to our $\varphi_n(x)$) are not orthogonal, but instead the $Q_n(-x)$ are.

5. EMP-fraction. We now turn to the general situation where the elements of the MP-fraction can be written in the form (3.6), (3.8)–(3.9). We shall in this section assume that the sequence $\{\varphi_n\}$ is *strongly regular*, which means that in addition to (3.10) we also have

$$G_{2m} \neq 0, \quad m = 1, 2, \dots, \qquad H_{2m+1} \neq 0, \quad m = 0, 1, 2, \dots. \tag{5.1}$$

Note in particular that a positive MP-fraction satisfies this condition.

In [**8**] we did not use orthonormal functions φ_n but orthogonal functions normalized such that $H_n = 1$ for all n. This normalization is not consistent with the sign normalization (2.3) used here in the Stieltjes case, (cf. (4.18)). In the extension process that we are going to outline we shall therefore treat the general situation given by (3.6), (3.8)–(3.9). The formulas we obtain will for $H_n = 1$ reduce to formulas found in [**8**, Section 3].

We shall construct an extension

$$1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{b_n} \tag{5.2}$$

of the continued fraction (3.5) such that (3.5) is the *even contraction* of (5.2). From general formulas (see, e.g., [**21, 22**]) we find that $\{a_n\}$, $\{b_n\}$ must satisfy the equations

$$\begin{aligned} b_0 &= \kappa_0, \\ b_2 a_1 &= \theta_1, \\ \frac{b_{2n} a_{2n-1} a_{2n-2}}{b_{2n-2}} &= -\theta_n, \quad n = 2, 3, \dots \\ a_2 + b_1 b_2 &= \kappa_1, \\ a_{2n} + b_{2n} b_{2n-1} + a_{2n-1} \frac{b_{2n}}{b_{2n-2}} &= \kappa_n, \quad n = 2, 3, \dots. \end{aligned}$$

These may in our situation (cf. (3.5), (3.8)–(3.9)), be written in the following form

$$b_2 a_1 = \frac{F_1 z}{z - \alpha_1} \tag{5.3}$$

$$\frac{b_{4m}a_{4m-1}a_{4m-2}}{b_{4m-2}} = -F_{2m}\frac{z-\alpha_{2m-2}}{z-\alpha_{2m}}, \tag{5.4}$$

$$\frac{b_{4m+2}a_{4m+1}a_{4m}}{b_{4m}} = -F_{2m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}, \tag{5.5}$$

$$a_2 + b_1 b_2 = \frac{H_1}{z-\alpha_1} + \frac{G_1 z}{z-\alpha_1}, \tag{5.6}$$

$$\begin{aligned} a_{4m} + b_{4m}b_{4m-1} + a_{4m-1}\frac{b_{4m}}{b_{4m-2}} \\ = H_{2m} + G_{2m}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}, \quad m = 1, 2, \ldots \end{aligned} \tag{5.7}$$

$$\begin{aligned} a_{4m+2} + b_{4m+2}b_{4m+1} + a_{4m+1}\frac{b_{4m+2}}{b_{4m}} \\ = H_{2m+1}\frac{z-\alpha_{2m}}{z-\alpha_{2m+1}} + G_{2m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}, \quad m = 0, 1, 2, \ldots . \end{aligned} \tag{5.8}$$

We define λ_n, μ_n by

$$\lambda_{2n} = H_n, \quad n = 1, 2, \ldots \tag{5.9}$$

$$\lambda_1 = \frac{F_1}{G_1}, \quad \lambda_{2n+1} = -\frac{F_{n+1}}{G_{n+1}}, \quad n = 1, 2, \ldots \tag{5.10}$$

$$\mu_2 = G_1, \quad \mu_{2n} = \frac{G_1 G_2 \cdots G_n}{H_1 H_2 \cdots H_{n-1}}, \quad n = 2, 3, \ldots \tag{5.11}$$

