ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 3, Number 3, Fall 2003

ENTIRE FUNCTIONS OF EXPONENTIAL TYPE AND UNIQUENESS CONDITIONS ON THEIR REAL PART

RAPHAËLE SUPPER

ABSTRACT. This paper extends uniqueness results due to Boas and Trembinska, on entire functions with exponential growth whose real part vanishes on lattice points. Here the case is studied where the real part of the function satisfies given relations or assumes prescribed values at lattice points. These results are obtained thanks to such tools as analytic functionals, their Fourier-Borel transform and some operators acting in the space of entire functions in $\mathbf{C}^N$ of exponential type, including difference and differential operators of infinite order with constant coefficients. There are also applications to some difference equation studied by Buck, Boas and Yoshino.

0. Introduction.

Carlson's theorem for entire functions in $\mathbf{C}$ of exponential type $< \pi$ gives rise to various generalizations. It involves entire functions f such that

$$|f(z)| \le Ce^{\tau|z|}, \quad \text{for each } z \in \mathbf{C} \tag{1}$$

where $C > 0$ and $0 < \tau < \pi$ are two constants. This theorem states that such a function f is identically zero in $\mathbf{C}$ as soon as $f(n) = 0$ for each $n \in \mathbf{N}$ (see [**7, 21**]). This uniqueness theorem extends to entire functions in $\mathbf{C}^N$ (see [**4, 16**]) and to harmonic functions in $\mathbf{R}^N$ (see [**2**]), which vanish on $\mathbf{N}^N$ and grow exponentially.

In the case $N = 1$, [**8**] studies the following situation:

Theorem I [**8**]. *A function f entire in $\mathbf{C}$, of exponential type $< \pi$, whose real part vanishes on $\mathbf{Z}$ and $\mathbf{Z} + i$, is constant: $f \equiv ib$, $b \in \mathbf{R}$.*

AMS *Mathematics Subject Classification.* Primary 30D15, 32A15, 42B10, 46F15, Secondary 39A10, 11B68.
Key words and phrases. Entire functions of exponential type, analytic functionals, Fourier-Borel transform, Lagrange interpolation.
Received by the editors on January 18, 2001, and in revised form on April 9, 2001.

Since the conclusion is not "$f \equiv 0$ in $\mathbf{C}$," Theorem I is not properly speaking a uniqueness theorem for f but rather for $(f + \bar{f})/2$, with $\bar{f}$ defined by $\bar{f}(z) = \overline{f(\bar{z})}$ for all $z \in \mathbf{C}$. That is why the conditions "$\Re e f(n) = \Re e f(n+i) = 0$, for all $n \in \mathbf{Z}$" will nevertheless be named *uniqueness conditions*.

Theorem I may be read as a uniqueness theorem for *harmonic functions* of two variables where uniqueness conditions deal with the lattice points on two parallel lines of $\mathbf{C} \simeq \mathbf{R}^2$. More precisely, $\mathbf{Z} \times \{0, k\}$ is a uniqueness set for harmonic functions in $\mathbf{R}^2$ of exponential type $< \pi/k$ with $k \in \mathbf{N}$ (see [**8**]). For a version where uniqueness conditions deal with the lattice points on two intersecting lines: see [**8, 19**].

Favoring the reading in words of *harmonic functions*, Theorem I is the starting point of several works on interpolation of harmonic functions in $\mathbf{R}^2$ (see [**1, 10, 11, 12, 25, 29**]). Theorem I extends to harmonic functions in $\mathbf{R}^N$ where uniqueness conditions deal with the lattice points on two parallel hyperplanes (see [**24, 29, 34**]).

Favoring rather the reading of Theorem I in words of *real part of entire functions*, [**30**] studies the case of entire functions of two variables. This result remains of course valid for entire functions of N variables, $N \geq 2$:

Theorem II [**30**]. *Let f be an entire function in $\mathbf{C}^2$, with the growth*:

$$|f(z_1, z_2)| \leq C e^{\tau(|z_1|+|z_2|)}, \quad \text{for all } (z_1, z_2) \in \mathbf{C}^2 \tag{2}$$

for some constants $C > 0$ and $0 < \tau < \pi$. If the real part of f vanishes on $\mathbf{Z}^2$ and $(\mathbf{Z}+i)^2$, then f is identically zero in $\mathbf{C}^2$, provided that:

(i) *the restriction of f to $\mathbf{R}^2$ belongs to $L^2(\mathbf{R}^2)$*

(ii) $\sum_{(n_1,n_2)\in\mathbf{Z}^2} |f(n_1, n_2)| < +\infty$.

Hypotheses (i) and (ii) are required by the interpolation

$$f(z_1, z_2) = \sum_{(n_1,n_2)\in\mathbf{Z}^2} f(n_1, n_2) \frac{\sin \pi(z_1 - n_1)}{\pi(z_1 - n_1)} \frac{\sin \pi(z_2 - n_2)}{\pi(z_2 - n_2)},$$

see also [**32**]. Here with the technique of analytic functionals, it appears that the conclusion of Theorem II still holds without (i) and that (ii) may be weakened: see Theorem 3 below. Moreover, analytic functionals also allow to include situations where numbers $f(n_1, n_2)$ satisfy some infinite order recurrence relations (Theorem 4) or assume prescribed values (Theorem 6).

1. Statement of results.

1.1. For any $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_N) \in (\mathbf{C}^*)^N$, let M_γ denote the $N \times N$ matrix

$$M_\gamma = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \cdots & \gamma_N \\ 0 & \gamma_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \gamma_N \end{pmatrix}$$

and

$$M_\gamma^{-1} = \begin{pmatrix} \gamma_1^{-1} & -\gamma_1^{-1} & \cdots & \cdots & -\gamma_1^{-1} \\ 0 & \gamma_2^{-1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \gamma_N^{-1} \end{pmatrix}$$

For any compact $K \subset \mathbf{C}^N$, let $M_\gamma K = \{M_\gamma \zeta : \zeta \in K\}$. Let $I_{K,\gamma}$ be the set of those $k \in \mathbf{Z}$ such that the hyperplane $\{\zeta \in \mathbf{C}^N : \langle \gamma, \zeta \rangle = k\pi\}$ intersects K, with the notation $\langle \gamma, \zeta \rangle = \gamma_1\zeta_1 + \cdots + \gamma_N\zeta_N$. The support function H_K of K is defined by $H_K(z) = \max_{\zeta \in K} \Re e\langle z, \zeta \rangle$ for all $z \in \mathbf{C}^N$. Let $\| \cdot \|$ be a norm in $\mathbf{C}^N$ and $\mathrm{Exp}\,(\mathbf{C}^N, K)$ be the space of all entire functions f in $\mathbf{C}^N$ satisfying: for all $\varepsilon > 0$, $C_\varepsilon > 0$ exists such that $|f(z)| \le C_\varepsilon e^{H_K(z) + \varepsilon \|z\|}$ for all $z \in \mathbf{C}^N$. For every function h, entire in $\mathbf{C}^N$, let $\bar{h}$ be the entire function defined by $\bar{h}(z) = \overline{h(\overline{z_1}, \dots, \overline{z_N})}$ for all $z \in \mathbf{C}^N$. For entire functions h in $\mathbf{C}^{N-1}$, a similar definition of $\bar{h}$ holds.

Theorem 1. *Let $\alpha \in \mathbf{C}^N$, $N \ge 2$, such that $\gamma = (\Im m\, \alpha_1, \dots, \Im m\, \alpha_N) \in (\mathbf{R}^*)^N$ and K a (nonempty) convex compact set contained in U^N, where $U = \{u \in \mathbf{C} : |\Im m\, u| < \pi\}$. The functions $f \in \mathrm{Exp}\,(\mathbf{C}^N, K)$ such that*

$$\Re e f(u) = \Re e f(\mu + \alpha) = 0, \quad \forall\, \mu \in \mathbf{N}^N \tag{3}$$

are the functions

$$(4) \quad f(z) = \sum_{k\in I_{K,\gamma}} A_k\Big(\frac{z_2}{\gamma_2} - \frac{z_1}{\gamma_1}, \dots, \frac{z_N}{\gamma_N} - \frac{z_1}{\gamma_1}\Big) e^{k\pi z_1/\gamma_1}, \quad \forall\, z \in \mathbf{C}^N$$

where the functions $A_k \in \mathrm{Exp}\,(\mathbf{C}^{N-1}, L_k)$ *satisfy* $A_k = -\overline{A_k}$ *with compact sets* $L_k \subset \mathbf{C}^{N-1}$ *such that* $\{k\pi\} \times L_k \subset M_\gamma K$. *If* $I_{K,\gamma} = \varnothing$, *then* $f \equiv 0$ *in* $\mathbf{C}^N$. *If* $I_{K,\gamma} = \{0\}$, *then the functions* f *reduce to*

$$(5) \qquad f(z) = A\Big(\frac{z_2}{\gamma_2} - \frac{z_1}{\gamma_1}, \cdots, \frac{z_N}{\gamma_N} - \frac{z_1}{\gamma_1}\Big), \quad \forall\, z \in \mathbf{C}^N,$$

with functions $A \in \mathrm{Exp}\,(\mathbf{C}^{N-1}, L)$ *satisfying* $\bar{A} = -A$ *and compact sets* $L \subset \mathbf{C}^{N-1}$ *such that* $\{0\} \times L \subset M_\gamma K$.

As a situation where $I_{K,\gamma} = \{0\}$, we have for instance

Theorem 2. *Let* K, α *and* γ *be defined as in Theorem* 1. *Suppose that* K *is stable under the maps* $\zeta \mapsto \lambda\zeta$ *for all* $\lambda \in \mathbf{C}$, $|\lambda| \le 1$, *and that its support function* H_K *satisfies* $H_K(\gamma) < \pi$. *Then the functions* $f \in \mathrm{Exp}\,(\mathbf{C}^N, K)$ *satisfying* (3) *are the functions* (5).

This result applies in particular to entire functions f with such a growth as

$$|f(z)| \le C e^{\tau_1|z_1| + \cdots + \tau_N|z_N|}, \quad \forall\, z \in \mathbf{C}^N,$$

where the constants $C > 0$, $\tau_1 > 0, \dots, \tau_N > 0$, satisfy $\displaystyle\sum_{1\le j\le N} \tau_j|\gamma_j| < \pi$.

As for situations where $I_{K,\gamma} \ne \{0\}$, we have

Theorem 3. *Let* K, α *and* γ *be defined as in Theorem* 1.