(5.12)
$$\mu_1 = 1, \quad \mu_{2n+1} = \frac{(G_{n+1} + F_{n+1}/H_n)H_1 H_2 \cdots H_n}{G_1 G_2 \cdots G_{n+1}}, \quad n = 1, 2, 3, \ldots .$$

With this notation we find the following solution of the system (5.3)–(5.8), cf. [**8**, Section 3],

$$a_{4m} = \lambda_{4m}, \quad m = 1, 2, \ldots \tag{5.13}$$

$$a_2 = \frac{\lambda_2}{z-\alpha_1}, \quad a_{4m+2} = \lambda_{4m+2}\frac{z-\alpha_{2m}}{z-\alpha_{2m+1}}, \quad m = 1,2,\dots \tag{5.14}$$

$$a_{4m-1} = \lambda_{4m-1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}, \quad m = 1,2,\dots \tag{5.15}$$

$$a_1 = \frac{\lambda_1 z}{z-\alpha_1}, \quad a_{4m+1} = \lambda_{4m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}, \quad m = 1,2,\dots \tag{5.16}$$

$$b_{4m} = \mu_{4m}, \quad m = 1,2,\dots \tag{5.17}$$

$$b_{4m+2} = \mu_{4m+2}, \quad m = 0,1,2,\dots \tag{5.18}$$

$$b_{4m-1} = \mu_{4m-1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m}}, \quad m = 1,2,\dots \tag{5.19}$$

$$b_1 = \frac{z}{z-\alpha_1}, \quad b_{4m+1} = \mu_{4m+1}\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}}, \quad m = 1,2,\dots . \tag{5.20}$$

The coefficients λ_n, μ_n satisfy the equality

$$\lambda_{2n+1} + \mu_{2n}\mu_{2n+1} = \lambda_{2n}, \quad n = 1,2,\dots . \tag{5.21}$$

We conclude from (3.10) and (5.1) together with (5.9)–(5.12) that

$$\lambda_n \neq 0, \quad \mu_n \neq 0, \quad n = 1,2,\dots . \tag{5.22}$$

A continued fraction $\kappa_0 + \mathbf{K}_{n=1}^{\infty} a_n/b_n$ where the elements are of the form (5.13)–(5.20) with the coefficients satisfying (5.21) is called an *EMP-fraction* (*Extended MP-fraction*). We have seen that an MP-fraction satisfying (3.10) and (5.1) is the even contraction of an EMP-fraction satisfying (5.22).

6. Positive EMP-fractions. We now consider a positive MP-fraction with elements given by (3.6), (3.8)–(3.9). Recall that positivity means that (4.15)–(4.19) are satisfied. The denominator sequence is then strongly regular and, according to Section 5, there exists an EMP-fraction with elements given by (5.9)–(5.20) whose even contraction is the MP-fraction.

Theorem 6.1. *The coefficients of the EMP-fraction obtained by extension of a positive MP-fraction satisfy the following inequalities*:

(6.1) $$\lambda_{4m} < 0, \quad m = 1, 2, 3, \dots, \qquad \lambda_{4m+2} > 0, \quad m = 0, 1, 2, \dots$$

(6.2) $$\lambda_1 < 0, \quad \lambda_{4m-1} < 0, \quad m = 1, 2, \dots, \qquad \lambda_{4m+1} > 0, \quad m = 1, 2, \dots$$

(6.3) $$\mu_{4m} < 0, \quad m = 1, 2, \dots, \qquad \mu_{4m+2} < 0, \quad m = 0, 1, 2, \dots$$

(6.4) $$\mu_{4m-1} < 0, \quad m = 1, 2, \dots, \qquad \mu_{4m+1} > 0, \quad m = 0, 1, 2, \dots .$$

Proof. These inequalities follow immediately from the defining formulas (5.9)–(5.12) together with (4.15)–(4.19). □

We shall call an EMP-fraction which satisfies (6.1)–(6.3) a *positive EMP-fraction*. Note that by (5.21) the inequalities (6.4) are automatically satisfied. Thus Theorem 6.1 states that an EMP-fraction obtained by extension of a positive MP-fraction is a positive EMP-fraction. We note that, in the two-point case, i.e., when $\alpha_{2m} = \alpha \to -\infty$ and $\alpha_{2m+1} = \beta \to 0$, the positive EMP-fraction becomes, with proper normalization, equivalent to a positive PC-fraction. They are multi-point generalizations of PC-fractions.

7. Contractions of EMP-fractions. When the elements of an EMP-fraction satisfy

$$\mu_{2n} \neq 0, \quad n = 1, 2, \dots,$$

we shall call the continued fraction *e-regular*. It follows easily from the discussion in Section 5 that an e-regular EMP-fraction has an even contraction which is an MP-fraction with elements

(7.1) $$F_1 = \lambda_1 \mu_2, \quad F_n = -\lambda_{2n-1} \lambda_{2n-2} \frac{\mu_{2n}}{\mu_{2n-2}}, \quad n = 2, 3, \dots$$

(7.2) $$G_1 = \mu_2, \quad G_n = \lambda_{2n-2}\,\frac{\mu_{2n}}{\mu_{2n-2}}, \quad n = 2, 3, \ldots$$

(7.3) $$H_n = \lambda_{2n}, \quad n = 1, 2, \ldots$$

In particular a positive EMP-fraction is e-regular, and thus the even contraction exists.

Theorem 7.1. *The even contraction of a positive EMP-fraction is a positive MP-fraction.*

Proof. The inequalities (4.15)–(4.19) follow immediately from (6.1)–(6.3) and (7.1)–(7.3). □

When the elements of an EMP-fraction satisfy

$$\mu_{2m+1} \neq 0, \quad n = 0, 1, 2, \ldots\,,$$

we shall call the EMP-fraction *o-regular*. An o-regular EMP-fraction $1 + \mathbf{K}_{n=1}^{\infty} a_n/b_n$ given by (5.13)–(5.21) has the odd contraction

$$\lambda_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{\xi_n}{\eta_n}$$

where the elements are given by (see [**21, 22**])

$$\lambda_0 = 1 + \lambda_1,$$

$$\xi_n = -a_{2n-1}a_{2n}\,\frac{b_{2n+1}}{b_{2n-1}}, \quad n = 1, 2, \ldots\,,$$

$$\eta_n = a_{2n+1} + b_{2n}b_{2n+1} + a_{2n}\,\frac{b_{2n+1}}{b_{2n-1}}, \quad n = 1, 2, \ldots\,.$$

Substituting from (5.13)–(5.20), we get

$$\xi_{2m} = Z_{2m}\,\frac{z - \alpha_{2m-1}}{z - \alpha_{2m+1}}, \quad m = 1, 2, \ldots$$

$$\xi_{2m+1} = Z_{2m+1}\,\frac{z-\alpha_{2m}}{z-\alpha_{2m+2}}, \quad m = 0,1,2\dots$$

$$\eta_{2m} = X_{2m}\,\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+1}} + Y_{2m}\,\frac{z-\alpha_{2m}}{z-\alpha_{2m+1}}, \quad m = 1,2,\dots$$

$$\eta_{2m+1} = X_{2m+1}\,\frac{z-\alpha_{2m+1}}{z-\alpha_{2m+2}} + Y_{2m+1}\,\frac{(z-\alpha_{2m})(z-\alpha_{2m+1})}{(z-\alpha_{2m-1})(z-\alpha_{2m+2})}, \quad m = 0,1,2,\dots,$$

where

$$Z_n = -\lambda_{2n}\lambda_{2n-1}\,\frac{\mu_{2n+1}}{\mu_{2n-1}}, \quad n = 1,2,\dots \tag{7.4}$$

$$X_n = \lambda_{2n}, \quad n = 1,2,\dots \tag{7.5}$$

$$Y_n = \lambda_{2n}\,\frac{\mu_{2n+1}}{\mu_{2n-1}}, \quad n = 1,2,\dots \tag{7.6}$$