Suppose that $0 \in I_{K,\gamma}$. *The functions* $f \in \mathrm{Exp}\,(\mathbf{C}^N, K)$ *such that*

(3) $\Re e f(u) = \Re e f(\mu + \alpha) = 0$ *for each* $\mu \in \mathbf{N}^N$

(6) *for any* $\nu = (0, \nu_2, \dots, \nu_N) \in \{0\} \times \mathbf{N}^{N-1}$, *in each of the two cases* $t \to +\infty$ *and* $t \to -\infty$, *the function* $|f(\nu + t\gamma)|$ *does not tend towards* $+\infty$,

are the functions of the form (5) *above.*

Under the additional hypothesis that, for any $\nu = (0, \nu_2, \dots, \nu_N) \in \{0\} \times \mathbf{N}^{N-1}$*, the set of values* $\{f(\nu + t\gamma) : t \in \mathbf{C}\}$ *contains* 0 (*at least in its adherence*), *then* $f \equiv 0$ *in* $\mathbf{C}^N$.

When $0 \notin I_{K,\gamma} \neq \varnothing$: *if* $f \in \operatorname{Exp}(\mathbf{C}^N, K)$ *satisfies* (3) *and* (6), *then* $f \equiv 0$ *in* $\mathbf{C}^N$.

Theorems I and II as special cases of the previous results:

• When $N = 2$, these results include Theorem II [**30**]: for $\alpha = (i, i)$, $\gamma = (1, 1)$ and K the polydisk $B^2 = \{(\zeta_1, \zeta_2) : |\zeta_i| \leq \tau < \pi\} \subset U^2$. Notice that $0 \in I_{K,\gamma} \subset \{-1, 0, 1\}$. Hypothesis (6) "for each $n_2 \in \mathbf{N}$ in each of the two cases $t \to +\infty$ and $t \to -\infty$, the function $|f(t, n_2 + t)|$ does not tend towards $+\infty$" is fulfilled, because the series $\sum_{n_1 \in \mathbf{Z}} |f(n_1, n_2 + n_1)|$ converges, therefore $f(n_1, n_2 + n_1) \to 0$ when $n_1 \to +\infty$ (idem when $n_1 \to -\infty$). Moreover, the set $\{|f(t, n_2 + t)| : t \in \mathbf{C}\}$ contains 0 in its adherence. Hence $f \equiv 0$ in $\mathbf{C}^2$.

• When $N = 1$, Theorem 3 still holds and hypothesis (6) becomes: "in each of the two cases $t \to +\infty$ and $t \to -\infty$, the function $|f(t\gamma)|$ does not tend towards $+\infty$." According to Theorem 2, assumption (6) may even be suppressed when the compact set K is a disk $\{u \in \mathbf{C} : |u| \leq \tau < \pi\}$ (namely when the growth of f is of the kind (1) with $\tau < \pi$) and $|\gamma|\tau < \pi$. Functions A in (5) are now constants. With $\gamma = 1$ we thus include Theorem I.

1.2. In condition (3), the numbers $r_\mu = \Re e f(\mu)$ and $s_\mu = \Re e f(\mu + \alpha)$, $\mu \in \mathbf{N}^N$, are allowed to satisfy more general relations than $r_\mu = s_\mu = 0$ for each $\mu \in \mathbf{N}^N$. For instance, they may satisfy recurrence relations (even of infinite order) such as:

$$\sum_{\nu \in \mathbf{N}^N} a_\nu r_{\nu+\mu} = \sum_{\nu \in \mathbf{N}^N} b_\nu s_{\nu+\mu} = 0, \quad \text{for all } \mu \in \mathbf{N}^N,$$

where the functions $\zeta \mapsto \sum_{\nu \in \mathbf{N}^N} a_\nu e^{\langle \nu, \zeta \rangle}$ and $\zeta \mapsto \sum_{\nu \in \mathbf{N}^N} b_\nu e^{\langle \nu, \zeta \rangle}$ are holomorphic in a neighborhood of $\operatorname{Conv}(K \cup \overline{K})$ and do not vanish at any point of $\operatorname{Conv}(K \cup \overline{K})$. Here $\operatorname{Conv}(K \cup \overline{K})$ denotes the convex hull of $K \cup \overline{K}$ with $\overline{K} = \{z : (\overline{z_1}, \dots, \overline{z_N}) \in K\}$. Analytic functionals provide a natural expression for such relations. We refer to Section 2 for more precisions about analytic functionals T and the Fourier-Borel transform $\mathcal{FB}$.

Definition 1. Given a (nonidentically zero) function φ, holomorphic in a neighborhood of a compact convex set K, let

$$\varphi(D) : \operatorname{Exp}(\mathbf{C}^N, K) \longrightarrow \operatorname{Exp}(\mathbf{C}^N, K)$$

$$f \longmapsto \varphi(D)f = \mathcal{FB}(\varphi T) \quad \text{where } T = \mathcal{FB}^{-1}(f).$$

For any $f \in \operatorname{Exp}(\mathbf{C}^N, K)$ and any $\alpha \in \mathbf{C}^N$, let f_0 and f_α denote the functions of $\operatorname{Exp}(\mathbf{C}^N, K \cup \overline{K})$ defined by $f_0 : z \mapsto [f(z) + \bar{f}(z)]/2$ and $f_\alpha : z \mapsto [f(z+\alpha) + \bar{f}(z+\bar{\alpha})]/2$.

Theorem 4. *Let K, α, γ be defined as in Theorem* 1*. Let φ and ψ be two functions holomorphic in a neighborhood of* $\operatorname{Conv}(K \cup \overline{K})$*, nonzero at any point of* $\operatorname{Conv}(K \cup \overline{K})$*. The functions $f \in \operatorname{Exp}(\mathbf{C}^N, K)$ such that*:

(7) $\quad [\varphi(D)f_0](\mu) = [\psi(D)f_\alpha](\mu) = 0 \quad$ for each $\mu \in \mathbf{N}^N$

are the functions of the form (4)*.*

When $0 \in I_{K,\gamma} \neq \{0\}$, the functions $f \in \operatorname{Exp}(\mathbf{C}^N, K)$ satisfying (6) *and* (7) *are the functions* (5)*. When $0 \notin I_{K,\gamma} \neq \varnothing$: if $f \in \operatorname{Exp}(\mathbf{C}^N, K)$ satisfies* (6) *and* (7)*, then $f \equiv 0$.*

1.3. These results will follow from Theorem 5 below. We refer to Sections 2.3 and 2.6 for the definitions of the analytic functionals $\overline{T}, T^{M_\gamma^{-1}}$ and $T \times S$.

Theorem 5. *Let K be a convex compact set of $\mathbf{C}^N$ and $\gamma \in (\mathbf{C}^*)^N$. Then the analytic functionals T carried by K such that $(e^{2i\langle\gamma,\zeta\rangle} - 1)T_\zeta = 0$ are: $T = \sum_{k \in I_{K,\gamma}} (\delta_{k\pi} \times B_k)^{M_\gamma^{-1}}$ where $\delta_{k\pi}$ is the Dirac's mass at the point $k\pi \in \mathbf{C}$ and the B_k are analytic functionals carried by compacts $L_k \subset \mathbf{C}^{N-1}$ such that $\{k\pi\} \times L_k \subset M_\gamma K$. If $I_{K,\gamma} = \varnothing$, then $T = 0$. Suppose $\gamma \in (\mathbf{R}^*)^N$. If the analytic functionals T are moreover required to satisfy $T = -\overline{T}$, then the corresponding B_k also satisfy $B_k = -\overline{B_k}$ for any $k \in I_{K,\gamma}$.*

1.4. Section 3.3 presents some applications of the operator $\varphi(D)$ to equations of the kind $f(z+1) - f(z) = b(z)$ in the space of entire functions of exponential growth, previously studied by [**7, 9, 33**]. This allows one to study the case where the numbers r_μ and s_μ assume prescribed values and leads to a generalization of Theorem 1:

Theorem 6. *Let K, α, γ be defined as in Theorem* 1. *Suppose that $K = M_\gamma^{-1}(K' \times K'')$ and $\overline{K} = K$, with K' and K'' some compact sets of $\mathbf{C}$ and $\mathbf{C}^{N-1}$ respectively. Let a and $b \in \operatorname{Exp}(\mathbf{C}^N, K)$ be such that $\bar{a} = a$ and $\bar{b} = b$.*

The functions $f \in \operatorname{Exp}(\mathbf{C}^N, K)$ such that $\Re e f(\mu) = a(\mu)$ and $\Re e f(\mu + \alpha) = b(\mu)$ for all $\mu \in \mathbf{N}^N$ are the functions obtained from those of the kind (4) *by adding to them the function*

$$\begin{aligned} z \longmapsto 2\Big\langle B_\zeta, e^{\langle z,\zeta\rangle} \, \frac{1 - Q_{z_1/\gamma_1}(\langle \gamma, \zeta\rangle) e^{-\langle \gamma,\zeta\rangle z_1/\gamma_1}}{e^{\langle \alpha,\zeta\rangle} - e^{\langle \bar{\alpha},\zeta\rangle}} \Big\rangle \\ + 2\Big\langle A_\zeta, e^{\langle z,\zeta\rangle} \, \frac{1 - Q_{z_1/\gamma_1}(\langle \gamma, \zeta\rangle) e^{-\langle \gamma,\zeta\rangle z_1/\gamma_1}}{1 - e^{2i\langle \gamma,\zeta\rangle}} \Big\rangle \\ + \langle A_\zeta, e^{\langle z,\zeta\rangle} Q_{z_1/\gamma_1}(\langle \gamma, \zeta\rangle) e^{-\langle \gamma,\zeta\rangle z_1/\gamma_1} \rangle \end{aligned}$$

where $A = \mathcal{FB}^{-1}(a)$ and $B = \mathcal{FB}^{-1}(b)$ are the analytic functionals whose Fourier-Borel transforms are a and b, respectively, and $Q_u(v)$ is the Lagrange interpolation polynomial of the function $v \mapsto e^{uv}$, $u, v \in \mathbf{C}$, interpolated at the points $v = k\pi$, with $k \in I_{K,\gamma}$.

1.5. The paper is organized as follows. Section 2 gathers various results on analytic functionals and the Fourier-Borel transform. Section 3 is devoted to the operator $\varphi(D)$ introduced in Definition 1. The proofs of Theorems 1–6 are developed in Section 4.

2. Analytic functionals.

2.1. We denote by $\mathcal{H}(\mathbf{C}^N)$ the space of entire functions in $\mathbf{C}^N$, equipped with the topology of the uniform convergence on each compact subset of $\mathbf{C}^N$.

Definition 2. The analytic functionals are the linear forms $T : \mathcal{H}(\mathbf{C}^N) \to \mathbf{C}$ continuous on this space.

Definition 3. Given a (nonempty) compact subset K of $\mathbf{C}^N$, an analytic functional T is said to be carried by K if, for each (relatively compact) neighborhood V of K, there exists a constant $C_V > 0$ such that $|\langle T, h\rangle| \le C_V \sup_V |h|$ for all $h \in \mathcal{H}(\mathbf{C}^N)$.

Notation. $\mathcal{H}'(K)$ stands for the space of analytic functionals carried by K, written $\mathcal{H}'_N(K)$ when there is some ambiguity on the dimension.

For more details on analytic functionals, on the Fourier-Borel transform, i.e. on Sections 2.1 and 2.4 respectively, see [**4, 13, 17, 18, 20, 22**] and [**21**] in the case $N = 1$.

Given an open neighborhood V of K, any analytic functional carried by K is extendable into a continuous linear form on $\mathcal{H}(V)$ (the space of holomorphic functions in V, equipped with the topology of the uniform convergence on each compact of V). This extension is unique provided that V is a Runge's domain. Further a convex compact set has got a system of neighborhoods which are Runge's domains.

2.2. Multiplication by a holomorphic function.

Definition 4. Given $T \in \mathcal{H}'(K)$ and $\varphi \in \mathcal{H}(V)$ where K is a convex compact set of $\mathbf{C}^N$ and V a (Runge's domain) neighborhood of K, the product φT is the linear form $\varphi T : \mathcal{H}(\mathbf{C}^N) \to \mathbf{C}$ defined by $\langle \varphi T, h\rangle = \langle T, \varphi h\rangle$ for all $h \in \mathcal{H}(\mathbf{C}^N)$. Here T is identified in the right member with its (unique) extension to $\mathcal{H}(V)$.

Lemma 1. *With K and φ defined as above, then $\varphi T \in \mathcal{H}'(K)$ for any $T \in \mathcal{H}'(K)$. Letting $B \in \mathcal{H}'(K)$, we have*

(i) *if φ does not vanish at any point of K, then there exists a unique analytic functional $T \in \mathcal{H}'(K)$ such that $\varphi T = B$. It is $T = (1/\varphi)B$.*

(ii) *in the case $N = 1$, let $\alpha \in K$ and $\varphi(\zeta) = \zeta - \alpha$. The analytic functionals $T \in \mathcal{H}'_1(K)$ such that $\varphi T = B$ are: $T = \lambda\delta_\alpha + \theta_\alpha B$ where $\lambda \in \mathbf{C}$, δ_α is the Dirac mass at α, the analytic functional $\theta_\alpha B$ is defined* *by $\langle \theta_\alpha B, h \rangle = \langle B, \theta_\alpha h \rangle$ for all $h \in \mathcal{H}(\mathbf{C})$ and the entire function $\theta_\alpha h$ by $(\theta_\alpha h)(\zeta) = (h(\zeta) - h(\alpha))/(\zeta - \alpha)$ for each $\zeta \neq \alpha$ and $(\theta_\alpha h)(\alpha) = h'(\alpha)$.*

Proofs of Lemmas 1–4 will be omitted for brevity. The interested reader could easily make a proof.

Remark on (i). When $K \subset U^N$, [**4**] provides an integral representation for $(1/\varphi)B$.

Remark on (ii). For $\varphi(\zeta) = (\zeta - \alpha)(\zeta - \beta)$, $\alpha, \beta \in K$, the solutions of $\varphi T = B$ are $T = \mu\delta_\beta + \lambda\theta_\beta\delta_\alpha + \theta_\beta\theta_\alpha B$ $(\lambda, \mu \in \mathbf{C})$ with $\theta_\beta\delta_\alpha = -\delta'_\alpha$ if $\beta = \alpha$ (for the derivative of δ_α, see the notations of Lemma 5), and $\theta_\beta\delta_\alpha = (\delta_\beta - \delta_\alpha)/(\beta - \alpha)$ if $\beta \neq \alpha$.

2.3. Let M be an $N \times N$ matrix with coefficients in $\mathbf{C}$.

Notations. For each $z = (z_1, \dots, z_N) \in \mathbf{C}^N$ and each $E \subset \mathbf{C}^N$, we note $\bar{z} = (\overline{z_1}, \dots, \overline{z_N})$, $\overline{E} = \{\bar{z} : z \in E\}$ and $ME = \{Mz : z \in E\}$. For any function φ defined on E, let $\bar{\varphi}$ be the function defined on $\overline{E}$ by $\bar{\varphi}(z) = \overline{(\varphi(\bar{z}))}$ for all $z \in \overline{E}$. For any function φ defined on ME, let φ^M be defined on E by $\varphi^M : z \mapsto \varphi(Mz)$.

Remarks. a) For example, with $h(z) = e^{\langle \alpha, z \rangle}$, $\alpha \in \mathbf{C}^N$, we have $\bar{h}(z) = e^{\langle \bar{\alpha}, z \rangle}$.

b) If φ expands around the origin into $\varphi(z) = \sum_{\nu \in \mathbf{N}^N} a_\nu z^\nu$, $a_\nu \in \mathbf{C}$, $z^\nu = z_1^{\nu_1} \cdots z_N^{\nu_N}$ for all $\nu = (\nu_1, \dots, \nu_N) \in \mathbf{N}^N$, then $\bar{\varphi}$ has the Taylor development $\bar{\varphi}(z) = \sum_{\nu \in \mathbf{N}^N} \overline{a_\nu} z^\nu$.

c) If $h \in \mathcal{H}(\mathbf{C}^N)$, then $\bar{h} \in \mathcal{H}(\mathbf{C}^N)$ and $(h + \bar{h})/2 \in \mathcal{H}(\mathbf{C}^N)$ coincides on $\mathbf{R}^N$ with $\Re e\, h$.

d) For a function φ holomorphic in a neighborhood of ME, notice that φ^M is holomorphic in a neighborhood of E.

Definition 5. Given an analytic functional T, let $\overline{T} : \mathcal{H}(\mathbf{C}^N) \to \mathbf{C}$ and $T^M : \mathcal{H}(\mathbf{C}^N) \to \mathbf{C}$ be defined by $\langle \overline{T}, h \rangle = \overline{\langle T, \bar{h} \rangle}$ and $\langle T^M, h \rangle = \langle T, h^M \rangle$ for all $h \in \mathcal{H}(\mathbf{C}^N)$.

Lemma 2. (i) *$\overline{T}$ and T^M are analytic functionals and, moreover, if $T \in \mathcal{H}'(K)$ with K a compact of $\mathbf{C}^N$, then $\overline{T} \in \mathcal{H}'(\overline{K})$ and $T^M \in \mathcal{H}'(MK)$;*

(ii) *We have $\bar{h}^M = \overline{h^{\overline{M}}}$ and $\overline{T^M} = \overline{T}^{\overline{M}}$, where $\overline{M}$ stands for the $N \times N$ matrix in which each coefficient is the conjugate of the corresponding coefficient of M;*

(iii) *If $T \in \mathcal{H}'(K)$ with K a compact convex set in $\mathbf{C}^N$, then $\overline{\varphi T} = \bar{\varphi}\overline{T}$ for any function φ holomorphic in some neighborhood of K and $(\varphi^M T)^M = \varphi T^M$ for any function φ holomorphic in some neighborhood of MK;*

(iv) *If M is invertible, we have $(h^M)^{M^{-1}} = h$ and $(T^M)^{M^{-1}} = T$.*

2.4. The link with entire functions of exponential growth.

Definition 6. The Fourier-Borel transform of an analytic functional T is the entire function, written $\hat{T}$ or $\mathcal{FB}(T)$ defined by: $\hat{T}(z) = \langle T_\zeta, e^{\langle z,\zeta\rangle}\rangle$ for all $z \in \mathbf{C}^N$ with $\langle z, \zeta\rangle = \sum_{1\le j\le N} z_j\zeta_j$ (see for instance [**4, 17, 20, 22**]).

In case of ambiguity on the dimension, we will denote this function by $\mathcal{FB}_N(T)$.

Lemma 3. *With the notations of the preceding section, we have:*

(i) $\hat{\overline{T}} = \overline{\hat{T}}$

(ii) *$\widehat{T^M}(z) = \hat{T}({}^tMz)$ for each $z \in \mathbf{C}^N$ with ${}^tM$ the transpose of the matrix M.*

(iii) *For $N = 1$ and $\alpha \in \mathbf{C}$, $\mathcal{FB}_1(\theta_0 T)$ is the primitive (vanishing at $z = 0$) of $\hat{T}$.*

More generally, $\mathcal{FB}_1(\theta_\alpha T)(z) = e^{\alpha z}\int_0^z e^{-\alpha\omega}\hat{T}(\omega)\,d\omega$ for all $z \in \mathbf{C}$.

The entire function $\hat{T}$ is of exponential growth. More precisely, if $T \in \mathcal{H}'(K)$, where K is a compact subset of $\mathbf{C}^N$, then $\hat{T}$ belongs to the space $\mathrm{Exp}\,(\mathbf{C}^N, K)$ defined at the beginning of Section 1.1.

Theorem. *The Fourier-Borel transform* $\mathcal{FB}:\mathcal{H}'(K)\to\mathrm{Exp}(\mathbf{C}^N,K)$ *is injective*; *it is moreover bijective if the compact* K *is convex.*

This fundamental result was established in the case $N=1$ by Pólyà [**23**] and, for any N, by Martineau and Ehrenpreis [**15, 22**] (see also [**17**]).

2.5. Let us also point out this uniqueness theorem of Carlson type due to [**4**] (see also [**16**]).

Theorem. *Let* K *be a convex compact set contained in* U^N *where* U *stands for the horizontal strip* $\{u \in \mathbf{C} : |\Im m\, u| < \pi\}$. *Then* $\mathbf{N}^N$ *is a uniqueness set for the functions of* $\mathrm{Exp}\,(\mathbf{C}^N, K)$, *i.e. every* $f \in \mathrm{Exp}\,(\mathbf{C}^N, K)$ *such that* $f(\nu) = 0$ *for each* $\nu \in \mathbf{N}^N$ *is identically zero in* $\mathbf{C}^N$.