By a simple transformation, see [**8**, Section 6], this continued fraction is seen to be equivalent to an MP-fraction $\lambda_0 + \mathbf{K}_{n=1}^{\infty} u_n/v_n$ (corresponding to the interpolation sequence $\{\alpha_2, \alpha_1, \alpha_4, \alpha_3, \dots\}$), where

$$u_{2m} = Z_{2m}\,\frac{z-\alpha_{2m-3}}{z-\alpha_{2m-1}}, \quad m = 1,2,\dots$$

$$u_{2m+1} = Z_{2m+1}\,\frac{z-\alpha_{2m}}{z-\alpha_{2m+2}}, \quad m = 0,1,2,\dots$$

$$v_{2m} = X_{2m} + Y_{2m}\,\frac{z-\alpha_{2m}}{z-\alpha_{2m-1}}, \quad m = 1,2,\dots$$

$$v_{2m+1} = X_{2m+1}\,\frac{z-\alpha_{2m-1}}{z-\alpha_{2m+2}} + Y_{2m+1}\frac{z-\alpha_{2m}}{z-\alpha_{2m+2}}, \quad m = 1,2,\dots.$$

A positive EMP-fraction is clearly o-regular, and thus the odd contraction exists.

Theorem 7.2. *The odd contraction of a positive EMP-fraction is equivalent to an MP fraction of the form* $\lambda_0 + \mathbf{K}_{n=1}^{\infty} u_n/v_n$ *and* $\lambda_0 - \mathbf{K}_{n=1}^{\infty} u_n/v_n$ *is a positive MP-fraction.*

Proof. From (5.21), (6.1)–(6.3) and (7.4)–(7.6), we find that

$$\begin{aligned} Z_1 &< 0, \quad Z_n > 0, \quad n = 2, 3, 4, \dots \\ X_2 &> 0, \quad X_{2m} < 0, \quad m = 1, 2, \dots \\ X_{2m+1} &> 0, \qquad m = 0, 1, 2, \dots \\ Y_{2m} &> 0, \qquad m = 1, 2, \dots \\ Y_{2m+1} &< 0, \qquad m = 0, 1, 2, \dots . \end{aligned}$$

So the theorem is proved. □

Remark 7.3. A remark similar to the one given in Remark 4.2 is in order here. The alternating sign for the X_n and Y_n can be avoided if the nth moment is defined with an additional factor $(-1)^n$.

Acknowledgments. The work of the first author is partially supported by the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with the author. The work of the second author was partially supported by the scientific research project PB 96-1029 of the Spanish D.G.E.S.

REFERENCES

1. G.A. Baker, Jr. and P.R. Graves-Morris, *Padé approximants*, vol. 59, *Encyclopedia Math. Appl.*, 2nd ed., Cambridge Univ. Press, 1996.

2. C. Bonan-Hamada, W.B. Jones and O. Njåstad, *Natural solutions of strong Stieltjes moment problems derived from PC-fractions*, in *Orthogonal functions, moment theory and continued fractions* (W.B. Jones and A.S. Ranga, eds.), Lecture Notes in Pure and Appl. Math., vol. 199, Dekker, New York, 1998, pp. 15–30.

3. A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad, *Orthogonal rational functions with poles on the unit circle*, J. Math. Anal. Appl. **182** (1994), 221–243.

4. ———, *Orthogonality and boundary interpolation*, in *Nonlinear numerical methods and rational approximation* II (A.M. Cuyt, ed.), Kluwer Acad. Publ., Dordrecht, 1994, pp. 37–48.

5. ———, *Recurrence relations for orthogonal functions*, in *Continued fractions and orthogonal functions* (S.C. Cooper and W.J. Thron, eds.), Lecture Notes in Pure and Appl. Math., vol. 154, Dekker, New York, 1994, pp. 24–46.

6. ———, *A Favard theorem for rational functions with poles on the unit circle*, East J. Approx. **3** (1997), 21–37.