Remarks. a) In this uniqueness result, $\mathbf{N}^N$ may be deprived of a finite number of points, as well as in conditions (3) and (7) in Theorems 1–4.

b) For functions of $\mathrm{Exp}\,(\mathbf{C}^N, MK)$, where M is an $N\times N$ invertible matrix and K a convex compact set of U^N, a uniqueness set is ${}^tM^{-1}\mathbf{N}^N$.

2.6. Juxtaposition of two analytic functionals $T \in \mathcal{H}_1'(K)$ and $S \in \mathcal{H}_{N-1}'(L)$, where K and L are two nonempty compact sets of $\mathbf{C}$ and $\mathbf{C}^{N-1}$, respectively.

For every $z = (z_1, \dots, z_N) \in \mathbf{C}^N$ we write $z_{(1)} = (z_2, \dots, z_N) \in \mathbf{C}^{N-1}$, that is, $z = (z_1, z_{(1)})$, as well as $E_1 = \{z_1 : z \in E\}$ and $E_{(1)} = \{z_{(1)} : z \in E\}$ for every subset $E \subset \mathbf{C}^N$. Note that $H_{K\times L}(z) = H_K(z_1) + H_L(z_{(1)})$.

Definition 7. The juxtaposition of T and S is the linear form $T \times S : \mathcal{H}(\mathbf{C}^N) \mapsto \mathbf{C}$ defined by $\langle T\times S, h\rangle = \langle T_{\zeta_1}, \langle S_{\zeta_{(1)}}, h(\zeta_1, \zeta_{(1)})\rangle\rangle$ for any $h \in \mathcal{H}(\mathbf{C}^N)$.

It is easy to verify that $T\times S$ is an analytic functional with $T\times S \in$

$\mathcal{H}'_N(K \times L)$, that $\mathcal{FB}_N(T \times S)(z) = \mathcal{FB}_1(T)(z_1)\mathcal{FB}_{N-1}(S)(z_{(1)})$ for all $z \in \mathbf{C}^N$ and that $\overline{T \times S} = \overline{T} \times \overline{S}$.

Lemma 4. *Let K be a (nonempty) compact subset of $\mathbf{C}^N$, let $a_1, a_2, \ldots, a_n$ be n distinct points in K_1 and $B_1, B_2, \ldots, B_n \in \mathcal{H}'_{N-1}(K_{(1)})$. If $\sum_{1 \le k \le n} \delta_{a_k} \times B_k = 0$ in $\mathcal{H}'_N(K)$, then $B_j \equiv 0$ in $\mathcal{H}'_{N-1}(K_{(1)})$ for each $j \in \{1, 2, \ldots, n\}$.*

3. An operator in the space $\operatorname{Exp}(\mathbf{C}^N, K)$.

3.1. Some observations about Definition 1.

a) Notation $\varphi(D)$ is explained by the fact that, in the case where $N = 1$ and $\varphi(\zeta) = \zeta$, then $\varphi(D)f = f'$.

b) Since K is convex, there exists, for any $f \in \operatorname{Exp}(\mathbf{C}^N, K)$ a (unique) analytic functional $T \in \mathcal{H}'(K)$ such that $f = \hat{T} = \mathcal{FB}(T)$. Since analytic functional φT is carried by K, see Section 2.2, the function $\varphi(D)(f)$ does belong to $\operatorname{Exp}(\mathbf{C}^N, K)$.

c) If φ does not vanish at any point of K, then transformation $\varphi(D)$ is a bijection.

d) If φ and ψ are holomorphic in a neighborhood of K, then $(\varphi\psi)(D) = \varphi(D) \circ \psi(D)$.

Example 1. With $\varphi(\zeta) = e^{\langle \alpha, \zeta \rangle}$, $\alpha \in \mathbf{C}^N$, $\varphi(D)$ is the translation operator.

$$[\varphi(D)f](z) = f(z + \alpha), \quad \text{for all } z \in \mathbf{C}^N.$$

Example 2. Let φ be the sum of a power series $\sum_{\nu \in \mathbf{N}^N} a_\nu \zeta^\nu$ such that the compact set K is contained in a convergence polydisk $V = \{\zeta \in \mathbf{C}^N : |\zeta_j| < r_j, j = 1, \ldots, N\}$, i.e. the real numbers $r_1 > 0$, $r_2 > 0, \ldots, r_N > 0$ satisfy $\limsup_{|\nu| \to +\infty} (|a_\nu| r_1^{\nu_1} \ldots r_N^{\nu_N})^{1/|\nu|} = 1$. Then $\varphi(D)$ provides in a natural way the differential operator of infinite order in $\operatorname{Exp}(\mathbf{C}^N, K)$:

$$\varphi(D)f = \sum_{\nu \in \mathbf{N}^N} a_\nu D^\nu f$$

for all $f \in \mathrm{Exp}\,(\mathbf{C}^N, K)$, where

$$D^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \cdot \partial z_2^{\nu_2} \cdots \partial z_N^{\nu_N}}$$

and $|\nu| = \nu_1 + \nu_2 + \cdots + \nu_N$.

3.2. Other examples in the case $N = 1$, see [**7**, 247–248, 250–251].

Example 3. When φ is an exponential polynomial, then $\varphi(D)$ defines a difference-differential operator with constant coefficients.

Example 4. $\varphi(D)$ may provide a notion of derivative of order $\alpha \in \mathbf{C} \setminus \mathbf{Z} : f^{(\alpha)} = \mathcal{FB}(\varphi T)$, with $\varphi(\zeta) = \zeta^\alpha = e^{\alpha \log \zeta}$ (if there is a half-line of $\mathbf{C}$ from the origin not intersecting K).

Example 5. When φ is sum of a Dirichlet's series $\sum_{k \in \mathbf{N}} a_k e^{\lambda_k \zeta}$, $\lambda_k > 0$, which converges absolutely in a neighborhood of K, $\varphi(D)$ provides in a natural way a notion of difference operator "of infinite order" in $\mathrm{Exp}\,(\mathbf{C}, K)$.

Example 6. When φ is the sum of a Laurent series $\sum_{k \in \mathbf{Z}} a_k \zeta^k$ whose convergence annulus $V = \{r < |\zeta| < R\}$ contains K, then $\varphi(D)$ defines the differential operator "of infinite order" with constant coefficients: $\varphi(D) f = \sum_{k \in \mathbf{Z}} a_k f^{(k)}$, $f \in \mathrm{Exp}\,(\mathbf{C}, K)$.

Illustration. Let K be a convex compact set contained in $\mathbf{C}^*$, $b \in \mathrm{Exp}\,(\mathbf{C}, K)$ and t fixed in $\mathbf{C}$. The differential equation of infinite order

$$J_0(t) f(z) + \sum_{n \geq 1} J_n(t) [f^{(n)}(z) + (-1)^n f^{(-n)}(z)] = b(z),$$

(J_n standing for the Bessel's function of first kind of order n), has a unique solution in $\mathrm{Exp}\,(\mathbf{C}, K) : f(z) = \langle B_\zeta, e^{\zeta(z-(t/2))+(t/2\zeta)} \rangle$ where $B = \mathcal{FB}^{-1}(b) \in \mathcal{H}'(K)$.

Particular case. If φ is an entire function of exponential type, then $\varphi(D)$ extends to a differential operator acting in $\mathcal{H}(\mathbf{C})$, not only in $\operatorname{Exp}(\mathbf{C}, K)$ (see [**5, 6**]).

Lemma 5. *Let K be a convex compact subset of $\mathbf{C}$ and φ a holomorphic function in a neighborhood of K. Let $\alpha_1, \alpha_2, \dots, \alpha_r$ be the zeros of φ contained in K and $m_1, m_2, \dots, m_r$ their respective multiplicities.*

(i) *The solutions in* $\operatorname{Exp}(\mathbf{C}, K)$ *of $\varphi(D)f = 0$ are the functions*

$$f(z) = \sum_{1 \le j \le r} P_j(z) e^{\alpha_j z} \tag{8}$$
$$(P_j \in \mathbf{C}[z] \text{ of degree } < m_j;\ j = 1, 2, \dots, r).$$

(ii) *The solutions in $\mathcal{H}'(K)$ of $\varphi T = 0$ are the analytic functionals of the kind*

$$T = \sum_{1 \le j \le r} \sum_{0 \le k < m_j} c_{jk} \delta_{\alpha_j}^{(k)}, \quad c_{jk} \in \mathbf{C},$$

where δ_{α_j} denotes the Dirac mass at the point α_j and $\delta_{\alpha_j}^{(k)}$ its kth derivative

$$\langle \delta_{\alpha_j}^{(k)}, h \rangle = (-1)^k \langle \delta_{\alpha_j}, h^{(k)} \rangle = (-1)^k h^{(k)}(\alpha_j), \quad \forall\, h \in \mathcal{H}(\mathbf{C}).$$

Proof. If we denote $P(\zeta) = \prod_{1 \le j \le r} (\zeta - \alpha_j)^{m_j}$ and $\varphi_r(\zeta) = (\varphi(\zeta)/P(\zeta))$, then φ_r is holomorphic in a neighborhood of K and does not vanish at any point of K, hence $\varphi(D)f = 0 \Leftrightarrow \varphi_r(D)[P(D)f] = 0 \Leftrightarrow P(D)f = 0$ whose solutions are the functions (8).

3.3. In the case $N = 1$, application to some difference equations of the kind $f(z+1) - f(z) = b(z)$ (see [**9, 33**]).

Proposition 1. *Let K be a convex compact set of $\mathbf{C} \setminus \{2ik\pi : k \in \mathbf{Z}^*\}$ and $B \in \mathcal{H}'(K)$. The solutions $T \in \mathcal{H}'(K)$ of $(e^\zeta - 1)T = B$ are the analytic functionals defined by*

$$H(\mathbf{C}) \ni h \longmapsto \langle T, h \rangle = c.h(0) + \Big\langle B_\zeta, \frac{h(\zeta) - h(0)}{e^\zeta - 1} \Big\rangle, \quad c \in \mathbf{C}.$$

If K does not contain 0, there is a unique solution $T=B/(e^{\zeta}-1)$, in other words, $c=\langle B_{\zeta}, 1/(e^{\zeta}-1)\rangle$ in both Proposition 1 and Corollary 1 below.

Proof. Let $\varphi(\zeta) = e^{\zeta} - 1$ and $\varphi_1(\zeta) = (e^{\zeta} - 1)/\zeta$, thus $(1/\varphi_1)$ is holomorphic in a neighborhood of K, and $C = (1/\varphi_1)B \in \mathcal{H}'(K)$. According to Lemma 1, the solutions of $\varphi T = B$ are $T = c\delta_0 + \theta_0 C$, $c \in \mathbf{C}$, and $\langle \theta_0 C, h\rangle = \langle (1/\varphi_1)B, \theta_0 h\rangle = \langle B, (1/\varphi_1(\zeta))(h(\zeta) - h(0))/\zeta\rangle = \langle B, (h(\zeta) - h(0))/(e^{\zeta} - 1)\rangle$ for any $h \in \mathcal{H}(\mathbf{C})$. $\square$

Corollary 1. *Let K be a convex compact subset of the disk $\{\zeta \in \mathbf{C} : |\zeta| < 2\pi\}$ and $b \in \operatorname{Exp}(\mathbf{C}, K)$. The solutions $f \in \operatorname{Exp}(\mathbf{C}, K)$ of $f(z+1) - f(z) = b(z)$ are the functions*

$$f(z) = c + \sum_{n\geq 1} b^{(n-1)}(0)\, \frac{\beta_n(z) - \beta_n(0)}{n!}, \quad c \in \mathbf{C}, \quad c = f(0)$$

where the β_n denote the Bernoulli's polynomials.

When $0 \notin K$, there is a unique solution f defined by $f(z) = \langle B_{\zeta}, (1/\zeta)\rangle + \sum_{n\geq 1} b^{(n-1)}(0)(\beta_n(z)/n!)$.

Remark. This proposition applies to entire functions of exponential type $< 2\pi$ and thus extends a result of [**9**, p. 555].

Proof of Corollary 1. With $B = \mathcal{FB}^{-1}(b) \in \mathcal{H}'(K)$, let us apply Proposition 1: $\mathcal{FB}(\theta_0 C)(z) = \langle B_{\zeta}, (e^{z\zeta}-1)/(e^{\zeta}-1)\rangle$ whence Corollary 1 follows, introducing Bernoulli's polynomials and their generating function: $e^{z\zeta}/(e^{\zeta} - 1) = (1/\zeta) + \sum_{n\geq 1}(\beta_n(z)/n!)\zeta^{n-1}$, $0 < |\zeta| < 2\pi$, and noting that $\langle B, \zeta^{n-1}\rangle = b^{(n-1)}(0)$. The series on the right side converges uniformly on each compact subset of the disk $\{\zeta \in \mathbf{C} : |\zeta| < 2\pi\}$ (see [**14**, pp. 297–299]).

Lemmas 1 and 5 also allow including the following result:

Theorem [**7**, p. 111]. *Let f be an entire function of exponential type τ, satisfying $f(z+2\pi) - f(z) = b(z)$, where b is an entire function of* *zero exponential type. Then* $f(z) = \sum_{-n \leq k \leq n} c_k e^{ikz} + g(z)$ *with* $n \leq \tau$ *and* g *entire of zero exponential type.*

This statement extends to entire functions b of exponential type $\sigma \leq \tau$. Then g will be of exponential type $\leq \sigma$. More precisely:

Proposition 2. *Let* K *and* L *be two convex compact sets of* $\mathbf{C}$ ($L \subset K$) *and* $B \in \mathcal{H}'(L)$. *The solutions* $T \in \mathcal{H}'(K)$ *of* $(e^{2\pi\zeta} - 1)T = B$ *are the analytic functionals*:

$$\mathcal{H}(\mathbf{C}) \ni h \longmapsto \langle T, h\rangle = \left\langle B_\zeta, \frac{h(\zeta) - Q_h(\zeta)}{e^{2\pi\zeta} - 1}\right\rangle + \sum_{k \in \mathbf{Z}: ik \in K} c_k h(ik), \quad c_k \in \mathbf{C},$$

where Q_h *is the Lagrange interpolation polynomial of the function* h, *interpolated at the points* $ik \in L \cap i\mathbf{Z}$.

Corollary 2. *Let* K *and* L *be two convex compact sets of* $\mathbf{C}$ ($L \subset K$) *and* $b \in \mathrm{Exp}\,(\mathbf{C}, L)$. *There exists a function* $g \in \mathrm{Exp}\,(\mathbf{C}, L)$, *depending only on* b *and* L, *such that the solutions in* $\mathrm{Exp}\,(\mathbf{C}, K)$ *of* $f(z + 2\pi) - f(z) = b(z)$ *are given by*

$$f(z) = g(z) + \sum_{k \in \mathbf{Z}: ik \in K} c_k e^{ikz}, \quad c_k \in \mathbf{C}.$$

Such a function g *is given by* $g(z) = \langle B_\zeta, (e^{z\zeta} - Q_z(\zeta))/(e^{2\pi\zeta} - 1)\rangle$, *for all* $z \in \mathbf{C}$, *where* $B = \mathcal{FB}^{-1}(b) \in \mathcal{H}'(L)$ *and* Q_z *is the Lagrange interpolation polynomial of the function* $\zeta \mapsto e^{z\zeta}$, *interpolated at the points* $\zeta = ik \in L \cap i\mathbf{Z}$.

Proof of Proposition 2. Here $\varphi(\zeta) = e^{2\pi\zeta} - 1$. The solutions in $\mathcal{H}'(K)$ of $\varphi T = 0$ are the $\sum_{k \in \mathbf{Z}: ik \in K} c_k \delta_{ik}$, according to Lemma 5.

Let $\alpha_1, \alpha_2, \dots, \alpha_r$ denote the elements of $L \cap i\mathbf{Z}$. For any $s \in \{1, 2, \dots, r\}$, let $P_s(\zeta) = \prod_{1 \leq j \leq s} (\zeta - \alpha_j)$ and, for any $h \in \mathcal{H}(\mathbf{C})$ let $Q_{h,s}$ be the Lagrange interpolation polynomial of the function h, interpolated at the points $\alpha_1, \alpha_2, \dots, \alpha_s$.

Let $\varphi_r(\zeta) = \varphi(\zeta)/P_r(\zeta)$ (hence $(1/\varphi_r)$ is holomorphic in a neighborhood of L) and $C = (1/\varphi_r)B \in \mathcal{H}'(L)$. A particular solution of

$\varphi T = B$ in $\mathcal{H}'(L)$ is $\theta_{\alpha_1}\theta_{\alpha_2}\cdots\theta_{\alpha_r}C \in \mathcal{H}'(L)$, see Lemma 1. Since $\langle\theta_{\alpha_1}\theta_{\alpha_2}\cdots\theta_{\alpha_r}C, h\rangle = \langle B, (1/\varphi_r)\theta_{\alpha_r}\ldots\theta_{\alpha_2}\theta_{\alpha_1}h\rangle$, it remains to check that

$$\frac{h(\zeta) - Q_{h,r}(\zeta)}{e^{2\pi\zeta} - 1} = \frac{1}{\varphi_r(\zeta)}\,(\theta_{\alpha_r}\ldots\theta_{\alpha_1}h)(\zeta)$$

for all $\zeta \in L \cup \mathbf{C} \setminus i\mathbf{Z}$. Let us prove by induction that, for any $s \leq r$

$$h(\zeta) - Q_{h,s}(\zeta) = P_s(\zeta)(\theta_{\alpha_s}\ldots\theta_{\alpha_1}h)(\zeta).$$

It is obvious for $s = 1$, since $Q_{h,1}(\zeta) = h(\alpha_1)$. From the recurrence relation

$$Q_{h,s+1}(\zeta) = Q_{h,s}(\zeta) + \frac{P_s(\zeta)}{P_s(\alpha_{s+1})}\,[h(\alpha_{s+1}) - Q_{h,s}(\alpha_{s+1})]$$

it arises that

$$\begin{aligned}(\theta_{\alpha_{s+1}}\theta_{\alpha_s}\ldots\theta_{\alpha_1}h)(\zeta) &= \left[\theta_{\alpha_{s+1}}\left(\frac{h - Q_{h,s}}{P_s}\right)\right](\zeta)\\ &= \frac{1}{\zeta - \alpha_{s+1}}\,\frac{1}{P_s(\zeta)}\,[h(\zeta) - Q_{h,s+1}(\zeta)].\end{aligned}$$

3.4. An analogous result in N variables, $N \geq 2$.

Definition 8. Let $\alpha \in \mathbf{C}$. For any $h \in \mathcal{H}(\mathbf{C}^N)$, let $\vartheta_\alpha h \in \mathcal{H}(\mathbf{C}^N)$ be defined by

$$(\vartheta_\alpha h)(\zeta) = \frac{h(\zeta_1, \zeta_{(1)}) - h(\alpha, \zeta_{(1)})}{\zeta_1 - \alpha}$$

for any $\zeta \in \mathbf{C}^N$ (notation $\zeta_{(1)}$ was introduced in Section 2.6). If $\zeta_1 = \alpha$, then $(\vartheta_\alpha h)(\zeta) = ((\partial h)/(\partial\zeta_1))(\alpha, \zeta_{(1)})$. Let K be a compact subset of $\mathbf{C}^N$. For any $T \in \mathcal{H}'(K)$, let $\vartheta_\alpha T \in \mathcal{H}'(K_1 \times K_{(1)})$ be defined by $\langle\vartheta_\alpha T, h\rangle = \langle T, \vartheta_\alpha h\rangle$ for every $h \in \mathcal{H}(\mathbf{C}^N)$.

Proposition 3. *Let K be a convex compact subset of $\mathbf{C}^N$ and $B \in \mathcal{H}'(K)$. For any $h \in \mathcal{H}(\mathbf{C}^N)$ and any $\omega \in \mathbf{C}^{N-1}$, let $Q_{h,\omega}$ denote here the Lagrange interpolation polynomial of the function $v \mapsto h(v, \omega)$,*

$v \in \mathbf{C}$, interpolated at the points $v = k\pi$, $k \in \mathbf{Z}$ such that $k\pi \in K_1$. Then the analytic functional

$$\mathcal{H}(\mathbf{C}^N) \ni h \mapsto \left\langle B_\zeta, \frac{h(\zeta) - Q_{h,\zeta_{(1)}}(\zeta_1)}{e^{2i\zeta_1} - 1} \right\rangle$$

is carried by $K_1 \times K_{(1)}$.