7. ———, *A rational moment problem on the unit circle*, Methods Appl. Anal. **4** (3) (1997), 283–310.

8. ———, *Continued fractions and orthogonal rational functions*, in *Orthogonal functions, moment theory and continued fractions*: *Theory and applications* (W.B. Jones and A.S. Ranga, eds.), Lecture Notes in Pure and Appl. Math., vol. 199, Dekker, New York, 1998, pp. 69–100.

9. ———, *Interpolation of Nevanlinna functions by rationals with poles on the real line*, in *Orthgonal functions, moment theory and continued fractions*: *Theory and applications* (W.B. Jones and A.S. Ranga, eds.), Lecture Notes in Pure and Appl. Math., vol. 199, Dekker, New York, 1998, pp. 101–110.

10. ———, *Orthogonal rational functions*, Cambridge Monographs Appl. Comput. Math., vol. 5, Cambridge Univ. Press, Cambridge, 1999.

11. ———, *A rational Stieltjes moment problem*, Appl. Math. Comput. **128** (2-3), (2002), 217–235.

12. ———, *Determinacy of a rational moment problem*, J. Comput. Appl. Math. **133** (1-2) (2000), 241–252.

13. ———, *Monotonicity of multi-point Padé approximants*, Comm. Anal. Theory Continued Fractions **8** (2000), 12–27.

14. M.A. Gallucci and W.B. Jones, *Rational approximations corresponding to Newton series* (*Newton-Padé approximants*), J. Approx. Theory **17** (1976), 366–392.

15. A. Gonchar and G. López, *On Markov's theorem for multipoint Padé approximants for functions of Stieltjes type*, Math. USSR-Sb. **105** (1978), 512–524. English transl.: Math. USSR-Sb **34** (1978), 449–459.

16. E. Hendriksen and O. Njåstad, *A Favard theorem for rational functions*, J. Math. Anal. Appl. **142** (2) (1989), 508–520.

17. ———, *Positive multipoint Padé continued fractions*, Proc. Edinburgh Math. Soc. **32** (1989), 261–269.

18. M.E.H. Ismail and D.R. Masson, *Generalized orthogonality and continued fractions*, J. Approx. Theory **83** (1995), 1–40.

19. W.B. Jones, O. Njåstad and W.J. Thron, *Continued fractions associated with the trigonometric moment problem and other strong moment problems*, Constr. Approx. **2** (1986), 197–211.

20. ———, *Perron-Carathéodory continued fractions*, in *Rational approximation and its applications in mathematics and physics* (J. Gilewicz, M. Pindor and W. Siemaszko, eds.), Lecture Notes in Math., vol. 1237, Springer-Verlag, Berlin, 1987, pp. 188–206.

21. W.B. Jones and W.J. Thron, *Continued fractions. Analytic theory and applications*, Addison-Wesley, Reading, Mass., 1980.

22. L. Lorentzen and H. Waadeland, *Continued fractions with applications*, Stud. Comput. Math., vol. 3, North-Holland, Amsterdam, 1992.

23. J.H. McCabe and J.A. Murphy, *Continued fractions which correspond to power series expansions at two points*, J. Inst. Math. Appl. **17** (1976), 233–247.

24. O. Njåstad, *Multipoint Padé approximation and orthogonal rational functions*, in *Nonlinear numerical methods and rational approximation* (A. Cuyt, ed.), D. Reidel Publ. Co., Dordrecht, 1988, pp. 259–270.

Department of Computer Science, K.U. Leuven, Belgium
E-mail address: Adhemar.Bultheel@cs.kuleuven.ac.be

Department of Mathematical Analysis, La Laguna University, Tenerife, Spain
E-mail address: pglez@ull.es

Department of Mathematics, University of Amsterdam, The Netherlands
E-mail address: erik@wins.uva.nl

Department of Math. Sci., Norwegian Univ. of Science and Technology, Trondheim, Norway
E-mail address: njastad@math.ntnu.no