In particular, with $h(\zeta) = e^{\langle z,\zeta\rangle}$, $z \in \mathbf{C}^N$, then $Q_{h,\zeta_{(1)}}(\zeta_1) = e^{z_2\zeta_2 + \cdots + z_N\zeta_N} Q_{z_1}(\zeta_1) = e^{\langle z,\zeta\rangle} Q_{z_1}(\zeta_1) e^{-\zeta_1 z_1}$ with Q_{z_1} the Lagrange interpolation polynomial of the function $v \mapsto e^{z_1 v}$, $v \in \mathbf{C}$, interpolated at the points $v = k\pi \in K_1 \cap \pi\mathbf{Z}$. Therefore, we have

Corollary 3. *The entire function defined on $\mathbf{C}^N$ by*

$$z \mapsto \left\langle B_\zeta, e^{\langle z,\zeta\rangle} \, \frac{1 - Q_{z_1}(\zeta_1) e^{-\zeta_1 z_1}}{e^{2i\zeta_1} - 1} \right\rangle$$

belongs to $\operatorname{Exp}(\mathbf{C}^N, K_1 \times K_{(1)})$.

It follows that the same holds for the entire functions

$$z \longmapsto \langle B_\zeta, e^{\langle z,\zeta\rangle} (1 - Q_{z_1}(\zeta_1) e^{-\zeta_1 z_1}) \rangle$$

and

$$z \longmapsto \langle B_\zeta, e^{\langle z,\zeta\rangle} Q_{z_1}(\zeta_1) e^{-\zeta_1 z_1} \rangle.$$

Proof of Proposition 3. It works as in the previous proof

$$\frac{h(\zeta) - Q_{h,\zeta_{(1)}}(\zeta_1)}{e^{2i\zeta_1} - 1} = \frac{1}{\varphi_r(\zeta_1)} \, (\vartheta_{\alpha_r} \ldots \vartheta_{\alpha_1} h)(\zeta)$$

with $\alpha_1, \ldots, \alpha_r$ the elements of $K_1 \cap \pi\mathbf{Z}$ and

$$\varphi_r(v) = \frac{e^{2iv} - 1}{(v - \alpha_1) \cdots (v - \alpha_r)}, \quad v \in \mathbf{C}.$$

4. Proof of the theorems.

4.1. Notations K_1 and $K_{(1)}$ were defined in Section 2.6.

Lemma 6. *Given a convex compact set K of $\mathbf{C}^N$ and ψ a function of one variable, holomorphic in a neighborhood of the compact set $K_1 \subset \mathbf{C}$, let $\alpha_1, \alpha_2, \dots, \alpha_r$ be the distinct zeros of ψ contained in K_1 and $m_1, m_2, \dots, m_r$ their respective multiplicities. Let φ be the function (holomorphic in a neighborhood of $K_1 \times \mathbf{C}^{N-1}$) defined by $\varphi(\zeta) = \psi(\zeta_1)$ for all $\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbf{C}^N$. Then*

(i) *the solutions in* $\operatorname{Exp}(\mathbf{C}^N, K)$ *of $\varphi(D)f = \psi(\partial/\partial z_1)f = 0$ are the functions*

$$f(z) = \sum_{1 \le j \le r} \sum_{0 \le k < m_j} C_{jk}(z_2, \dots, z_N) z_1^k e^{\alpha_j z_1},$$

with functions $C_{jk} \in \operatorname{Exp}(\mathbf{C}^{N-1}, L_j)$ for some compact $L_j \subset K_{(1)}$ such that $\{\alpha_j\} \times L_j \subset K$

(ii) *the solutions in $\mathcal{H}'(K)$ of $\varphi T = 0$ are the analytic functionals*

$$T = \sum_{1 \le j \le r} \sum_{0 \le k < m_j} \delta_{\alpha_j}^{(k)} \times B_{jk} \quad \textit{where } B_{jk} \in \mathcal{H}'_{N-1}(L_j).$$

Proof of Lemma 6. It is enough to perform it in two particular cases.

(a) Let $\alpha \in K_1$. The functions $f \in \operatorname{Exp}(\mathbf{C}^N, K)$ satisfying $(\partial f/\partial z_1) - \alpha f = 0$ are the functions $f(z_1, z_2, \dots, z_N) = C(z_2, \dots, z_N)e^{\alpha z_1}$ where $C \in \operatorname{Exp}(\mathbf{C}^{N-1}, L)$ for compact sets L such that $\{\alpha\} \times L \subset K$. Indeed, $C(z_2, \dots, z_N) = f(0, z_2, \dots, z_N)$ and $(z_2, \dots, z_N) \mapsto H_K(0, z_2, \dots, z_N)$ is the support function, in $\mathbf{C}^{N-1}$, of the compact set $K_{(1)} \subset \mathbf{C}^{N-1}$.

(b) Let $\alpha \in K_1$, $\beta \in K_1$, L as in (a), $A \in \operatorname{Exp}(\mathbf{C}^{N-1}, L)$ and P a polynomial of one variable. The solutions in $\operatorname{Exp}(\mathbf{C}^N, K)$ of $(\partial f/\partial z_1) - \beta f = A(z_2, \dots, z_N)P(z_1)e^{\alpha z_1}$ are the functions $f(z_1, z_2, \dots, z_N) = A(z_2, \dots, z_N)Q(z_1)e^{\alpha z_1} + C(z_2, \dots, z_N)e^{\beta z_1}$ where $C \in \operatorname{Exp}(\mathbf{C}^{N-1}, L')$ for compact sets L' such that $\{\alpha\} \times L' \subset K$. The polynomial Q may be defined explicitly in function of P, α and β (it doesn't depend on

A, nor L nor L') with $\deg Q = \deg P$ if $\alpha \neq \beta$ and $\deg Q = \deg P + 1$ if $\alpha = \beta$.

Translation in words of analytic functionals is immediate via the Fourier-Borel transform.

4.2. Proof of Theorem 5.

When $I_{K,\gamma} = \varnothing$, we immediately deduce from Lemma 1(i) that $T = 0$. Let us abbreviate here $M_\gamma = M$. Observe that $e^{2i\langle\gamma,\zeta\rangle} - 1 = \varphi^M(\zeta) = \varphi(M\zeta)$ where

$$\begin{aligned} \varphi : \mathbf{C}^N &\longrightarrow \mathbf{C} \\ \zeta &\longmapsto \psi(\zeta_1) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \psi : \mathbf{C} &\longrightarrow \mathbf{C} \\ u &\longmapsto e^{2iu} - 1. \end{aligned}$$

The question is thus to solve

$$\varphi^M T = 0 \quad \text{in } \mathcal{H}'(K), \tag{9}$$

but since $(\varphi^M T)^M = \varphi T^M$ according to Lemma 2, this is equivalent to solving

$$\varphi S = 0 \quad \text{in } \mathcal{H}'(MK), \tag{10}$$

whose solutions are related to those of (9) by $S = T^M$. Lemma 6 is applied to the compact set MK. The zeros of ψ contained in $(MK)_1 = \{\langle\gamma,\zeta\rangle : \zeta \in K\}$ are the $k\pi$, where $k \in I_{K,\gamma}$, with multiplicity 1. The solutions of (10) are thus the analytic functionals

$$S = \sum_{k \in I_{K,\gamma}} \delta_{k\pi} \times B_k$$

where the B_k are analytic functionals carried by some compact $L_k \subset (MK)_{(1)}$ such that $\{k\pi\} \times L_k \subset MK$. Finally the solutions of (9) are the analytic functionals $T = \sum_{k \in I_{K,\gamma}} (\delta_{k\pi} \times B_k)^{M^{-1}}$. These analytic functionals $(\delta_{k\pi} \times B_k)^{M^{-1}}$ are carried by $M^{-1}(\{k\pi\} \times L_k) \subset K$.

In the case $\gamma \in (\mathbf{R}^*)^N$, the additional condition $\overline{T} = -T$ leads to $T^M = -\overline{T}^M = -\overline{T^M}$ (because $M = M_\gamma$ has real coefficients). As $T^M = \sum_{k \in I_{K,\gamma}} \delta_{k\pi} \times B_k$, it follows that

$$\sum_{k \in I_{K,\gamma}} \delta_{k\pi} \times (B_k + \overline{B_k}) = 0 \quad \text{in } \mathcal{H}'_N(K).$$

According to Lemma 4, $B_k = -\overline{B_k}$ for any k.

Corollary 4. *Let K, γ and $I_{K,\gamma}$ be defined as in Theorem 5. Those $f \in \operatorname{Exp}(\mathbf{C}^N, K)$ which satisfy $f(z+2i\gamma) = f(z)$, for all $z \in \mathbf{C}^N$, are the functions*

$$f(z) = \sum_{k \in I_{K,\gamma}} A_k \left(\frac{z_2}{\gamma_2} - \frac{z_1}{\gamma_1}, \dots, \frac{z_N}{\gamma_N} - \frac{z_1}{\gamma_1} \right) e^{k\pi z_1/\gamma_1}$$

where $A_k \in \operatorname{Exp}(\mathbf{C}^{N-1}, L_k)$, for some compact L_k such that $\{k\pi\} \times L_k \subset M_\gamma K$. (If $I_{K,\gamma} = \varnothing$, then $f \equiv 0$.) If f is moreover compelled to $\bar{f} = -f$ and $\gamma \in (\mathbf{R}^*)^N$, then A_k must satisfy $\overline{A_k} = -A_k$ for any $k \in I_{K,\gamma}$.

Proof. Let $T = \mathcal{FB}^{-1}(f)$ and $S = T^M$ as in the previous proof. The translation of (9) in words of entire functions of exponential growth is written $f(z+2i\gamma) - f(z) = 0$ in $\operatorname{Exp}(\mathbf{C}^N, K)$. With the notation $Z = {}^tM^{-1}z$, it follows from Lemma 3 that

$$\begin{aligned} f(z) = \mathcal{FB}_N(S^{M^{-1}})(z) &= \sum_{k \in I_{K,\gamma}} \mathcal{FB}_N(\delta_{k\pi} \times B_k)(Z) \\ &= \sum_{k \in I_{K,\gamma}} \mathcal{FB}_1(\delta_{k\pi})(Z_1) \mathcal{FB}_{N-1}(B_k)(Z_{(1)}) \end{aligned}$$

hence Corollary 4, with $A_k = \mathcal{FB}_{N-1}(B_k) \in \operatorname{Exp}(\mathbf{C}^{N-1}, L_k)$. □

4.3. Proof of Theorems 1, 3 and 4.

The entire function $f_0 : z \mapsto [f(z) + \overline{f(\bar{z})}]/2$ coincides with $\Re e f$ on $\mathbf{R}^N$ and belongs to $\operatorname{Exp}(\mathbf{C}^N, K \cup \overline{K})$; notice that f_0 is the Fourier-Borel transform of $(T + \overline{T})/2$, where T stands for $\mathcal{FB}^{-1}(f) \in \mathcal{H}'(K)$.

Similarly, the entire function $f_\alpha : z \mapsto [f(z+\alpha) + \overline{f(\bar{z}+\alpha)}]/2$ coincides on $\mathbf{R}^N$ with $x \mapsto \Re e f(x+\alpha)$ and belongs to $\operatorname{Exp}(\mathbf{C}^N, K \cup \overline{K})$; observe that f_α is the Fourier-Borel transform of

$$\frac{e^{\langle \alpha, \zeta \rangle} T_\zeta + e^{\langle \bar{\alpha}, \zeta \rangle} \overline{T}_\zeta}{2} = \frac{e^{\langle \bar{\alpha}, \zeta \rangle} (e^{2i\langle \gamma, \zeta \rangle} T_\zeta + \overline{T}_\zeta)}{2}.$$

According to the uniqueness theorem [**4**] (see Section 2.5) applied to the compact $\mathrm{Conv}\,(K \cup \overline{K})$, these functions are both identically zero in $\mathbf{C}^N$ since they vanish on $\mathbf{N}^N$ (maybe deprived of a finite number of points).

For the proof of Theorem 4, the same argument applies to the functions $\varphi(D)f_0$ and $\psi(D)f_\alpha$ and leads to $\varphi(D)f_0 \equiv 0$ and $\psi(D)f_\alpha \equiv 0$; it follows then from Lemma 1 that $f_0 \equiv 0$ and $f_\alpha \equiv 0$ in $\mathbf{C}^N$. We obtain $\bar{f} = -f$ and $f(z + \alpha - \bar{\alpha}) = f(z)$ for any $z \in \mathbf{C}^N$. According to Corollary 4,

$$f(z) = \sum_{k \in I_{K,\gamma}} A_k\left(\frac{z_2}{\gamma_2} - \frac{z_1}{\gamma_1}, \dots , \frac{z_N}{\gamma_N} - \frac{z_1}{\gamma_1}\right) e^{k\pi z_1/\gamma_1}$$

with functions $A_k \in \mathrm{Exp}\,(\mathbf{C}^{N-1}, L_k)$ satisfying $A_k = -\overline{A_k}$ and compact sets L_k such that $\{k\pi\} \times L_k \subset M_\gamma K = MK$. Theorem 1 is now established.

If $I_{K,\gamma} \neq \varnothing$ and $I_{K,\gamma} \neq \{0\}$, it remains to prove that $A_k \equiv 0$ in $\mathbf{C}^{N-1}$ for each $k \neq 0$. Let $M_{(1)}$ denote the square matrix, extracted from M, built with its last $N-1$ lines and columns. As $M_{(1)}^{-1}\mathbf{N}^{N-1}$ is a uniqueness set for A_k, we will show that $A_k(\nu') = 0$ for each $\nu' = [(\nu_2/\gamma_2), \dots , (\nu_N/\gamma_N)]$, where $(\nu_2, \dots , \nu_N) \in \mathbf{N}^{N-1}$. On the line in $\mathbf{C}^N$ of equation

$$\begin{cases} z_2 = \nu_2 + z_1(\gamma_2/\gamma_1) \\ \vdots \\ z_N = \nu_N + z_1(\gamma_N/\gamma_1) \end{cases}$$

we have $f(z) = \sum_{k \in I_{K,\gamma}} A_k(\nu')e^{k\pi z_1/\gamma_1}$. This line of directing vector γ, drawn through the point $\nu = (0, \nu_2, \dots , \nu_N) \in \{0\} \times \mathbf{N}^{N-1}$, also has equation $z = \nu + t\gamma$, $t \in \mathbf{C}$.

Since K is convex, it is easy to verify that $I_{K,\gamma}$ is an "interval" of $\mathbf{Z}$. Let us fix $k_0 \in \mathbf{N}^*$ such that $I_{K,\gamma} \subset [-k_0, k_0]$ and write $A_k(\nu') = 0$ for $k \notin I_{K,\gamma}$. Thus

$$\begin{aligned} f(\nu + t\gamma) &= \sum_{-k_0 \le k \le k_0} A_k(\nu')e^{k\pi t} \\ &= e^{k_0\pi t}\left[A_{k_0}(\nu') + \sum_{-k_0 \le k < k_0} A_k(\nu')e^{(k-k_0)\pi t}\right]. \end{aligned}$$

When $t \in \mathbf{R}$ and tends toward $+\infty$, the second term in the square brackets tends towards zero. Thus $A_{k_0}(\nu') = 0$, otherwise $|f(\nu + t\gamma)|$ would tend towards $+\infty$ as $t \to +\infty$. Similarly, with $e^{(k_0-1)\pi t}$ factored out, we obtain $A_{k_0-1}(\nu') = 0$, and so on until $A_1(\nu') = 0$. Hence

$$f(\nu + t\gamma) = e^{-k_0\pi t}\left[A_{-k_0}(\nu') + \sum_{-k_0<k\leq 0} A_k(\nu')e^{(k+k_0)\pi t}\right].$$

When t tends towards $-\infty$, $t \in \mathbf{R}$, the second term in the square brackets tends towards zero. Thus $A_{-k_0}(\nu') = 0$, otherwise $|f(\nu + t\gamma)|$ would tend towards $+\infty$ when $t \to -\infty$. Similarly

$$A_{-k_0+1}(\nu') = \cdots = A_{-1}(\nu') = 0.$$

When the "interval" $I_{K,\gamma}$ of $\mathbf{Z}$ does not contain 0, then $I_{K,\gamma} \subset \mathbf{N}^*$ or $I_{K,\gamma} \subset -\mathbf{N}^*$ so that $A_k(\nu') = 0$ for all $k \in I_{K,\gamma}$.

4.4. The above proof also shows that, when $I_{K,\gamma} = \{0\}$, the condition (6) becomes useless in the statement of Theorem 3. Proof of Theorem 2 reduces to

Lemma 7. *Let K be a compact of $\mathbf{C}^N$, stable under the maps $\zeta \mapsto \lambda\zeta$ for all $\lambda \in \mathbf{C}$, $|\lambda| \leq 1$. Let $\gamma \in \mathbf{C}^N$ and $r \in \mathbf{C}$. Then K intersects the hyperplane $\{\zeta \in \mathbf{C}^N : \langle\zeta, \gamma\rangle = r\}$ if and only if $H_K(\gamma) \geq |r|$.*

Proof of Lemma 7. Let us write $r = |r|e^{i\alpha}$, $\alpha \in \mathbf{R}$.

Let $\zeta \in K$ be such that $\langle\zeta, \gamma\rangle = r$ and $\zeta' = e^{-i\alpha}\zeta \in K$, then $\langle\zeta', \gamma\rangle = |r| = \Re e\langle\zeta', \gamma\rangle \leq H_K(\gamma)$. Conversely, let $\zeta \in K$ be such that $\Re e\langle\zeta, \gamma\rangle \geq |r|$ and $\theta \in \mathbf{R}$ such that $e^{i\theta}\langle\zeta, \gamma\rangle \in \mathbf{R}^+$. We have $\zeta' = e^{i\theta}\zeta \in K$ and $\langle\zeta', \gamma\rangle \geq \Re e\langle\zeta, \gamma\rangle \geq |r|$. There exists $\zeta_r \in K$ such that $\langle\zeta_r, \gamma\rangle = r$; for instance, $\zeta_r = 0$ if $\langle\zeta', \gamma\rangle = 0$. Otherwise, let $\lambda = |r|/\langle\zeta', \gamma\rangle \in [0, 1]$; then $\zeta_r = \lambda e^{i\alpha}\zeta' \in K$ and $\langle\zeta_r, \gamma\rangle = r$. □

Proof of Theorem 2. Since $H_K(\gamma) < \pi$, K does not intersect any hyperplane $\{\zeta \in \mathbf{C}^N : \langle\zeta, \gamma\rangle = k\pi\}$ where $k \in \mathbf{Z}^*$. Therefore $I_{K,\gamma} = \{0\}$. □

Remark. Lemma 7 and Theorem 2 apply in particular when K is complete multicircular (cf. Reinhardt's domains, see [**31**, pp. 47–48]), for instance, when K is a polydisk $K=\{\zeta\in\mathbf{C}^N : |\zeta_j|\le r_j, j=1,\dots,N\}$ whose radii satisfy $\sum_{1\le j\le N} r_j|\gamma_j| < \pi$, since its support function H_K is known to be defined by $H_K(z) = \sum_{j=1}^N r_j|z_j|$ for all $z\in\mathbf{C}^N$.

4.5. Proof of Theorem 6.

Let f_1 and f_2 be the entire functions defined by

$$f_1(z) = 2\Big\langle B_\zeta, e^{\langle z,\zeta\rangle}\, \frac{1-Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}}{e^{\langle\alpha,\zeta\rangle}-e^{\langle\bar\alpha,\zeta\rangle}}\Big\rangle$$

and

$$\begin{aligned} f_2(z) = 2\Big\langle A_\zeta, e^{\langle z,\zeta\rangle}\, &\frac{1-Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}}{1-e^{2i\langle\gamma,\zeta\rangle}}\Big\rangle \\ &+ \langle A_\zeta, e^{\langle z,\zeta\rangle} Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}\rangle \end{aligned}$$

for any $z\in\mathbf{C}^N$. First verify that they belong to $\operatorname{Exp}(\mathbf{C}^N, K)$ or rather verify that $f_1^{{}^tM}$ and $f_2^{{}^tM}$ belong to $\operatorname{Exp}(\mathbf{C}^N, MK)$. If one remembers that $\langle {}^tMz,\zeta\rangle = \langle z, M\zeta\rangle$, see Lemma 3, this leads to

$$f_1^{{}^tM}(z) = f_1({}^tMz) = 2\Big\langle B_\zeta^M, e^{-\langle {}^tM^{-1}\bar\alpha,\zeta\rangle} e^{\langle z,\zeta\rangle}\, \frac{1-Q_{z_1}(\zeta_1)e^{-\zeta_1 z_1}}{e^{2i\zeta_1}-1}\Big\rangle$$

and

$$\begin{aligned} f_2^{{}^tM}(z) = f_2({}^tMz) = 2\Big\langle A_\zeta^M, e^{\langle z,\zeta\rangle}\, &\frac{1-Q_{z_1}(\zeta_1)e^{-\zeta_1 z_1}}{1-e^{2i\zeta_1}}\Big\rangle \\ &+ \langle A_\zeta^M, e^{\langle z,\zeta\rangle} Q_{z_1}(\zeta_1)e^{-\zeta_1 z_1}\rangle \end{aligned}$$

for all $z\in\mathbf{C}^N$. It follows from Proposition 3 and Corollary 3 that

$$f_1^{{}^tM} \in \operatorname{Exp}(\mathbf{C}^N, (MK)_1\times(MK)_{(1)}).$$

The same holds for $f_2^{{}^tM}$.

Next verify that f_1 satisfies $f_1(z)+\overline{f_1(\bar z)} = 0$ and $f_1(z+\alpha)+\overline{f_1(\bar z+\alpha)} = 2b(z)$ for all $z\in\mathbf{C}^N$. The last relation may also be written as

$$f_1(z)+\overline{f_1(\bar z+2i\gamma)} = 2b(z-\alpha), \quad \text{for all } z\in\mathbf{C}^N. \tag{11}$$

Both relations follow from the fact that $\overline{Q_{\bar{u}}} = Q_u$ for each $u \in \mathbf{C}$ and that $\overline{\langle B, h\rangle} = \langle \overline{B}, \bar{h}\rangle = \langle B, \bar{h}\rangle$ for every function h holomorphic in a neighborhood of K. Since

$$f_1(z) = 2\Big\langle B_\zeta, e^{\langle z-\alpha,\zeta\rangle}\, \frac{1 - Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}}{1 - e^{-2i\langle\gamma,\zeta\rangle}}\Big\rangle$$

and $Q_{u+2i} = Q_u$, it follows that

$$f_1(z + 2i\gamma) = 2\Big\langle B_\zeta, e^{\langle z-\bar{\alpha},\zeta\rangle}\, \frac{1 - e^{-2i\langle\gamma,\zeta\rangle}Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}}{1 - e^{-2i\langle\gamma,\zeta\rangle}}\Big\rangle$$

which leads to

$$\begin{aligned}\overline{f_1(\bar{z} + 2i\gamma)} &= 2\Big\langle B_\zeta, e^{\langle z-\alpha,\zeta\rangle}\, \frac{1 - e^{2i\langle\gamma,\zeta\rangle}Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}}{1 - e^{2i\langle\gamma,\zeta\rangle}}\Big\rangle \\ &= -2\Big\langle B_\zeta, e^{\langle z-\alpha,\zeta\rangle}\, \frac{e^{-2i\langle\gamma,\zeta\rangle} - Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}}{1 - e^{-2i\langle\gamma,\zeta\rangle}}\Big\rangle\end{aligned}$$

hence (11) is fulfilled.

It remains to show that $f_2(z) + \overline{f_2(\bar{z})} = 2a(z)$ and $f_2(z+\alpha) + \overline{f_2(\bar{z}+\alpha)} = 0$ for all $z \in \mathbf{C}^N$. The second relation may also be written as

$$f_2(z) + \overline{f_2(\bar{z} + 2i\gamma)} = 0, \quad \text{for all } z \in \mathbf{C}^N. \tag{12}$$

Since

$$\begin{aligned}\overline{f_2(\bar{z})} = -2\Big\langle A_\zeta, e^{\langle z,\zeta\rangle}(1 - Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1})\, \frac{e^{2i\langle\gamma,\zeta\rangle}}{1 - e^{2i\langle\gamma,\zeta\rangle}}\Big\rangle \\ + \langle A_\zeta, e^{\langle z,\zeta\rangle}Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}\rangle\end{aligned}$$

we obtain:

$$\begin{aligned}f_2(z) + \overline{f_2(\bar{z})} &= 2\Big\langle A_\zeta, e^{\langle z,\zeta\rangle}\Big(1 - Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}\Big)\Big\rangle \\ &\quad + (1+1)\langle A_\zeta, e^{\langle z,\zeta\rangle}Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}\rangle = 2a(z).\end{aligned}$$

The relation (12) follows similarly from

$$\overline{f_2(\bar{z}+2i\gamma)} = 2\Big\langle A_\zeta, e^{\langle z,\zeta\rangle} e^{-2i\langle\gamma,\zeta\rangle}\, \frac{1-e^{2i\langle\gamma,\zeta\rangle}Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}}{1-e^{-2i\langle\gamma,\zeta\rangle}}\Big\rangle$$

$$+ \langle A_\zeta, e^{\langle z,\zeta\rangle} Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}\rangle$$

$$= -2\Big\langle A_\zeta, e^{\langle z,\zeta\rangle}\, \frac{1-e^{2i\langle\gamma,\zeta\rangle}Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}}{1-e^{2i\langle\gamma,\zeta\rangle}}\Big\rangle$$

$$+ \langle A_\zeta, e^{\langle z,\zeta\rangle} Q_{z_1/\gamma_1}(\langle\gamma,\zeta\rangle)e^{-\langle\gamma,\zeta\rangle z_1/\gamma_1}\rangle.$$

REFERENCES

1. K.F. Andersen, *On the representation of harmonic functions by their values on lattice points*, J. Math. Anal. Appl. **49** (1975), 692–695.

2. V. Avanissian, *Cellules d'harmonicité et prolongement analytique complexe*, Travaux en cours, Hermann, Paris, 1985.

3. ———, *Sur les fonctionnnelles analytiques et la transformation G, Applications*, Exposition. Math. **14** (1996), 193–225.

4. V. Avanissian and R. Gay, *Sur une transformation des fonctionnelles analytiques et ses applications aux fonctions entières de plusieurs variables*, Bull. Soc. Math. France **103** (1975), 341–384.

5. L. Bernal-Gonzalez, *On universal entire functions with zero-free derivatives*, Arch. Math. **68** (1997), 145–150.

6. J.P. Bézivin and F. Gramain, *Solutions entières d'un système d'équations aux différences*, Ann. Inst. Fourier (Grenoble) **43** (1993), 791–814.

7. R.P. Boas, Jr., *Entire functions*, Academic Press, Inc., New York, 1954.

8. ———, *An uniqueness theorem for harmonic functions*, J. Approx. Theory **5** (1972), 425–427.

9. R.C. Buck, *A class of entire functions*, Duke Math. J. **13** (1946), 541–559.

10. C.H. Ching, *An interpolation formula for harmonic functions*, J. Approx. Theory **15** (1975), 50–53.

11. C.H. Ching and C.K. Chui, *A representation formula for harmonic functions*, Proc. Amer. Math. Soc. **39** (1973), 349–352.

12. C.K. Chui and G.A. Roberts, *An interpolation formula for harmonic functions on the set of integers*, J. Approx. Theory **29** (1980), 144–150.

13. P.D. Cordaro and F. Trèves, *Hyperfunctions on hypo-analytic manifolds*, Princeton Univ. Press, Princeton, 1994.

14. J. Dieudonné, *Calcul infinitésimal*, Hermann, Paris, 1968.

15. L. Ehrenpreis, *A fundamental principle for systems of linear differential equations with constant coefficients and some of its applications*, Proc. Internat. Symp. on Linear Spaces (Jerusalem, 1961), pp. 161–174.

16. F. Gramain, *Fonctions entières arithmétiques*, Lecture Notes in Math. **694** (1976/77), 96–125.

17. L. Hörmander, *An introduction to complex analysis in several variables*, D. van Nostrand Co., Princeton, 1966.

18. ———, *Notions of convexity*, Basel, Boston, 1994.

19. Y.A. Kazmin, *On a theorem of F. Carlson*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **101** (1992), 37–43, in Russian; transl. in Moscow Univ. Math. Bull. **47** (1992), 32–37.

20. P. Lelong and L. Gruman, *Entire functions of several complex variables*, Grundlehren Math. Wiss. **282**, Springer, New York, 1986.

21. B.Y. Levin, *Lectures on entire functions* (in collaboration with Yu. Lyubarskii, M. Sodin and V. Tkachenko), Transl. Math. Monographs, Amer. Math. Soc., Providence, 1996.

22. A. Martineau, *Sur les fonctionnelles analytiques et la transformation de Fourier-Borel*, J. Anal. Math. de Jérusalem **11** (1963), 1–164.

23. G. Pólyà, *Untersuchungen über Lücken und Singularitäten von Potenzreihen*, Math. Z. **19** (1929), 549–640.

24. N.V. Rao, *Carlson theorem for harmonic functions in* $\mathbf{R}^n$, J. Approx. Theory **12** (1974), 309–314.

25. Q.I. Rahman and G. Schmeisser, *Representation of entire harmonic functions by given values*, J. Math. Anal. Appl. **115** (1986), 461–469.

26. R. Supper, *Exemples d'application des fonctionnelles analytiques*, Complex Variables Theory Appl. **18** (1992), 201–212.

27. ———, *Fonctionnelles analytiques liées aux fonctions spéciales et fonctions arithmétiques au sens d'Abel*, Thèse 508/TS-35, Publication de l'I.R.M.A., I.S.S.N. 0755-3390, 1992.

28. ———, *Quelques applications liées aux fonctionnelles analytiques*, Gaz. Math. **57** (1993), 65–79.

29. ———, *Formules d'interpolation et théorèmes d'unicité pour des fonctions harmoniques de type exponentiel*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), 299–310.

30. A.M. Trembinska, *A uniqueness theorem for entire functions of two complex variables*, J. Math. Anal. Appl. **158** (1991), 456–465.

31. V.S. Vladimirov, *Methods of the theory of functions of many complex variables*, English transl. (L. Ehrenpreis, ed.), The M.I.T. Press, Cambridge, MA, 1966.

32. K. Yoshino, *Liouville type theorems for entire functions of exponential type*, Complex Variables Theory Appl. **5** (1985), 21–51.

33. ———, *Difference equation in the space of holomorphic functions of exponential type and Ramanujan summation*, reprint September 1996.

34. D. Zeilberger, *Uniqueness theorems for harmonic functions of exponential type*, Proc. Amer. Math. Soc. **61** (1976), 335–340.

Université Louis Pasteur, UFR de Mathématique et Informatique, URA CNRS 001, 7 rue René Descartes, F-67 084 Strasbourg Cedex, France
E-mail address: supper@math.u-strasbg.fr